ISyE 4803

Advanced Supply Chain Logistics

Fall 2009

Homework 1 solution

Question 1 (25 points)

Let $N = \{1,2,3,4,5,6\}$ denote the given set of nodes, and let $E = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$ denote the given set of edges. Let $G = (N,E)$ denote the given undirected network. Let c_{ij} be the cost associated with edge $\{i,j\}$.

1. Write down a formulation to efficiently compute a good lower bound for the optimal objective value of the traveling salesman problem on the network.
   Answer: Choose any node $r \in N$ as the special "root" node. Then the following is a relaxation, that gives a lower bound for the optimal objective value of the traveling salesman problem on the network.
   Let $x_{ij} = \begin{cases} 1 & \text{if edge } \{i,j\} \text{ is used} \\ 0 & \text{o/w} \end{cases}$

$$
\begin{array}{lrcll}
\min & \sum_{\{i,j\}\in E} c_{ij}x_{ij} & & & \\
\text{s.t.} & & & & \\
& \sum_{\{i\in N:\{i,r\}\in E\}} x_{ir} + \sum_{\{i\in N:\{r,i\}\in E\}} x_{ri} & = & 2 & \\
& \sum_{\{j\in N\setminus\{r\}:\{j,i\}\in E\}} x_{ji} + \sum_{\{j\in N\setminus\{r\}:\{i,j\}\in E\}} x_{ij} & \geq & 1 & \forall\, i \in N\setminus\{r\} \\
& \sum_{\{\{i,j\}\in E:i\in S, j\notin S\cup\{r\}\}} x_{ij} + \sum_{\{\{i,j\}\in E:i\notin S\cup\{r\}, j\in S\}} x_{ij} & \geq & 1 & \forall\, S \subset N\setminus\{r\} \;:\; 2 \leq |S| \leq \left\lfloor \frac{|N-1|}{2} \right\rfloor \\
& x_{ij} & \in & \{0,1\} & \forall\, \{i,j\} \in E
\end{array}
$$

For example, if you choose $r = 1$, then the formulation is

$$
\begin{array}{lrcll}
\min & \{20x_{1,2} + 27x_{1,3} + 22x_{1,4} + 25x_{1,5} + 21x_{1,6} & & & \\
 & +26x_{2,3} + 31x_{2,4} + 29x_{2,5} + 23x_{2,6} + 28x_{3,4} & & & \\
 & +32x_{3,5} + 33x_{3,6} + 30x_{4,5} + 34x_{4,6} + 24x_{5,6}\} & & & \\
\text{s.t.} & & & & \\
 & x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} & = & 2 & \\
 & x_{2,3} + x_{2,4} + x_{2,5} + x_{2,6} & \geq & 1 & \\
 & x_{2,3} + x_{3,4} + x_{3,5} + x_{3,6} & \geq & 1 & \\
 & x_{2,4} + x_{3,4} + x_{4,5} + x_{4,6} & \geq & 1 & \\
 & x_{2,5} + x_{3,5} + x_{4,5} + x_{5,6} & \geq & 1 & \\
 & x_{2,6} + x_{3,6} + x_{4,6} + x_{5,6} & \geq & 1 & \\
 & x_{2,4} + x_{2,5} + x_{2,6} + x_{3,4} + x_{3,5} + x_{3,6} & \geq & 1 & S = \{2,3\} \\
 & x_{2,3} + x_{2,5} + x_{2,6} + x_{3,4} + x_{4,5} + x_{4,6} & \geq & 1 & S = \{2,4\} \\
 & x_{2,3} + x_{2,4} + x_{2,6} + x_{3,5} + x_{4,5} + x_{5,6} & \geq & 1 & S = \{2,5\} \\
 & x_{2,3} + x_{2,4} + x_{2,5} + x_{3,6} + x_{4,6} + x_{5,6} & \geq & 1 & S = \{2,6\} \\
 & x_{2,3} + x_{2,4} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6} & \geq & 1 & S = \{3,4\} \\
 & x_{2,3} + x_{2,5} + x_{3,4} + x_{3,6} + x_{4,5} + x_{5,6} & \geq & 1 & S = \{3,5\} \\
 & x_{2,3} + x_{2,6} + x_{3,4} + x_{3,5} + x_{4,6} + x_{5,6} & \geq & 1 & S = \{3,6\} \\
 & x_{2,4} + x_{2,5} + x_{3,4} + x_{3,5} + x_{4,5} + x_{5,6} & \geq & 1 & S = \{4,5\} \\
 & x_{2,4} + x_{2,6} + x_{3,4} + x_{3,6} + x_{4,5} + x_{5,6} & \geq & 1 & S = \{4,6\} \\
 & x_{2,5} + x_{2,6} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6} & \geq & 1 & S = \{5,6\} \\
 & x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}, x_{1,6}, x_{2,3}, x_{2,4}, x_{2,5}, x_{2,6}, x_{3,4}, x_{3,5}, x_{3,6}, x_{4,5}, x_{4,6}, x_{5,6} & \in & \{0,1\} &
\end{array}
$$

