Robust Parameter Design with Feedback Control

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Abstract

Robust parameter design (or, briefly, parameter design) has been widely used as a cost effective tool to reduce process variability by appropriate selection of control factors to make the process insensitive to noise. However, when there are strong noise factors in the process, use of parameter design alone may not be effective and an on-line control strategy can be used to compensate for the effect of noise. In this paper, a parameter design methodology in the presence of feedback control is developed. Systems with pure-gain dynamic model are considered and the best proportional-integral (PI) and minimum mean squared error (MMSE) control strategies are developed by using parameter design. The proposed method is illustrated using a real life example from a urea packing plant.

KEY WORDS: Experiments; Quality Engineering; Process control; Proportional-Integral control; Minimum mean squared error control.

1 Introduction

There are many processes in which there exist strong noise the effect of which cannot be nullified by using parameter design. The use of control is inevitable in these situations.

Feedback control involves measurement of the output at regular intervals and compensation for the effect of the uncontrollable disturbance through a controllable process parameter. To understand the combined role of parameter design and feedback control in reducing process variation, consider a simple model

$$Y_t = -2 + 2x - q_t + 0.5xq_t + 2C_{t-1} + z_t, \tag{1}$$

where x is a control factor that is not changed during production, q is a noise factor and C is a control factor that is adjusted to compensate for the strong unobservable disturbance z. Changing C by one unit at time $t-1$ produces a 2 units change in Y at time t. We assume that q and z are random variables with mean 0 and variance 1. Suppose the target value of Y is 10. Clearly, if we can set $x = 4$ and $C = 2$, then the target is achieved on average, and we have $Var(Y) = 2$. Instead, if we set x to 2 and C to 4, then the effect of q on Y is removed, and the target is still achieved with a much lower variance of 1.

Now suppose that instead of being a white noise process, z_t is a non-stationary disturbance which makes the output Y unstable. In such a case, one can set C to an initial value $C_0 = 4$, and keep on adjusting C_t with a view to compensate for the disturbance z, make the process stable and consequently minimize the variation of Y around the target. In actual practice, this can be achieved by obtaining a forecast of z_t at time point $t-1$ from the past observations and adjusting C based on such a forecast.

However, similar to the case of parameter design with feedforward control (Joesph 2003), a two-stage approach for quality improvement using first parameter design and then control systems may not work well. For example, in model (1), if x interacts with C, and we have a model of the form

$$Y_t = -2 + 2x - q_t + 0.5xq_t + (2 - 0.75x)C_{t-1} + z_t, \tag{2}$$

then obviously the choice of x would have an impact on the effect of q on y (i.e., robustness of the process) as well as on the control law. Thus the control law would depend on the parameter design solution and vice-versa. Another situation where this may happen is when the variance component associated with z_t depends on x. For example, if z_t is an autoregressive process of order 1 such that $z_t = a_t + \phi z_{t-1}$ and $var(a_t) = \sigma^2(x) = (1-0.5x)^2$, then the performance of the control law as well robustness of the process depends on the choice of x.

Joseph (2003) developed a general parameter design methodology for systems with feed-forward control. In this article, we propose an integrated approach to conduct a parameter design experiment for systems with feedback control. In Section 2, we describe an industrial scenario as a motivating example. In Section 3 we give an overview of some common process inertia models and feedback control schemes. In Section 4, a framework for parameter design with feedback control is proposed for a specific class of process inertia models (pure gain) and the discrete proportional-integral (PI) control scheme. In Section 5 we discuss the extension of the proposed framework to minimum mean squared error (MMSE) feedback control scheme. Section 6 illustrates the proposed approach with an example from a packing plant. Section 7 contains some concluding remarks and future research directions.

2 A Motivating Example

As a motivating example we consider the packing experiment described by Dasgupta, Sarkar and Tamankar (2002). The paper describes an automated packing process in which the input material flows into the machine from a hopper. The target weight can be pre-set. There are several control factors $\mathbf{X}$, which are set at the beginning of production and usually not altered.

