

# A Friendly Approach to Studying Aliasing Relations of Mixed Factorials in the Form of Product Arrays\*

Abhyuday Mandal<sup>†</sup>

*Georgia Institute of Technology, Atlanta, GA 30318, USA*

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## Abstract

Mixed-level fractional factorial designs are commonly used in industries but its aliasing relations have not been studied in full rigor. These designs take the form of a product array. Aliasing patterns of mixed level factorial designs are discussed here.

*Keywords:* Factorial design; Fractional factorial design; Product arrays.

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## 1 Introduction

Two- and three-level factorial and fractional factorial designs are widely used in industrial experimentations and are discussed in detail in design of experiments textbooks (Box, Hunter and Hunter (1978); Cochran and Cox (1950)). The literature on symmetric designs is already voluminous. For example, the theory of regular fractions for symmetric factorials is given by Dey and Mukerjee (1999). Wu and Hamada (2000) devote a full chapter in their applied design of experiment textbook, on analysis techniques for mixed-level factorial plans. Although these are important designs, their aliasing patterns have not been studied explicitly.

Mixed-levels designs typically occur when there are both qualitative and quantitative factors in the experiment, and the qualitative factors have more than two levels and the

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<sup>†</sup>Corresponding author. *E-mail address:* amandal@isye.gatech.edu

quantitative factors have two levels. Consider an experiment by Hale-Bennett and Lin (1997) and reported in Wu and Hamada (2000) that was performed to improve a painting process of charcoal grill parts. A mixed-level 36-run design (Table 1) was used to study six factors: three of them ( $A, B, C$ ) were at two levels and the other three ( $D, E, F$ ) were at three levels. It is a  $2^{3-1} \times 3^{3-1}$  design which consists of  $4 \times 9 = 36$  runs and is a “product” of a 4-run  $2^{3-1}$  and a 9-run  $3^{3-1}$  design. Now it is not evident that the factorial effect  $ABD^2E$  is same as that of  $ABDE^2$ . If all the factors of a factorial effect are at two-levels,  $AB$  for example, a modulo 2 operation should be performed. Similarly, modulo 3 operations are used when all of them are have three level, as in the case for  $DE^2$ . But what about  $ABDE^2$ ? It is not obvious whether modulo 2 or modulo 3 operations should be done in calculating the aliasing relationship of a mixed level factorial effect. In fact, there is no simple answer to this question, as will be clear from the discussions of Section 2.

In Section 2, we develop the general theory for  $s_1^{n_1} \times s_2^{n_2}$  factorial designs and illustrate it in the context of a  $2^3 \times 3^3$  design. In Section 3, we discuss  $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$  factorial designs and discuss the Paint experiment as an example of  $2^{3-1} \times 3^{3-1}$  design.

## 2 $s_1^{n_1} \times s_2^{n_2}$ factorial designs

An experiment involving  $n_1$  factors each at  $s_1$  levels and  $n_2$  factors each at  $s_2$  levels is an  $s_1^{n_1} \times s_2^{n_2}$  asymmetrical factorial experiment. Suppose the levels of the  $s_i$ -level factor are coded as  $s_i$  elements of Galois field  $GF(s_i)$  where  $s_i$  is a prime or prime power. With levels as  $0, 1, \dots, s_i - 1$ , a typical treatment combination, i.e., a combination of the levels of the  $n_1 + n_2 = n$  factors will be represented by an ordered  $n$ -tuple  $i_1 \dots i_{n_1} j_1 \dots j_{n_2}$  where  $i_k \in \{0, 1, \dots, s_1 - 1\}, 1 \leq k \leq n_1$  and  $j_k \in \{0, 1, \dots, s_2 - 1\}, 1 \leq k \leq n_2$ . Clearly,

altogether there are  $s_1^{n_1} s_2^{n_2}$  treatment combinations.

In what follows,  $(a, b)$  and  $(a', b)'$  will be used interchangeably for the sake of notational simplicity where  $a$  and  $b$  are column vectors of dimension  $n_1$  and  $n_2$ , respectively.