The relaxation above is known as the minimum-cost spanning r-tree problem, also known as the 1-tree problem.

2. Use an algorithm of your choice and compute a good lower bound for the optimal objective value of the traveling salesman problem on the network. Report your lower bound and the solution with which you obtained the lower bound.
Answer: Use the Prim algorithm or the Kruskal algorithm to find a minimum spanning tree on the network with nodes $N \setminus \{r\}$. Then add the two cheapest edges attached to node r. For example, if you choose $r = 1$, then the minimum spanning tree on $N \setminus \{r\} = \{2, 3, 4, 5, 6\}$ has edges $\{2, 6\}, \{5, 6\}, \{2, 3\}, \{3, 4\}$ and the two cheapest edges attached to node $r = 1$ is $\{1, 2\}, \{1, 6\}$, giving a lower bound of $23 + 24 + 26 + 28 + 20 + 21 = 142$.

3. Write down a complete integer linear programming formulation for the traveling salesman problem on the network. With "complete" is meant that you should write the objective function and each individual constraint with the given data substituted in the appropriate places.
Answer: Let $x_{ij} = \begin{cases} 1 & \text{if edge } \{i,j\} \text{ is used} \\ 0 & \text{o/w} \end{cases}$

$$
\begin{array}{lrcll}
\min & \sum_{\{i,j\}\in E} c_{ij}x_{ij} & & & \\
\text{s.t.} & & & & \\
 & \sum_{\{j\in N\,:\,\{j,i\}\in E\}} x_{ji} + \sum_{\{j\in N\,:\,\{i,j\}\in E\}} x_{ij} & = & 2 & \forall\, i \in N \\
 & \sum_{\{\{i,j\}\in E\,:\,i\in S, j\notin S\}} x_{ij} + \sum_{\{\{i,j\}\in E\,:\,i\notin S, j\in S\}} x_{ij} & \geq & 2 & \forall\, S \subset N \;:\; 3 \leq |S| \leq \left\lfloor \frac{|N|}{2} \right\rfloor \\
 & x_{ij} & \in & \{0,1\} & \forall\, \{i,j\} \in E
\end{array}
$$