Let Y denote the response (weight of packed bag) and T denote the target weight. When a bag is packed, the material flows into the bag in two stages, viz. main (coarse) feed stage (when the material flows into the bag thick and fast) and dribble (fine) feed stage (when the material just trickles down into the bag). In-flight material compensation C determines how early the main feed will be cut off. The main-feed cut-off value is $T-(C+\text{Dribble feed quantity})$. For example, if C is set to zero, the target weight is 50 lb, and the dribble feed quantity is 12 lb, then the main feed will be cut off at (50-12)=38 lb. But after the main feed is cut off, there will still be some material flow, which will result in Y being greater than 50 kgs. If C is now increased to 1 lb, then the main feed will be cut off at (50-12-1)

= 37 lb, and Y will consequently be reduced. C is therefore used as an on-line adjustment factor to compensate for the effect of noise. The noise is strong and is a manifestation of a multitude of small effects none of which can be measured individually. However, an off-line noise factor that can be controlled to some extent for experimental purposes is the material composition (course/fine/lumpy).

Among the set of control factors $\mathbf{X}$, some are likely to interact with noise and/or with C. Further, the variance of the unobservable noise is also expected to depend on some of the control factors. This is thus a case of robust parameter design with feedback control. The actual experiment and analysis of data will be discussed in Section 6.

3 Feedback control schemes, process inertia models and role of DOE

3.1 Feedback control schemes and process inertia

Suppose, at time t, the effect of present and past adjustments to C is experienced in the output Y as a compensation of y_t units of weight. The need for control arises because of disturbances whose overall effect on Y is represented at the output by z_t, i.e., z_t represents what would happen to Y if no control action was taken. We consider the noise z_t and the compensation y_t as deviations from the target value. Then, the output error, i.e., the deviation from the target of the controlled process will be

$$e_t = y_t + z_t. \tag{3}$$

There is a vast literature on feedback control schemes (e.g., Astrom (1970); Davis and Vinter (1985); Box, Jenkins and Reinsel (1994)). Among various control schemes, the *discrete proportional-integral* (PI) control schemes have received particular attention because of structural simplicity and ease of implementation. In a discrete PI control scheme, the control equation is of the form

$$-C_t = k_0 + k_p e_t + k_I \sum_{i=1}^{t} e_i, \tag{4}$$

where k_p and k_I are positive constants that determine the amount of proportional and integral control. In the example cited in Section 2, the controller is a special case with $k_p = 0$ (integral control).

Another commonly used feedback control scheme is the *minimum mean squared error* (MMSE) scheme. Under certain model assumptions and choice of parameters, the discrete PI control scheme and MMSE schemes can be shown to be equivalent (Box and Luceno 1997). However, in general, the PI schemes are seen to be quite efficient over a broad range of the parameter space. Furthermore, as shown by Tsung, Wu and Nair (1998), the PI schemes are more robust to model misspecification than MMSE schemes.

A simple first-order dynamic model that characterizes many processes of practical interest is given by the difference equation

$$y_t = \alpha + \delta y_{t-1} + g(1-\delta)C_{t-1}, \tag{5}$$

where $0 < \delta < 1$

The inertial properties of the equation can be appreciated from the consideration that, t time periods after a unit step change is made in C, the change in y will be $g(1-\delta^t)$. Thus, for this dynamic model, the output change asymptotically approaches g units. Details can be found in Box and Kramer (1992).

A further simplicification of (5) can be achieved by assuming that essentially all the change induced by C will occur in a single time interval. This corresponds to setting $\delta = 0$ in (5), i.e.,

$$y_t = \alpha + gC_{t-1}. \tag{6}$$

This is called the *pure-gain model*, which is a realistic scenario in many run-to-run control applications (Del Castillo and Hurwitz 1997). Box and Kramer (1992) considered primarily the pure gain model in their discussion on feedback control. Under model (6), the PI control scheme reduces to integral control ($k_P = 0$ in (4)).

In Section 4, while developing a framework for parameter design with feedback control, we shall restrict attention to the pure-gain dynamics and the integral control scheme.