A treatment contrast  $L$  is said to belong to the pencil  $(a, b)$  if it is of the form

$$L = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\}, \quad (1)$$

where

$$V_{i,j}(a, b) = \{(x, y) = (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})' : a'x = \alpha_i, b'y = \beta_j\},$$

$0 \leq i \leq s_1 - 1$ ,  $0 \leq j \leq s_2 - 1$ ; the effect of a treatment combination represented by  $(x, y)$  will be denoted by  $\tau(x, y)$  and  $l(i, j)$ 's are real numbers, not all zeros, satisfying

$$\sum_{i=0}^{s_1-1} l(i, j) = \sum_{j=0}^{s_2-1} l(i, j) = 0. \quad (2)$$

In other words, a treatment contrast  $L$  belongs to  $(a, b)$  if for all  $(x, y)$  belonging to the same  $V_{i,j}(a, b)$ , the coefficient of  $\tau(x, y)$  in  $L$  is also the same.

In general, consider any two pencils  $(a, b)$  and  $(a^*, b^*)$ . These two pencils are distinct if  $a$  is distinct from  $a^*$  and  $b$  is distinct from  $b^*$ , in the sense of symmetric factorial designs. Recall that, in symmetric fractions, pencils with proportional entries are considered as identical.

For a  $s_1^{n_1} \times s_2^{n_2}$  product array, for the  $s_i^{n_i}$  factorial part, there are  $(s_i^{n_i} - 1)/(s_i - 1)$  distinct pencils which involve only  $s_i$ -level factors. The distinct pencils involving both  $s_1$ - and  $s_2$ -level factors are given by the products of those two sets of pencils involving only  $s_1$ - or only  $s_2$ -level factors. A simple counting of degrees of freedom justifies this formulation. Recall that the total number of factorial effects involving only  $s_i$ -level factors is  $(s_i^{n_i} - 1)/(s_i - 1)$ , each with  $(s_i - 1)$  d.f.. As the interactions involve both  $s_1$ -

and  $s_2$ -level factors are given by products of the other two sets of pencils, i.e., there are  $(s_1^{n_1} - 1)/(s_1 - 1) \times (s_2^{n_2} - 1)/(s_2 - 1)$  pencils of this kind, each with  $(s_1 - 1)(s_2 - 1)$  d.f.. Thus, the above description accounts for

$$\frac{s_1^{n_1} - 1}{s_1 - 1}(s_1 - 1) + \frac{s_2^{n_2} - 1}{s_2 - 1}(s_2 - 1) + \frac{s_1^{n_1} - 1}{s_1 - 1} \frac{s_2^{n_2} - 1}{s_2 - 1}(s_1 - 1)(s_2 - 1) = s_1^{n_1} s_2^{n_2} - 1$$

d.f. which agrees with the fact that there are  $s_1^{n_1} s_2^{n_2}$  in all.

Following Bose (1947), in the Appendix we give the definition for treatment contrasts belonging to factorial effects for the general case of an  $s_1 \times \dots \times s_n$  factorials. The next two results link pencils with factorial effects.

**Result 2.1** (a) *Treatment contrasts belonging to distinct pencils are orthogonal to each other.* (b) *Let  $(a, b)$  be a pencil such that  $a_i \neq 0$  if  $i \in \{i_1, \dots, i_g\}$ , and  $= 0$  otherwise,  $b_j \neq 0$  if  $j \in \{i_1, \dots, i_h\}$ , and  $= 0$  otherwise, where  $1 \leq i_1 < \dots < i_g \leq n_1$ ,  $1 \leq j_1 < \dots < i_h \leq n_2$  and  $1 \leq g \leq n_1$ ,  $1 \leq h \leq n_2$ . Then any treatment contrast belonging to  $(a, b)$  also belongs to the factorial effect  $F_{i_1} \dots F_{i_g} F'_{j_1} \dots F'_{j_h}$ .*

### Example

Let us consider the  $2^3 \times 3^3$  full factorial design with two-level factors  $A, B, C$  and three-level factors  $D, E, F$ . The levels of  $A, B, C$  are denoted by 0 and 1, and those of  $D, E, F$  are denoted by 0, 1 and 2. Then a typical treatment combination, i.e., the combination of the levels of six factors will be represented by  $x = (a, b, c, d, e, f)$ , where  $a, b, c \in \{0, 1\}$  and  $d, e, f \in \{0, 1, 2\}$ . For example, the factorial effect  $ABDE^2$  is denoted by  $(a, b, c, d, e, f) \equiv (1, 1, 0, 1, 2, 0)$ . Clearly there are  $2^3 \times 3^3 = 216$  possible treatment combinations.