With the given data, the formulation is

$$
\begin{array}{lrcll}
\min & \{20x_{1,2} + 27x_{1,3} + 22x_{1,4} + 25x_{1,5} + 21x_{1,6} & & & \\
 & +26x_{2,3} + 31x_{2,4} + 29x_{2,5} + 23x_{2,6} + 28x_{3,4} & & & \\
 & +32x_{3,5} + 33x_{3,6} + 30x_{4,5} + 34x_{4,6} + 24x_{5,6}\} & & & \\
\text{s.t.} & & & & \\
 & x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} & = & 2 & \\
 & x_{1,2} + x_{2,3} + x_{2,4} + x_{2,5} + x_{2,6} & = & 2 & \\
 & x_{1,3} + x_{2,3} + x_{3,4} + x_{3,5} + x_{3,6} & = & 2 & \\
 & x_{1,4} + x_{2,4} + x_{3,4} + x_{4,5} + x_{4,6} & = & 2 & \\
 & x_{1,5} + x_{2,5} + x_{3,5} + x_{4,5} + x_{5,6} & = & 2 & \\
 & x_{1,6} + x_{2,6} + x_{3,6} + x_{4,6} + x_{5,6} & = & 2 & \\
 & x_{1,4} + x_{1,5} + x_{1,6} + x_{2,4} + x_{2,5} + x_{2,6} + x_{3,4} + x_{3,5} + x_{3,6} & \geq & 2 & S = \{1,2,3\} \\
 & x_{1,3} + x_{1,5} + x_{1,6} + x_{2,3} + x_{2,5} + x_{2,6} + x_{3,4} + x_{4,5} + x_{4,6} & \geq & 2 & S = \{1,2,4\} \\
 & x_{1,3} + x_{1,4} + x_{1,6} + x_{2,3} + x_{2,4} + x_{2,6} + x_{3,5} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{1,2,5\} \\
 & x_{1,3} + x_{1,4} + x_{1,5} + x_{2,3} + x_{2,4} + x_{2,5} + x_{3,6} + x_{4,6} + x_{5,6} & \geq & 2 & S = \{1,2,6\} \\
 & x_{1,2} + x_{1,5} + x_{1,6} + x_{2,3} + x_{2,4} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6} & \geq & 2 & S = \{1,3,4\} \\
 & x_{1,2} + x_{1,4} + x_{1,6} + x_{2,3} + x_{2,5} + x_{3,4} + x_{3,6} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{1,3,5\} \\
 & x_{1,2} + x_{1,4} + x_{1,5} + x_{2,3} + x_{2,6} + x_{3,4} + x_{3,5} + x_{4,6} + x_{5,6} & \geq & 2 & S = \{1,3,6\} \\
 & x_{1,2} + x_{1,3} + x_{1,6} + x_{2,4} + x_{2,5} + x_{3,4} + x_{3,5} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{1,4,5\} \\
 & x_{1,2} + x_{1,3} + x_{1,5} + x_{2,4} + x_{2,6} + x_{3,4} + x_{3,6} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{1,4,6\} \\
 & x_{1,2} + x_{1,3} + x_{1,4} + x_{2,5} + x_{2,6} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6} & \geq & 2 & S = \{1,5,6\} \\
 & x_{1,2} + x_{1,3} + x_{1,4} + x_{2,5} + x_{2,6} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6} & \geq & 2 & S = \{2,3,4\} \\
 & x_{1,2} + x_{1,3} + x_{1,5} + x_{2,4} + x_{2,6} + x_{3,4} + x_{3,6} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{2,3,5\} \\
 & x_{1,2} + x_{1,3} + x_{1,6} + x_{2,4} + x_{2,5} + x_{3,4} + x_{3,5} + x_{4,6} + x_{5,6} & \geq & 2 & S = \{2,3,6\} \\
 & x_{1,2} + x_{1,4} + x_{1,5} + x_{2,3} + x_{2,6} + x_{3,4} + x_{3,5} + x_{4,6} + x_{5,6} & \geq & 2 & S = \{2,4,5\} \\
 & x_{1,2} + x_{1,4} + x_{1,6} + x_{2,3} + x_{2,5} + x_{3,4} + x_{3,6} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{2,4,6\} \\
 & x_{1,2} + x_{1,5} + x_{1,6} + x_{2,3} + x_{2,4} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6} & \geq & 2 & S = \{2,5,6\} \\
 & x_{1,3} + x_{1,4} + x_{1,5} + x_{2,3} + x_{2,4} + x_{2,5} + x_{3,6} + x_{4,6} + x_{5,6} & \geq & 2 & S = \{3,4,5\} \\
 & x_{1,3} + x_{1,4} + x_{1,6} + x_{2,3} + x_{2,4} + x_{2,6} + x_{3,5} + x_{4,5} + x_{5,6} & \geq & 2 & S = \{3,4,6\} \\
 & x_{1,3} + x_{1,5} + x_{1,6} + x_{2,3} + x_{2,5} + x_{2,6} + x_{3,4} + x_{4,5} + x_{4,6} & \geq & 2 & S = \{3,5,6\} \\
 & x_{1,4} + x_{1,5} + x_{1,6} + x_{2,4} + x_{2,5} + x_{2,6} + x_{3,4} + x_{3,5} + x_{3,6} & \geq & 2 & S = \{4,5,6\} \\
 & x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}, x_{1,6}, x_{2,3}, x_{2,4}, x_{2,5}, x_{2,6}, x_{3,4}, x_{3,5}, x_{3,6}, x_{4,5}, x_{4,6}, x_{5,6} & \in & \{0,1\} &
\end{array}
$$