3.2 Choice of control scheme parameters and role of DOE

It is clear that under the discrete PI control scheme, the control can be poor or unstable if the constant k_I is incorrectly chosen. One way of selection of k_I is to study the nature of the underlying time series model for z_t and use this information for optimum selection of k. For example, if z_t is an ARIMA(0,1,1) process with parameter λ, then under model(5), $k_I = \lambda/g$ results in minimum output variation (Box, Jenkins and Reinsel (1994), Chapter 13).

Suppose a controller has been hooked up to a system, and is approximately of right design but is mistuned. One may try to tune it by formally identifying and fitting models for the process disturbance and dynamics. However, such an approach may be too tedious for routine use. Thus, as pointed out by Box and Kramer (1992), considerable improvement may be achieved by adopting a careful experimental approach, using k_p and k_I as factor levels. However, in case a controller has to be set up from scratch, or there is already a controller whose basic design is inappropriate, one has to design and conduct a more elaborate experiment to identify the appropriate models for process disturbance and dynamics. We shall consider both these situations in our proposed framework described in the following section.

4 A framework for parameter design with first order pure-gain dynamic models and discrete PI control scheme

4.1 Framework and statistical model

Fig 1 depicts a model for feedback control in the presence of control and noise factors. Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ denote the set of control factors that can only be changed at the process set up, and $\mathbf{N} = (N_1, N_2, \ldots, N_q)'$ denote the set of noise factors that can be varied during the experiment but cannot be measured during actual production. In addition, we have a control factor C that is adjusted on-line during production. Y and C are linked by a transfer function of the form

$$Y_t = \beta(\mathbf{X}, \mathbf{N}, C_{t-1}, C_{t-2}, \ldots) + z_t.$$

At time t, a correction is given to C_t on the basis of the observed output error e_t through a control equation $C_t = f(e_t, e_{t-1}, \ldots)$. As discussed in Section 3.1, we shall consider the following forms for the functions β and $f(e_t, e_{t-1}, \ldots)$:

$$\begin{aligned}\beta(\mathbf{X}, \mathbf{N}, C_{t-1}, C_{t-2}, \ldots) &= \alpha_0(\mathbf{X}, \mathbf{N}) + g(\mathbf{X})C_{t-1}, \\ f(e_t, e_{t-1}, \ldots) &= -k_0 - k_I \sum_{i=1}^{t} e_i.\end{aligned}$$

We thus postulate the following first-order pure-gain dynamic model

$$Y_t = \alpha_0(\mathbf{X}, \mathbf{N}) + g(\mathbf{X})C_{t-1} + z_t, \tag{7}$$

where $\{z_t\}$ is the disturbance due to unobservable noise factors and may be a stationary or a non-stationary process.

If T denotes the target, and $e_t = Y_t - T$ denotes the deviation from the target, then

$$e_t = \alpha(\mathbf{X}, \mathbf{N}) + g(\mathbf{X})C_{t-1} + z_t, \tag{8}$$

where $\alpha(\mathbf{X}, \mathbf{N}) = \alpha_0(\mathbf{X}, \mathbf{N}) - T$.

Note that this is essentially the same as (3), where y_t is given by (6). The added aspect is the dependence of the dynamics on $\mathbf{X}$ and $\mathbf{N}$. It is thus imperative that if an integral control scheme is employed to such a process, k_I necessarily has to be a function of $\mathbf{X}$ and $\mathbf{X}$ has to be such that the output is least sensitive to the effect of $\mathbf{N}$.