The pencils involving only the two-level factors or only the three-level factors can be described as usual. Thus the pencil  $AB$  is given by the contrasts between the two sets of treatment combinations for which  $a + b = 0$  or  $1 \pmod{2}$ . More explicitly, these two

sets are  $\{x : x = (a, b, c, d, e, f), a + b = 0 \pmod{2}\}$  and  $\{x : x = (a, b, c, d, e, f), a + b = 1 \pmod{2}\}$ . Clearly, there are 108 treatment combinations in each of these sets, e.g., the first set consists of the treatment combinations  $(0, 0, c, d, e, f)$  and  $(1, 1, c, d, e, f)$ , where  $c \in \{0, 1\}$  and  $d, e, f \in \{0, 1, 2\}$ , leading to  $54+54=108$  treatment combinations in all. In a similar manner, the pencil  $DEF^2$ , involving exclusively the three-level factors, is given by contrasts among three sets of treatment combinations for which  $d+e+2f = 0, 1$  or  $2 \pmod{3}$ . As before, there are  $8 \times 9 = 72$  treatment combinations in each of these sets. It is clear that any pencil involving  $A, B, C$  will have 1 d.f. while any pencil involving only  $D, E, F$  will have 2 d.f..

Now consider the interactions that involve both two- and three-level factors. Recall that there are 7 pencils  $A, B, C, AB, AC, BC$  and  $ABC$  involving only the two-level factors. Similarly there are 13 distinct pencils  $D, E, F, DE, DE^2, \dots, DE^2F^2$  involving only the three-level factors. The pencils representing interactions that involve both two- and three-level factors are given by the products of these two sets of pencils, i.e., there are  $7 \times 13 = 91$  pencils of this kind, namely,  $AD, AE, \dots, ADE^2F^2, BD, BE, \dots, BDE^2F^2, \dots, ABCD, ABCE, \dots, ABCDE^2F^2$ . Each of these 91 pencils carries 2 d.f.. Clearly, for example,  $AD$  and  $AD^2$  mean the same thing in this formulation (so we write only  $AD$ ). Similarly  $ABDE^2 = ABD^2E = AB(DE^2)^2$ . Taking care of the 7 pencils involving only the two-level factors and the 13 pencils involving only the three-level factors, the above description accounts for  $7 \times 1 + 13 \times 2 + 91 \times 2 = 215$  d.f., which agrees with the fact that there are  $2^3 \times 3^3 = 216$  treatment combinations in all.

How does one actually define contrasts belonging to pencils as considered in the last paragraph? Consider, for example, the pencil  $ABDE^2$ . For  $i = 0, 1$  and  $j = 0, 1, 2$ , define  $V_{i,j} = V_{i,j}(110, 120) = \{x : x = (a, b, c, d, e, f), a + b = i \pmod{2}, d + 2e = j \pmod{3}\}$ .

Note that  $a + b = i \pmod 2$  corresponds to  $AB$ , and  $d + 2e = j \pmod 3$  corresponds to  $DE^2$ . Clearly, each of the six sets  $V_{i,j}$  has cardinality  $4 \times 9 = 36$ . Let  $T(i, j)$  be the total of the treatment effects for the treatment combinations in  $V_{i,j}$ . Then a typical contrast belonging to the pencil  $ABDE^2$  will be of the form  $\sum_i \sum_j l(i, j)T(i, j)$ , where the scalars  $l(i, j)$ , not all zeros, must satisfy  $l(0, j) + l(1, j) = 0$  for every  $j$  and  $l(i, 0) + l(i, 1) + l(i, 2) = 0$  for every  $i$ . Thus there will be two such independent treatment contrasts, namely,  $L_1 = T(0, 0) - T(1, 0) - T(0, 2) + T(1, 2)$  and  $L_2 = T(0, 0) - T(1, 0) - 2T(0, 1) + 2T(1, 1) + T(0, 2) - T(1, 2)$ .

### 3 $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$ fractional factorial designs

A regular fraction of an  $s^n$  symmetrical factorial, where  $s (\geq 2)$  is a prime or prime power, is specified by any  $k$  ( $1 \leq k < n$ ) linearly independent pencils, say  $b^{(1)}, \dots, b^{(k)}$ , and consists of treatment combinations  $z$  satisfying  $Bz = c$ , where  $B$  is a  $k \times n$  matrix with rows  $(b^{(i)})'$ ,  $1 \leq i \leq k$ , and  $c$  is a fixed  $k \times 1$  vector over  $GF(s)$ . The specific choice of  $c$  is inconsequential. Hence, without loss of generality, it is assumed that  $c = 0$ , the  $k \times 1$  null vector over  $GF(s)$ . Then a regular fractional factorial plan is given by, say,

