

**ISyE 4111**  
**Advanced Supply Chain Logistics**  
**Fall 2010**

**Reading Assignment 5**  
**Due Date: October 7, 2010**

Read the materials below in preparation for discussion in class. The questions are intended to guide your reading and thoughts, but are not the only aspects that will be discussed in class.

1. Bell, W., Dalberto, L., Fisher, M., Greenfield, A., Jaikumar, R., Kedia, P., Mack, R., and Prutzman, P., "Improving the Distribution of Industrial Gases with an On-Line Computerized Routing and Scheduling Optimizer", *Interfaces*, vol. 13, nr. 6, pp. 4–23, 1983.
  - (a) Describe the company's products.
  - (b) Describe the company's distribution practice.
  - (c) Why is distribution efficiency crucial to companies in the industry?
  - (d) How does the company's vehicle routing problem differ from regular vehicle routing problems, and why?
  - (e) Describe the daily decisions that the scheduler has to make.
  - (f) Describe the data that formed part of the new vehicle scheduling system.
  - (g) What do the schedulers have to forecast?
  - (h) Why is forecasting so important in this setting?
  - (i) How did the scheduler forecast customer usage?
  - (j) What complicated the scheduler's forecasting?
  - (k) What implementation issues did they face with the vehicle scheduling system?
  - (l) What benefits did the vehicle scheduling system provide?
  - (m) Consider a customer  $i$ , with safety stock level  $s_i$ , initial inventory at time 0 equal to  $I_{i,0} > s_i$ , average usage rate of  $u_i$  per unit time, and tank capacity  $Q_i > I_{i,0}$ . Write down the expression for the minimum amount  $d_{i,t}$  and the maximum amount  $D_{i,t}$  that should be delivered to customer  $i$  up to time  $t$ .
  - (n) Explain the definition of  $V_k$  in the mixed integer linear programming formulation of the problem.
  - (o) Explain the mixed integer linear programming formulation of the problem.
  - (p) Formulate the problem with simpler and fewer  $x$ -variables, namely  $x_{i,t}$  instead of  $x_{i,k,t,v}$ .

(q) Consider the following Lagrangian relaxation of the mixed integer linear programming formulation of the problem:

$$\begin{aligned}
& \max_{x,y} \quad \sum_k \sum_{t \in T_k} \sum_{v \in V_k} \left[ \sum_{i \in S_k} (vd_i - c_v)x_{i,k,t,v} - FC_{k,v}y_{k,t,v} \right] \\
& \quad + \sum_{i \in A_1} \sum_t \left[ \lambda_{i,t} \left( \sum_{k \in R_i} \sum_{v \in V_k} \sum_{\{\tau \in T_k : \tau \leq t - tc_{i,k}\}} x_{i,k,\tau,v} - d_{i,t} \right) \right. \\
& \quad \left. + \pi_{i,t} \left( D_{i,t} - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{\{\tau \in T_k : \tau \leq t - tc_{i,k}\}} x_{i,k,\tau,v} \right) \right] \\
& \quad + \sum_{i \in A_2} \nu_i \left( 1 - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{t \in T_k} y_{k,t,v} \right) + \sum_{i \in A_3} \eta_i \left( 1 - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{t \in T_k} y_{k,t,v} \right) \\
\text{subject to} \quad & \sum_{i \in S_k} x_{i,k,t,v} \leq CAP_v y_{k,t,v} \quad \text{for all } k, t \in T_k, v \in V_k \\
& \sum_{k \in K_v} \sum_{\{\tau \in \{t, \dots, t+TL_k-1\} : \tau \in T_k\}} y_{k,\tau,v} \leq 1 \quad \text{for all } t, v \\
& 0 \leq x_{i,k,t,v} \leq \min\{Q_i, D_{i,t+tc_{i,k}}\} \quad \text{for all } k, t \in T_k, v \in V_k, i \in S_k \\
& x_{i,k,t,v} \geq d_{i,T} y_{k,t,v} \quad \text{for all } k, t \in T_k, v \in V_k, i \in S_k \cap A_2 \\
& y_{k,t,v} \in \{0, 1\} \quad \text{for all } k, t \in T_k, v \in V_k
\end{aligned}$$

where “penalties”  $\lambda_{i,t} \geq 0$  and  $\pi_{i,t} \geq 0$  for all  $i \in A_1$  and all  $t$ ,  $\nu_i$  unrestricted for all  $i \in A_2$ , and  $\eta_i \geq 0$  for all  $i \in A_3$ . Note that the constraints that link the  $y_{k,t,v}$  variables for different  $v$  together have been moved to the objective function. Next, recall that for any function  $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$  and any feasible set  $XY \subset \mathcal{X} \times \mathcal{Y}$ , it holds that

$$\max_{(x,y) \in XY} f(x,y) = \max_{x \in X} \{g(x) := \max_{y \in Y(x)} f(x,y)\} = \max_{y \in Y} \{h(y) := \max_{x \in X(y)} f(x,y)\}$$

where  $X(y) := \{x \in \mathcal{X} : (x,y) \in XY\}$  denotes the set of  $x$ -values that are feasible with the given  $y$ -value, and  $Y(x) := \{y \in \mathcal{Y} : (x,y) \in XY\}$  denotes the set of  $y$ -values that are feasible with the given  $x$ -value; that is, simultaneous maximization is equivalent to sequential optimization. Specifically, for our Lagrangian relaxation, for each  $y$ , let