4.2 Performance measure

Let us assume that $z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}$, where $\{a_t\}$ is a white noise process with zero mean and variance $\sigma^2(\mathbf{X})$. Recalling that in an integral control scheme C_{t-1} is set to $-k_0 - k_I \sum_{i=1}^{t-1} e_i$, we have from (8),

$$\begin{aligned}e_t &= \alpha(\mathbf{X}, \mathbf{N}) - g(\mathbf{X})\Big(k_0 + k_I \sum_{i=1}^{t-1} e_i\Big) + z_t \\ &= \Big(\alpha(\mathbf{X}, \mathbf{N}) - g(\mathbf{X})k_0\Big) - g(\mathbf{X})k_I \sum_{i=1}^{t-1} e_i + \sum_{j=0}^{\infty} \psi_j a_{t-j}.\end{aligned} \tag{9}$$

Clearly, the objective is to select $\mathbf{X}$ and the control law in such a way that the variance of e_t is minimized. Thus, it is reasonable to consider $Var(e_t)$ as the appropriate performance measure. We have,

$$\begin{aligned} Var(e_t) &= Var_a E_{\mathbf{N}}(e_t|a) + E_a Var_{\mathbf{N}}(e_t|a) \\ &= Var(u_t) + \pi(\mathbf{X}), \end{aligned} \tag{10}$$

where

$$\begin{aligned} u_t &= E_{\mathbf{N}}(e_t|a) \\ &= \Big(\alpha(\mathbf{X}) - g(\mathbf{X})k_0\Big) - g(\mathbf{X})k_I \sum_{i=1}^{t-1} u_i + \sum_{j=0}^{\infty} \psi_j a_{t-j} \end{aligned}$$

and $\quad \alpha(\mathbf{X}) = E_{\mathbf{N}}(\alpha(\mathbf{X}, \mathbf{N})), \qquad \pi(\mathbf{X}) = Var_{\mathbf{N}}(\alpha(\mathbf{X}, \mathbf{N}))$.

It is seen that if model (8) and the distribution of $\mathbf{N}$ is known, then we can compute the performance measure by substituting $Var(u_t)$ in (10). For a given process disturbance model (i.e., given the weights ψ_j) $Var(u_t)$ can be obtained through routine but tedious derivations (Tsung, Wu and Nair 1996). For example, consider the case where z_t is an ARIMA(0,1,1) process with parameter λ. Obtaining an expression for $Var(u_t)$ (Box and Kramer 1992) and its substitution in (10) yields

$$\begin{aligned} Var(e_t) &= \frac{1+\theta^2 - 2\phi(\mathbf{X})\theta}{1-\phi^2(\mathbf{X})}\sigma^2(\mathbf{X}) + \pi(\mathbf{X}), \quad && \text{if } -1 < \phi(\mathbf{X}) \leq 1, \\ &= \infty && \text{otherwise,} \end{aligned} \tag{11}$$

where $\phi(\mathbf{X}) = 1 - g(\mathbf{X})k_I$ and $\theta = 1 - \lambda$.

To illustrate the computation of the performance measure, let us revisit the earlier example in (2):

$$Y_t = -2 + 2x - N + 0.5xN + (2 - 0.75x)C_{t-1} + z_t.$$

We assume that $z_t = a_t + \lambda \sum_{i=1}^{t-1} a_i$ is an ARIMA(0,1,1) process with $\lambda = 0.2$, $E(a_t) = 0$, $Var(a_t) = (1 - 0.5x)^2$ and N is a random variable with mean 0 and variance 1. It can easily be seen that the performance measure is given by

$$PM(\mathbf{X}, k_I) = \frac{1.64 - 1.6\{1 - k_I(2 - 0.75x)\}}{1 - \{1 - k_I(2 - 0.75x)\}^2}(1 - 0.5x)^2 + (1 + 0.25x^2).$$

The problem is thus to

$$\text{minimize } PM(\mathbf{X}, k_I) \text{ subject to}$$
$$x^L \leq x < 2.67,$$
$$0 < k_I < \tfrac{0.2}{2-0.75x}.$$

The upper bound of $\mathbf{X}$ comes from the constraint $g(\mathbf{X}) > 0$ and x_L is a suitable lower bound beyond which the experimenter is not willing to go. Figure 2 shows the surface of the performance measure function. Setting $x_L = 0.5$, we find that the minimum is obtained at $x^* = 1$, $k_I^* = 0.16$ and the corresponding value of the performance measure is $PM(x^*, k_I^*) = 1.5$.