$$d(B) = \{z : Bz = 0\}.$$

In the same line, for a  $s_1^{n_1} \times s_2^{n_2}$  design, a regular fractional factorial plan,  $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$  is given by

$$d(B) = \{z : Bz = 0\} = \{(x, y) : B_1x = 0, B_2y = 0\}$$

where

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

Note that  $d(B_i)$  gives a regular  $s_i^{n_i-k_i}$  fractional factorial plan. For a symmetric fractional factorial, a pencil is called a defining pencil if it belongs to the row space of  $B$ . Equivalently, a defining pencil of a  $s_1^{n_1-k_1} \times s_2^{n_2-k_2}$  design is of the form  $(b_1, b_2)$  where  $b_i$  is a defining pencil of  $s_i^{n_i-k_i}$ .

Consider now any defining pencil  $(a, b)$ . Then  $a' = \lambda' B_1$  and  $b' = \xi' B_2$  for suitable  $\lambda$  and  $\xi$  with entries from  $GF(s_1)$  and  $GF(s_2)$ , respectively. Now it is not difficult to see that  $d(B) \subset V_{0,0}(a, b)$ . Recalling the definition of a treatment contrast, the following result is evident.

**Result 3.1** *No treatment contrast belonging to any defining pencil is estimable in  $d(B)$ .*

Two pencils are aliases of each other if their difference belongs to the row space of  $B$ . Let  $\mathcal{C}$  be the set of distinct pencils which are not defining pencils. Then we get the following Lemma.

**Lemma 3.1** *Let the pencils  $(a, b), (a^*, b^*) \in \mathcal{C}$  be aliases of each other and*

$$L = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\},$$

and

$$L^* = \sum_{k=0}^{s_1-1} \sum_{l=0}^{s_2-1} l^*(k, l) \left\{ \sum_{(x,y) \in V_{k,l}(a^*,b^*)} \tau(x, y) \right\},$$

be the treatment contrasts belonging to  $(a, b)$  and  $(a^*, b^*)$ , respectively. Then the parts of  $L$  and  $L^*$ , which involve only the treatment combinations included in  $d(B)$ , are identical.

Let  $L(B)$  be the part of  $L$  that involves only the treatment combination involved in the fraction  $d(B)$  and is often called the relevant part of  $L$ . Then the relevant parts of corresponding contrasts belonging to pencils that are aliases of each other, are identical. Let  $V_{i,j}((a, b), B) = V_{i,j}(a, b) \cap d(B)$ . Then for any pencil  $(a, b) \in \mathcal{C}$  and for its alias set  $\mathcal{A}$ , we get the following theorem.

**Theorem 3.1** For  $(a, b) \in \mathcal{A}$ , consider the corresponding treatment contrast

$$L = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\},$$

Then

$$\begin{aligned} & \sum_{(a,b)} \left[ \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\} \right] \\ &= s_1^{k_1} s_2^{k_2} \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}((a,b), B)} \tau(x, y) \right\} \end{aligned} \quad (3)$$

where  $\sum_{(a,b)}$  denote the sum over all  $(a, b) \in \mathcal{A}$ .

To prove this theorem, we need the following Lemma.

**Lemma 3.2** Consider any pencil  $(a, b) \in \mathcal{C}$  and let  $\mathcal{A}$  denote its alias set. Let

$$\phi_{i,j}((a, b)(x, y)) = \begin{cases} 1 & \text{if } a'x = \alpha_i, b'y = \beta_j, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Then for every treatment combination  $(x, y)$  and every  $(i, j)$ ,  $0 \leq i \leq s_1 - 1, 0 \leq j \leq s_2 - 1$ ,

$$\begin{aligned} & \sum_{(a,b)} \phi_{i,j}((a, b), (x, y)) = \\ & \begin{cases} s_1^{k_1} s_2^{k_2} & \text{if } (x, y) \in V_{i,j}((a, b), B), \\ 0 & \text{if } (x, y) \in d(B) - V_{i,j}((a, b), B), \\ s_1^{k_1-1} s_2^{k_2-1} & \text{if } (x, y) \notin d(B). \end{cases} \end{aligned} \quad (5)$$