$$\begin{aligned}
h(y) & := \max_x \quad \sum_k \sum_{t \in T_k} \sum_{v \in V_k} \sum_{i \in S_k} (vd_i - c_v)x_{i,k,t,v} \\
& \quad + \sum_{i \in A_1} \sum_t \left[ \lambda_{i,t} \left( \sum_{k \in R_i} \sum_{v \in V_k} \sum_{\{\tau \in T_k : \tau \leq t - tc_{i,k}\}} x_{i,k,\tau,v} - d_{i,t} \right) \right. \\
& \quad \left. + \pi_{i,t} \left( D_{i,t} - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{\{\tau \in T_k : \tau \leq t - tc_{i,k}\}} x_{i,k,\tau,v} \right) \right] \\
& \quad + \sum_{i \in A_2} \nu_i \left( 1 - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{t \in T_k} y_{k,t,v} \right) + \sum_{i \in A_3} \eta_i \left( 1 - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{t \in T_k} y_{k,t,v} \right)
\end{aligned}$$

$$\begin{aligned} \text{subject to } \sum_{i \in S_k} x_{i,k,t,v} &\leq CAP_v y_{k,t,v} && \text{for all } k, t \in T_k, v \in V_k \\ 0 \leq x_{i,k,t,v} &\leq \min\{Q_i, D_{i,t+tc_{i,k}}\} && \text{for all } k, t \in T_k, v \in V_k, i \in S_k \\ x_{i,k,t,v} &\geq d_{i,T} y_{k,t,v} && \text{for all } k, t \in T_k, v \in V_k, i \in S_k \cap A_2 \end{aligned}$$

Then, our Lagrangian relaxation can be written as follows:

$$\begin{aligned} \max_y \quad & h(y) \\ \text{subject to } \quad & \sum_{k \in K_v} \sum_{\{\tau \in \{t, \dots, t+TL_k-1\} : \tau \in T_k\}} y_{k,\tau,v} \leq 1 && \text{for all } t, v \\ & y_{k,t,v} \in \{0, 1\} && \text{for all } k, t \in T_k, v \in V_k \end{aligned}$$

- i. Show how to easily compute  $h(y)$  for any given  $y$ .
- ii. Show how to solve the Lagrangian relaxation problem  $\max_y h(y)$  by solving a number of knapsack problems, which in turn can be solved by solving a number of shortest path problems on an acyclic network. (Even if you cannot figure out all the details, I want you to be prepared to present what you have figured out.)

Hint: Note that the function  $h(y)$  decomposes by  $k, t, v$ . Specifically,

$$\begin{aligned} h(y) &= \sum_k \sum_{t \in T_k} \sum_{v \in V_k} h_{k,t,v}(y_{k,t,v}) \\ &+ \sum_{i \in A_2} \nu_i \left( 1 - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{t \in T_k} y_{k,t,v} \right) + \sum_{i \in A_3} \eta_i \left( 1 - \sum_{k \in R_i} \sum_{v \in V_k} \sum_{t \in T_k} y_{k,t,v} \right) \\ &+ \sum_{i \in A_1} \sum_t [\pi_{i,t} D_{i,t} - \lambda_{i,t} d_{i,t}] \end{aligned}$$

where

$$\begin{aligned} h_{k,t,v}(y_{k,t,v}) &:= \max_{\{x_{i,k,t,v} : i \in S_k\}} \sum_{i \in S_k} (v d_i - c_v) x_{i,k,t,v} \\ &+ \sum_{i \in S_k \cap A_1} \left[ \sum_{\{\tau' \in T_k : \tau' \geq t+tc_{i,k}\}} \lambda_{i,\tau'} x_{i,k,t,v} - \sum_{\{\tau' \in T_k : \tau' \geq t+tc_{i,k}\}} \pi_{i,\tau'} x_{i,k,t,v} \right] \\ \text{subject to } \quad & \sum_{i \in S_k} x_{i,k,t,v} \leq CAP_v y_{k,t,v} \\ & 0 \leq x_{i,k,t,v} \leq D_{i,t+tc_{i,k}} && \text{for all } i \in S_k \\ & x_{i,k,t,v} \geq d_{i,T} y_{k,t,v} && \text{for all } i \in S_k \cap A_2 \end{aligned}$$

If  $y_{k,t,v} = 0$ , then  $h_{k,t,v}(y_{k,t,v}) = \dots$ ? If  $y_{k,t,v} = 1$ , then simply group all the objective coefficients of each variable  $x_{i,k,t,v}$  together to write the objective in the form

$$h_{k,t,v}(1) = \max_{\{x_{i,k,t,v} : i \in S_k\}} \sum_{i \in S_k} \tilde{c}_{i,k,t,v} x_{i,k,t,v}$$

$$\begin{aligned}
\text{subject to } \quad & \sum_{i \in S_k} x_{i,k,t,v} \leq CAP_v \\
& 0 \leq x_{i,k,t,v} \leq D_{i,t+tc_{i,k}} \quad \text{for all } i \in S_k \\
& x_{i,k,t,v} \geq d_{i,T} \quad \text{for all } i \in S_k \cap A_2
\end{aligned}$$

Then, for each fixed  $k, t, v$ , sort the objective coefficients  $\{\tilde{c}_{i,k,t,v} : i \in S_k\}$  from largest to smallest. Now it should be obvious how to compute  $h_{k,t,v}(1)$ .