4. Find an optimal solution for the traveling salesman problem, using a solver of your choice. Report your solution and optimal objective value.
Answer: See Figure 1:

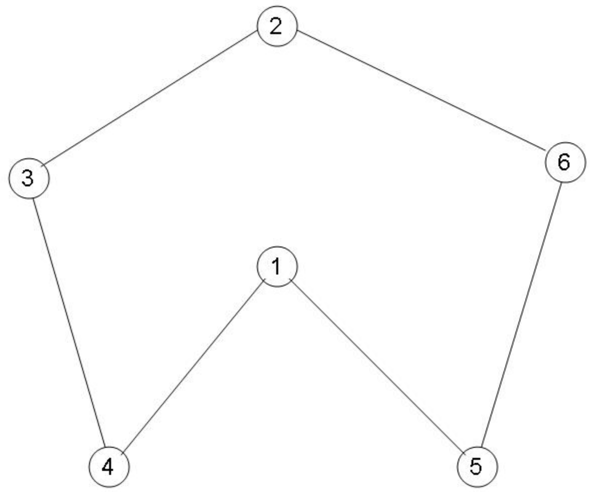

Figure 1: Total Distance = 148

Question 2

(25 points)

Let $N = \{1,2,3,4,5,6\}$ denote the given set of nodes, and let $A = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,3),(2,6),(3,1),(3,2),(3,4),(4,1),(4,3),(4,5),(5,1),(5,4),(5,6),(6,1),(6,2),(6,5)\}$ denote the given set of arcs. Let $G = (N, A)$ denote the given directed network. Let c_{ij} be the cost associated with arc (i, j).

1. Write down a formulation to efficiently compute a good lower bound for the optimal objective value of the traveling salesman problem on the network.
   Answer: To obtain a lower bound for the asymmetric traveling salesman problem just remove the sub-tour elimination constraints. The resulting relaxation is an assignment problem, also called a bipartite matching problem.
   Let $x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is used} \\ 0 & \text{o/w} \end{cases}$

$$\begin{array}{llll} \min & \sum_{(i,j)\in A} c_{ij}x_{ij} & & \\ \text{s.t.} & & & \\ & \sum_{\{j\in N\,:\,(j,i)\in A\}} x_{ji} & = & 1 & \forall\, i \in N \\ & \sum_{\{j\in N\,:\,(i,j)\in A\}} x_{ij} & = & 1 & \forall\, i \in N \\ & x_{ij} & \in & \{0,1\} & \forall\, (i,j) \in A \end{array}$$