**Proof.** A pencil in  $\mathcal{A}$  is of the form  $(p, q)$  where  $p = a + B_1' \lambda$  and  $q = b + B_2' \xi$  where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k_1})', \lambda_i \in GF(s_i)$  and  $\xi = (\xi_1, \xi_2, \dots, \xi_{k_2})', \xi_j \in GF(s_2)$ . For fixed  $(x, y)$  and  $(i, j)$ ,

$$\begin{aligned} & \sum_{(a,b)} \phi_{i,j}((a, b), (x, y)) = \\ & \# \left\{ \begin{aligned} & \lambda = (\lambda_1, \dots, \lambda_{k_1})' : x + \lambda' B_1 x = \alpha_i; \\ & \xi = (\xi_1, \dots, \xi_{k_2})' : b'y + \xi' B_2 y = \beta_j, \\ & \lambda_i \in GF(s_i), \xi_j \in GF(s_2) \forall i, j \end{aligned} \right\} \end{aligned}$$

where  $\#$  denotes the cardinality of a set.

- (i) If  $(x, y) \in V_{i,j}(a, b)$  then  $a'x + \lambda' B_1 x = \alpha_i$  for all  $k_1 \times 1$  vectors over  $GF(s_1)$  and  $b'y + \xi' B_2 y = \beta_j$  for all  $k_2 \times 1$  vectors over  $GF(s_2)$ . Hence the RHS of (5) is  $s_1^{k_1} s_2^{k_2}$ .
- (ii) If  $(x, y) \in d(B) - V_{i,j}((a, b), B)$ , then  $B_1 x = 0$ ,  $B_2 = 0$ . Also,  $a'x \neq \alpha_i$  and/or  $b'y \neq \beta_j$ . Then  $\sum_{(a,b)} \phi_{i,j}((a, b), (x, y)) = \#\{(\lambda, \xi) : a'x = \alpha_i, b'y = \beta_j\} = 0$ .
- (iii) If  $(x, y) \notin d(B)$ , then  $B_1 x \neq 0$ ,  $B_2 y \neq 0$ . Trivially  $a'x + \lambda' B_1 x = \alpha_i$  iff  $(B_1 x)' \lambda = \alpha_i - a'x$ . Since  $B_1 x \neq 0$ , exactly as in the proof of Lemma 2.1, one can freely choose  $(\lambda_2, \dots, \lambda_{k_1-1})$  in  $s_1^{k_1-1}$  ways to satisfy the above equation. Similarly  $b'y + \xi' B_2 y = \beta_j$  gives  $s_2^{k_2-1}$  choices of  $\xi_l$ 's. Combining the values of  $\lambda_k$ 's and  $\xi_l$ 's, the result follows.

**Proof of Theorem 3.1.** Let  $\Omega$  denote the set of all  $s_1^{n_1} s_2^{n_2}$  treatment combinations.

Using Lemma 3.2 and the indicator variable  $\phi_{i,j}((a, b)(x, y))$  in (4),

$$\begin{aligned}
& \sum_{(a,b)} \left[ \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\} \right] \\
&= \sum_{(a,b)} \left[ \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in \Omega} \phi_{i,j}((a, b)(x, y)) \tau(x, y) \right\} \right] \\
&= \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in \Omega} \left[ \sum_{(a,b)} \phi_{i,j}((a, b)(x, y)) \right] \tau(x, y) \right\} \\
&= s_1^{k_1} s_2^{k_2} \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}((a,b), B)} \tau(x, y) \right\},
\end{aligned}$$

since  $\sum l(i, j) = 0$ .

The RHS of (3) is a contrast involving only the treatment combinations included in  $d(B)$ . Therefore the RHS and hence the LHS of (3) will be estimable using the plan  $d(B)$ . In other words, while pencils belonging to the same alias set, are confounded with one another (Lemma A.3), the sum of corresponding contrasts, belonging to such pencils is estimable in  $d(B)$ . Thus any treatment contrast belonging to a pencil  $(a, b)$

which is not a defining pencil is estimable in  $d(B)$  if and only if corresponding contrasts belonging to all other pencils that are aliased with  $(a, b)$  are ignorable.

We say that a pencil is estimable in  $d(B)$  if so is every treatment contrast belonging to it. Similarly, if every treatment contrast belonging to a pencil is ignorable, then the pencil itself is called ignorable. Hence the following result is immediate.