The performance of the system without control may be evaluated by substituting $k_I = 0$ in the expression for performance measure. In the example discussed above, this would lead to an infinite variance, which is understood from the fact that the underlying disturbance is non-stationary and without control, the response would be unstable.

In order to express PM as a suitable function of $\mathbf{X}$ and k_I, one may conduct a suitable open-loop experiment with $\mathbf{X}, \mathbf{N}$ and C as experimental factors to estimate model (8), fit an appropriate time series model for z_t and then obtain an expression for PM by considering an appropriate distribution of $\mathbf{N}$. This approach is known as *response modelling* (Wu and Hamada 2000, Chapters 10 and 11). An alternative procedure is to directly model PM as a function of $\mathbf{X}$ and k_I by treating k_I as an experimental factor and conducting the experiment with the control loop. This is called *performance measure modelling*. In the following two subsections, we discuss the design and analysis of experiments under these two approaches.

4.3 Design of experiments and analysis of data

4.3.1 Response modelling

Recall that in Section 3.2, we had assumed the dependence of σ_a^2 (the variance of a_t) on $\mathbf{X}$. Thus, the response modelling may be thought of as a two-step approach:

1. Fitting a transfer function of the form $e_t = \alpha + gC_{t-1} + z_t$ for various combinations of $\mathbf{X}$ and $\mathbf{N}$.

2. Modelling α as a function of $(\mathbf{X}, \mathbf{N})$; g and σ_a^2 as functions of $\mathbf{X}$.

To achieve this objective, we may use a cross array design between $\mathbf{X}$ and $\mathbf{N}$ and nest all the levels of C within each $\mathbf{X}, \mathbf{N}$ combination. Thus, for each combination of $\mathbf{X}$ and $\mathbf{N}$, a time series in Y_t and hence e_t will be obtained by changing the levels of C. As in the simulated experiment described on p. 442 of Box et al. (1994), each level of C may be held constant for a fixed time, and τ observations may be generated. Figure 3 depicts such an experimental plan where three levels of C are considered. Instead of employing a cross array design, we may use a single array as well, ensuring that all the interactions between $\mathbf{X}$ and $\mathbf{N}$ are estimable. Details of cross array and single array designs may be found in Chapter 10 of Wu and Hamada (2000).

As indicated above, the analysis consists of the following two broad stages:

1. For each combination of $\mathbf{X}$ and $\mathbf{N}$,
   (a) Identification of the form of the disturbance $z_t = \sum_{j=0}^{\infty} \psi_j a_{t-j}$ using autocorrelation function (ACF) plots and partial autocorrelation function (PACF) plots;
   (b) Estimating the parameters (α, g, ψ_j's, σ_a^2) using a constrained iterative nonlinear least squares algorithm (Box et al. 1994, Chapter 7).
2. Treating α, g and σ_a^2 as three different responses, identify significant control factors and control-by-noise interactions and fit $\widehat{\alpha} = \alpha(\mathbf{X}, \mathbf{N})$, $\widehat{g} = g(\mathbf{X})$, $\widehat{\sigma_a^2} = \sigma_a^2(\mathbf{X})$.

Note that if we assume that σ_a^2 does not depend on $\mathbf{X}$, i.e., all the parameters associated with the disturbance z_t are free of $\mathbf{X}$, then the experiment can be considerably simplified. In such a case we may estimate the time series parameters from a single experimental run for fixed $\mathbf{X} = \mathbf{X}^*$ and $\mathbf{N} = \mathbf{N}^*$. Next, we may conduct a cross array design $D(\mathbf{X}) \otimes D(\mathbf{N}) \otimes D(C)$ and estimate model (3) directly from the experimental data.

Although the response modelling approach provides an in-depth understanding of the underlying phenomena, there is a possibility that the experiment will be very large and will require intensive computation. Further, it is obvious, that this experiment has to be run with an open loop. For systems in which controllers have already been installed, industrial personnel would usually be reluctant to run open-loop experiments. Thus, when the objective is to achieve robustness of a system that already has a feedback controller, the performance modelling approach discussed in the following section will be appropriate.