With the given data, the formulation is

$$
\begin{aligned}
\min \quad & \{21x_{1,2} + 33x_{1,3} + 40x_{1,4} + 23x_{1,5} + 25x_{1,6} \\
& +24x_{2,1} + 39x_{2,3} + 26x_{2,6} + 31x_{3,1} + 34x_{3,2} \\
& +29x_{3,4} + 22x_{4,1} + 27x_{4,3} + 30x_{4,5} + 32x_{5,1} \\
& +36x_{5,4} + 35x_{5,6} + 38x_{6,1} + 28x_{6,2} + 37x_{6,5}\} \\
\text{s.t.} \quad & \\
& x_{2,1} + x_{3,1} + x_{4,1} + x_{5,1} + x_{6,1} = 1 \\
& x_{1,2} + x_{3,2} + x_{6,2} = 1 \\
& x_{1,3} + x_{2,3} + x_{4,3} = 1 \\
& x_{1,4} + x_{3,4} + x_{5,4} = 1 \\
& x_{1,5} + x_{4,5} + x_{6,5} = 1 \\
& x_{1,6} + x_{2,6} + x_{5,6} = 1 \\
& x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} = 1 \\
& x_{2,1} + x_{2,3} + x_{2,6} = 1 \\
& x_{3,1} + x_{3,2} + x_{3,4} = 1 \\
& x_{4,1} + x_{4,3} + x_{4,5} = 1 \\
& x_{5,1} + x_{5,4} + x_{5,6} = 1 \\
& x_{6,1} + x_{6,2} + x_{6,5} = 1 \\
& x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}, x_{1,6}, x_{2,1}, x_{2,3}, x_{2,6}, x_{3,1}, x_{3,2} \in \{0,1\} \\
& x_{3,4}, x_{4,1}, x_{4,3}, x_{4,5}, x_{5,1}, x_{5,4}, x_{5,6}, x_{6,1}, x_{6,2}, x_{6,5} \in \{0,1\}
\end{aligned}
$$

2. Use a solver of your choice and compute a good lower bound for the optimal objective value of the traveling salesman problem on the network. Report your lower bound and the solution with which you obtained the lower bound.
   Answer: The solution of the relaxation is
   $x(1,5) = 1, x(2,6) = 1$
   $x(3,4) = 1, x(4,3) = 1$
   $x(5,1) = 1, x(6,2) = 1$
   with objective value 165, that is, the lower bound for the optimal objective value of the traveling salesman problem on the network is 165.

3. Use the solution with which you obtained the lower bound and a patching heuristic to obtain a pretty good feasible solution for the traveling salesman problem, and thus an upper bound for the optimal objective value. Draw a sequence of networks to show the steps you took in the heuristic. Report your solution and your upper bound.
   Answer: Use the patching heuristic from the textbook to get a feasible solution and an upper bound.

4. Write down a complete integer linear programming formulation for the traveling salesman problem on the network. With "complete" is meant that you should write the objective function and each individual constraint with the given data substituted in the appropriate places.
   Answer: Let $x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is used} \\ 0 & \text{o/w} \end{cases}$

$$
\begin{array}{lrcll}
\min & \sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\
\text{s.t.} & & & & \\
& \sum_{\{j\in N\,:\,(j,i)\in A} x_{ji} & = & 1 & \forall\ i \in N \\
& \sum_{\{j\in N\,:\,(i,j)\in A} x_{ij} & = & 1 & \forall\ i \in N \\
& \sum_{\{(i,j)\in A\,:\,i\in S, j\in S\}} x_{ij} & \leq & |S|-1 & \forall\ S \subset N\ :\ 2 \leq |S| \leq \left\lfloor \frac{|N|}{2} \right\rfloor \\
& x_{ij} & \in & \{0,1\} & \forall\ (i,j) \in A
\end{array}
$$