**Result 3.2** *A pencil  $b$ , which is not a defining pencil, is estimable in  $d(B)$  if and only if all other pencils that are aliased with  $b$  are ignorable.*

### Example

Now consider the fractional factorial design used for the Paint experiment. This kind of fraction treats the two- and three-level factors separately, leading to a product array. It is easy to see that these two-level factors form a  $2^{3-1}$  design with  $C = AB$ . The  $2^{3-1}$  design is used for the first four rows and repeated for the next eight groups of four rows. The three-level factors form a  $3^{3-1}$  design with  $F = DE$ . Each of the nine combinations of the  $3^{3-1}$  design appears in four consecutive entries in Table 1. The 36-run design in Table 1 consists of the  $4 \times 9$  combinations of the  $2^{3-1}$  design and the  $3^{3-1}$  design and is called a  $2^{3-1} \times 3^{3-1}$  design.

The defining relation of the  $2^{3-1} \times 3^{3-1}$  design can be obtained from those of its two component designs:  $\mathbf{I} = \mathbf{ABC}$  and  $\mathbf{I} = \mathbf{DEF}^2$ . So we decide to include the treatment combinations  $x = (a, b, c, d, e, f)$  satisfying  $a + b + c = 0 \pmod{2}$  and  $d + e + 2f = 0 \pmod{3}$ . There are 4 such choices of  $(a, b, c)$  and 9 such choices of  $(d, e, f)$ . Combining these, we will have  $4 \times 9 = 36$  treatment combinations in our plan which will be in the form of product array. The alias sets will again be of three types :

**Type I** (involving only two-level factors arising from  $I = ABC$ ): These are  $A = BC$ ,

$$B = AC, C = AB, \text{ each carrying 1 d.f..}$$

**Type II** (involving only three-level factors arising from  $I = DEF^2$ ): there will be 4 such alias sets, each carrying 2 d.f.. these are  $D = DE^2F = EF^2$ ;  $E = DF^2 = DE^2F^2$ ;  $F = DE = DEF$  and  $DE^2 = DF = EF$ .

**Type III** (involving the “mixed” pencils discussed earlier): These are obtained by combining each type I alias set with each type II alias set, e.g., a typical alias set of type III will be  $AD = ADE^2F = AEF^2 = BCD = BCDE^2 = BCEF^2$ . There will be  $3 \times 4 = 12$  such alias sets each carrying 2 d.f..

Thus the three types of alias sets mentioned above will together carry  $3 \times 1 + 4 \times 2 + 12 \times 2 = 35$  d.f.. This again agrees with the fact that there are 36 treatment combinations in the chosen fraction.

For any pencil in a type III alias set, it is not hard to see that each set  $V_{i,j}$  corresponding to that pencil will contain six of the treatment combinations included in our fraction. To see this, consider the pencil  $BCDE^2F$ . A treatment combination  $x = (a, b, c, d, e, f)$  in our fraction will then belong to the corresponding  $V_{i,j}$  if it satisfies  $b + c = i \pmod{2}$  and  $d + 2e + f = j \pmod{3}$ , in addition to satisfying  $a + b + c = 0 \pmod{2}$  and  $d + e + 2f = 0 \pmod{3}$  needed for inclusion in the fraction. Now the first and third of the equations just mentioned yield two solutions for  $(a, b, c)$  while the second and fourth of these equations yield three solutions for  $(d, e, f)$ . Combining these, we get six solutions altogether.

## 4 Summary

The designs discussed here are called product arrays. These are quite common in robust parameter designs and are named *cross arrays* (*inner-outer array* in Taguchi’s terminology). Although the product arrays have been used for mixed-level designs

(Shoemaker et. al., 1991), their aliasing relations have not been studied rigorously. In this paper, we have discussed the effect aliasing for fractional factorial designs of mixed level designs. The results obtained for mixed level factorial and fractional designs are similar to that of symmetric factorials. Here only  $s_1^{n_1} \times s_2^{n_2}$  factorials are discussed, although with heavier notations, and without any significant conceptual change, it is possible to obtain general results for  $s_1^{n_1} \times s_2^{n_2} \times \dots \times s_m^{n_m}$  factorials. One drawback of the above approach is that the product arrays may become too large. The rigorous study of mixed-level product arrays gives a deeper insight on the estimation properties of the design and paves the way for further research in extending the minimum aberration and estimation capacity criteria of such designs.