4.3.2 Performance measure modelling

Since this approach can be thought of as modelling the performance measure as a function of $\mathbf{X}$ and k_I, it would be reasonable to use a cross array design between $\mathbf{X}$ and k_I. The various noise combinations may be nested within each $(\mathbf{X}, k_I)$ combination. Thus, for each combination of $\mathbf{X}$ and k_I, a time series in Y_t and hence e_t will be obtained by changing the levels of $\mathbf{N}$. An illustration is given in Figure 4, where three levels of k_I and two noise combinations have been considered. In order to reduce the run-size, $\mathbf{X}$ and k_I can be accommodated in a single array. In this case, one must ensure that the important interactions between k_I and $\mathbf{X}$ are estimable.

Note that, in this set-up, the adjustment factor C need not be included in the experiment as it will be automatically changed during the course of the closed loop experiment. However, k_0 corresponds to the initial value of C (at time 0) and may be treated as 'another' control factor.

Selection of levels for k_I is a very important aspect. At least three levels should be chosen for k_I since for any given setting, the variation in the output is approximately a quadratic function of k_I and the quadratic effect of k_I should be important. However, keeping in mind the fact that grossly improper choice of k_I may make the output unstable and upset the entire process, some caution should be exercised in the selection of its levels. Thus, it may not be possible to hit the optimum with a single experiment and additional runs may be added later in the evolutionary operation mode as suggested by Box and Kramer (1992).

Let Y_{ijkt} be the t^{th} measured value of the characteristic at the i^{th} level of $\mathbf{X}$, j^{th} level of k_I, and k^{th} level of $\mathbf{N}$ $(i = 1, 2, \ldots, I,\ j = 1, 2, \ldots, J,\ k = 1, 2, \ldots, K,\ t = 1, 2, \ldots, \tau)$. Let $e_{ijkt} = Y_{ijkt} - T$. Then we compute the estimated value of the performance measure corresponding to the i^{th} level of $\mathbf{X}$ and j^{th} level of k_I as

$$\widehat{PM}_{ij} = \frac{1}{K\tau - 1}\sum_{t=1}^{K\tau}\Big(e_{ijkt} - \overline{e_{ij..}}\Big)^2,$$

$$\text{where } \overline{e_{ij..}} = \tfrac{1}{K\tau}\textstyle\sum_{k=1}^{K}\sum_{t=1}^{\tau} e_{ijkt}.$$

Next, we fit the linear regression model

$$\ln \widehat{PM} \quad = \quad f(\mathbf{X}, k_I), \tag{12}$$

and determine optimum values of $\mathbf{X}$ and k_I by optimizing the fitted function $f(\mathbf{X}, k_I)$.

The above analysis is based on the assumption that all the factor-level combinations would produce a stationary output around T, thereby ensuring the finiteness of $V(e_t)$. As seen in Section 4.2, this may not be the case if k_I is chosen at a level beyond the range within which it is capable of producing a stationary output. If it is found that for some level combinations (i, j, k), $\overline{e_{ij..}}$ is largely different from zero, it would be pragmatic to use the sample mean squared error $m_{ij}^2 = \frac{1}{K\tau-1}\sum_{t=1}^{K\tau}\Big(Y_{ijkt} - T\Big)^2$ instead of $\widehat{PM}_{ij}$.

5 Parameter design with the MMSE control scheme and the pure-gain model

Once again, consider model (8) with the same assumptions for the process disturbance z_t as mentioned at the beginning of Section 4.2. Under the MMSE control scheme, C_{t-1} is to be set in such a way that the one-step ahead forecast of Y_t is equal to the target T (or equivalently, the forecast of e_t is zero). An elaborate discussion can be found in Box and Luceno (1997).