With the given data, the formulation is

$$
\begin{array}{lrcll}
\min & \{21x_{1,2} + 33x_{1,3} + 40x_{1,4} + 23x_{1,5} + 25x_{1,6} & & & \\
& +24x_{2,1} + 39x_{2,3} + 26x_{2,6} + 31x_{3,1} + 34x_{3,2} & & & \\
& +29x_{3,4} + 22x_{4,1} + 27x_{4,3} + 30x_{4,5} + 32x_{5,1} & & & \\
& +36x_{5,4} + 35x_{5,6} + 38x_{6,1} + 28x_{6,2} + 37x_{6,5}\} & & & \\
\text{s.t.} & & & & \\
& x_{2,1} + x_{3,1} + x_{4,1} + x_{5,1} + x_{6,1} & = & 1 & \\
& x_{1,2} + x_{3,2} + x_{6,2} & = & 1 & \\
& x_{1,3} + x_{2,3} + x_{4,3} & = & 1 & \\
& x_{1,4} + x_{3,4} + x_{5,4} & = & 1 & \\
& x_{1,5} + x_{4,5} + x_{6,5} & = & 1 & \\
& x_{1,6} + x_{2,6} + x_{5,6} & = & 1 & \\
& x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} & = & 1 & \\
& x_{2,1} + x_{2,3} + x_{2,6} & = & 1 & \\
& x_{3,1} + x_{3,2} + x_{3,4} & = & 1 & \\
& x_{4,1} + x_{4,3} + x_{4,5} & = & 1 & \\
& x_{5,1} + x_{5,4} + x_{5,6} & = & 1 & \\
& x_{6,1} + x_{6,2} + x_{6,5} & = & 1 & \\
& x_{1,2} + x_{2,1} & \leq & 1 & S = \{1,2\} \\
& x_{1,3} + x_{3,1} & \leq & 1 & S = \{1,3\} \\
& x_{1,4} + x_{4,1} & \leq & 1 & S = \{1,4\} \\
& x_{1,5} + x_{5,1} & \leq & 1 & S = \{1,5\} \\
& x_{1,6} + x_{6,1} & \leq & 1 & S = \{1,6\} \\
& x_{2,3} + x_{3,2} & \leq & 1 & S = \{2,3\} \\
& x_{2,6} + x_{6,2} & \leq & 1 & S = \{2,6\} \\
& x_{3,4} + x_{4,3} & \leq & 1 & S = \{3,4\} \\
& x_{4,5} + x_{5,4} & \leq & 1 & S = \{4,5\} \\
& x_{5,6} + x_{6,5} & \leq & 1 & S = \{5,6\} \\
& x_{1,2} + x_{2,1} + x_{1,3} + x_{3,1} + x_{2,3} + x_{3,2} & \leq & 2 & S = \{1,2,3\} \\
& x_{1,2} + x_{2,1} + x_{1,6} + x_{6,1} + x_{2,6} + x_{6,2} & \leq & 2 & S = \{1,2,6\} \\
& x_{1,3} + x_{3,1} + x_{1,4} + x_{4,1} + x_{3,4} + x_{4,3} & \leq & 2 & S = \{1,3,4\} \\
& x_{1,4} + x_{4,1} + x_{1,5} + x_{5,1} + x_{4,5} + x_{5,4} & \leq & 2 & S = \{1,4,5\} \\
& x_{1,5} + x_{5,1} + x_{1,6} + x_{6,1} + x_{5,6} + x_{6,5} & \leq & 2 & S = \{1,5,6\} \\
& x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5}, x_{1,6}, x_{2,1}, x_{2,3}, x_{2,6}, x_{3,1}, x_{3,2} & \in & \{0,1\} & \\
& x_{3,4}, x_{4,1}, x_{4,3}, x_{4,5}, x_{5,1}, x_{5,4}, x_{5,6}, x_{6,1}, x_{6,2}, x_{6,5} & \in & \{0,1\} &
\end{array}
$$

5. Find an optimal solution for the traveling salesman problem, using a solver of your choice. Report your solution and optimal objective value.
   Answer: See Figure 2:

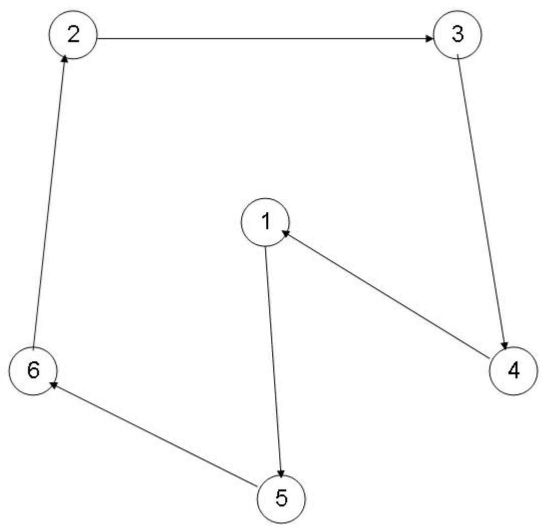

Figure 2: Total Distance = 176