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## Appendix

**Definition 1** Following Bose (1947), we give the definition for treatment contrasts belonging to factorial effects for the general case of an  $s_1 \times \dots \times s_n$  factorials. A treatment contrast

$$\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} \ell(j_1 \dots j_n) \tau(j_1 \dots j_n)$$

belongs to the *factorial effect*  $F_{i_1} \dots F_{i_g}$  ( $1 \leq i_1 < \dots < i_g \leq n; 1 \leq g \leq n$ ) if

- (i)  $\ell(j_1 \dots j_n)$  depends only on  $j_{i_1}, \dots, j_{i_g}$ , and
- (ii) writing  $\ell(j_1 \dots j_n) = \bar{\ell}(j_{i_1} \dots j_{i_g})$  in view of (i) above, the sum of  $\bar{\ell}(j_{i_1} \dots j_{i_g})$  separately over each of the arguments  $j_{i_1}, \dots, j_{i_g}$ , is zero.

The following two Lemmas are immediate.

**Lemma A.1** Let  $(a, b) = (a_1, \dots, a_{n_1}, b_1, \dots, b_{n_2})'$  be any fixed nonnull vector where  $a_i \in GF(s_1)$  and  $b_j \in GF(s_2)$ . Then each of the sets

$$V_{i,j}(a, b) = \{(x, y) = (x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})' : a'_i x = \alpha_i, b'_j y = \beta_j\} \quad (6)$$

$0 \leq i \leq s_1 - 1, 0 \leq j \leq s_2 - 1$  has cardinality  $s_1^{n_1-1} s_2^{n_2-1}$ .

**Lemma A.2** If  $(a^{(1)}, b^{(1)})$  and  $(a^{(2)}, b^{(2)})$  are distinct pencils, then for every  $(i, j), (i', j')$ , ( $0 \leq i, i' \leq s_1 - 1, 0 \leq j, j' \leq s_2 - 1$ ), the set  $V_{i,j}(a^{(1)}, b^{(1)}) \cap V_{i',j'}(a^{(2)}, b^{(2)})$  has cardinality  $s_1^{n_1-2} s_2^{n_2-2}$ .

**Proof of Result 2.1.** Consider distinct pencils,  $(a, b)$  and  $(a^*, b^*)$ . Let

$$L = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}(a,b)} \tau(x, y) \right\}, \quad (7)$$

and

$$L^* = \sum_{k=0}^{s_1-1} \sum_{l=0}^{s_2-1} l^*(k, l) \left\{ \sum_{(x,y) \in V_{k,l}(a^*,b^*)} \tau(x, y) \right\}, \quad (8)$$

be the treatment contrasts belonging to  $(a, b)$  and  $(a^*, b^*)$ , respectively. Here

$$\sum_{i=0}^{s_1-1} l(i, j) = \sum_{j=0}^{s_2-1} l(i, j) = 0, \quad (9)$$

and

$$\sum_{k=0}^{s_1-1} l^*(k, l) = \sum_{l=0}^{s_2-1} l^*(k, l) = 0. \quad (10)$$

Consider the scalar product of the coefficient vectors in (7) and (8). Observe that, for any  $(i, j)$  and  $(k, l)$ , if  $(x, y) \in V_{i,j}(a, b) \cap V_{i,j}(a^*, b^*)$ , then the contribution of  $\tau(x, y)$  to this scalar product equals  $l(i, j)l^*(k, l)$ . Hence the scalar product equals  $\sum \sum l(i, j)l^*(k, l) \# \{V_{i,j}(a, b) \cap V_{k,l}(a^*, b^*)\}$  which is zero by (9), (10) and Lemma A.2.

**Proof of Result 2.2.** Without loss of generality, let  $i_1 = 1, \dots, i_g = g$  and  $j_1 = 1, \dots, j_h = h$ . Then  $a_1, \dots, a_g$  are nonzero while  $a_{g+1} = \dots = a_{n_1} = 0$  and  $b_1, \dots, b_h$  are nonzero while  $b_{h+1} = \dots = b_{n_2} = 0$ , so that

$$V_{i,j}(a, b) = \left\{ (x, y) : \sum_{k=1}^g a_k x_k = \alpha_i, \sum_{l=1}^h b_l y_l = \beta_j \right\}$$

$0 \leq i \leq s_1 - 1$  and  $0 \leq j \leq s_2 - 1$ . Recalling the definition of a treatment contrast  $L$  in equations (1) and (2), it is easy to see that for any  $(x, y)$ , the coefficient of  $\tau(x, y)$  in  $L$  depends on  $(x, y)$  only through  $x_1, \dots, x_g$  and  $y_1, \dots, y_h$ . In fact, writing  $\bar{l}(x_1, \dots, x_g, y_1, \dots, y_h)$  for the coefficient of  $\tau(x, y)$  in  $L$ , one gets

$$\begin{aligned} \bar{l}(x_1, \dots, x_g, y_1, \dots, y_h) &= l(i, j) \\ \text{if } \sum_{k=1}^g a_k x_k &= \alpha_i, \sum_{l=1}^h b_l y_l = \beta_j, \end{aligned} \quad (11)$$

$0 \leq i \leq s_1 - 1$  and  $0 \leq j \leq s_2 - 1$ . Now, as  $a_1 \neq 0$ , the quantity  $\sum_{k=1}^g a_k x_k$  equals each of  $\alpha_0, \alpha_1, \dots, \alpha_{s_1-1}$  once as  $x_1$  assumes all possible values over  $GF(s_1)$ , each exactly once, for any fixed  $x_2, \dots, x_g, y_1, \dots, y_h$ . Hence by (11)

$$\sum_{x_1 \in GF(s_1)} \bar{l}(x_1, \dots, x_g, y_1, \dots, y_h) = 0,$$

for any fixed  $x_2, \dots, x_g, y_1, \dots, y_h$ . Similar arguments for other  $x_k$  and  $y_l$ 's complete the proof.

**Proof of Lemma 3.1.**

$$L(B) = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l(i, j) \left\{ \sum_{(x,y) \in V_{i,j}((a,b),B)} \tau(x, y) \right\},$$

and

$$L^*(B) = \sum_{i=0}^{s_1-1} \sum_{j=0}^{s_2-1} l^*(i, j) \left\{ \sum_{(x,y) \in V_{i,j}((a^*,b^*),B)} \tau(x, y) \right\}.$$

Therefore, it is enough to show that

$$V_{i,j}((a, b), B) = V_{i,j}((a^*, b^*), B) \forall i, j.$$

Since  $(a, b)$  and  $(a^*, b^*)$  are aliases of each other, we have  $a - a^* = B'_1 \lambda$  and  $b - b^* = B'_2 \xi$  for suitable  $\lambda \in GF(s_1)$  and  $\xi \in GF(s_2)$ . Now,

$$\begin{aligned} & V_{i,j}((a, b), B) \\ &= \{(x, y) : a'x = \alpha_i, b'y = \beta_j, B_1x = 0, B_2y = 0\} \\ &= \{(x, y) : (a^* + B'_1 \lambda)'x = \alpha_i, (b^* + B'_2 \xi)'y = \beta_j, \\ &\quad B_1x = 0, B_2y = 0\} \\ &= \{(x, y) : a'^x + \lambda' B_1x = \alpha_i, b'^y + \xi' B_2y = \beta_j, \\ &\quad B_1x = 0, B_2y = 0\} \\ &= \{(x, y) : a'^x = \alpha_i, b'^y = \beta_j, B_1x = 0, B_2y = 0\} \\ &= V_{i,j}((a^*, b^*), B) \end{aligned}$$

which completes the proof of the Lemma.

Table 1: Design Matrix, Paint Experiment

Run	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>
1	1	1	0	1	1	2
2	0	1	1	1	1	2
3	1	0	1	1	1	2
4	0	0	0	1	1	2
5	1	1	0	2	1	0
6	0	1	1	2	1	0
7	1	0	1	2	1	0
8	0	0	0	2	1	0
9	1	1	0	0	1	1
12	0	1	1	0	1	1
11	1	0	1	0	1	1
12	0	0	0	0	1	1
13	1	1	0	1	2	0
14	0	1	1	1	2	0
15	1	0	1	1	2	0
16	0	0	0	1	2	0
17	1	1	0	2	2	1
18	0	1	1	2	2	1
19	1	0	1	2	2	1
20	0	0	0	2	2	1
21	1	1	0	0	2	2
22	0	1	1	0	2	2
23	1	0	1	0	2	2
24	0	0	0	0	2	2
25	1	1	0	1	0	1
26	0	1	1	1	0	1
27	1	0	1	1	0	1
28	0	0	0	1	0	1
29	1	1	0	2	0	2
30	0	1	1	2	0	2
31	1	0	1	2	0	2
32	0	0	0	2	0	2
33	1	1	0	0	0	0
34	0	1	1	0	0	0
35	1	0	1	0	0	0
36	0	0	0	0	0	0