Thus, under the MMSE scheme for the pure gain model, C_{t-1} should be such that

$$\hat{e}_{t-1}(1) = \alpha(\mathbf{X}, \mathbf{N}) + g(\mathbf{X})C_{t-1} + \hat{z}_{t-1}(1) = 0. \tag{13}$$

Since $\alpha(\mathbf{X}, \mathbf{N})$ will not be known in reality, the MMSE control equation can be obtained from the above by taking the expectation over $\mathbf{N}$, i.e.,

$$C_{t-1} = \frac{1}{g(\mathbf{X})}\Big(- \alpha(\mathbf{X}) - \hat{z}_{t-1}(1)\Big). \tag{14}$$

Substituting (14) into (8), we get

$$e_t = \Big(\alpha(\mathbf{X}, \mathbf{N}) - \alpha(\mathbf{X})\Big) + (z_t - \hat{z}_{t-1}(1)). \tag{15}$$

Thus, we have

$$\begin{aligned} Var(e_t) &= Var_a E_{\mathbf{N}}(e_t|a) + E_a Var_{\mathbf{N}}(e_t|a) \\ &= Var\Big(z_t - \hat{z}_{t-1}(1)\Big) + \pi(\mathbf{X}) \\ &= \sigma^2(\mathbf{X}) + \pi(\mathbf{X}), \end{aligned} \tag{16}$$

where $\pi(\mathbf{X})$ is defined as before.

The last step of (16) follows from the properties of the MMSE forecast (Box et al. 1994, Chapter 5). The fact that under the ARIMA(0,1,1) disturbance, the discrete PI scheme with $k_I = \frac{\lambda}{g(\mathbf{X})}$ is the same as the MMSE scheme can be seen easily by substituting $k_I = \frac{\lambda}{g(\mathbf{X})}$ in (11) and observing that $Var(e_t)$ reduces to the form given by (16).

It is evident that with the MMSE control scheme, only the response modelling approach would work, since it is not possible to establish the control equation and observe the output error under control without estimating model (8). The design and analysis of experiments will be similar to that discussed in Section 4.3.

6 A case study

In Section 2 a study on optimization of a control scheme of the packing process of a urea manufacturing plant (Dasgupta et al., 2002) was mentioned as a motivating example. The underlying control scheme was a discrete PI scheme, slightly different from the classical one (see Appendix of Dasgupta et al. 2002). Among the 16 factors listed in the original case study, D (auto compensation proportional constant) and K (in-flight material compensation - start) correspond to k_I and k_0 respectively. The factors and levels of the experiment are given in Table 1.

The original experiment was conducted and analyzed somewhat superficially just like "another" parameter design exercise, not recognizing the specific roles of the factors k_I and

k_0. It may also be noted that the original experiment did not explicitly consider any noise factor. However, since we have three replicates, we can consider them to correspond to three levels of the noise factor material composition.

An L_{32} orthogonal array (OA) with an idle column was used to design the experiment. The idle column method is a technique to generate three-level columns by collapsing two columns in a two-level orthogonal array (Taguchi 1987; Grove and Davis 1991; Huwang, Wu and Yen 2002). Three-level columns for factors G, Q and k_I, as shown in Table 2, were obtained by this method. Besides the 16 main effects (19 degrees of freedom), provisions were made for estimation of five interaction effects, viz. $A \times B$, $G \times M$, $G \times J$, $H \times L$ and $B \times k_I$.

Sixty bags were filled for each of the $32 \times 3 = 96$ level combinations. The target weight was set at $T = 50.5$ kg. Although in our proposed framework, for each experimental combination we would generate a single time series over the different levels of noise, such was not the case in the original experiment. However, to demonstrate the proposed approach, we assume that the 180 observations corresponding to each of the 32 experimental runs constitute a single time series. The sample standard deviation of 180 observations corresponding to each run is shown in Table 2.

Using half-normal plots for the 27 treatment effects (Figure 5), we find that four effects viz. JG_q, k_0, Q_q and G_q stand apart from the rest. Using Lenth's method (Wu and Hamada 2000, Chapter 4) as a formal test of significance, it is seen that these 4 effects are significant at 1% level. Note that for any three-level factor X, we denote the linear and quadratic effects (orthogonal to each other) by X_l and X_q respectively (Wu and Hamada 2000, Chapter 5).

We notice that neither the main effect of k_I nor the $B \times k_I$ interaction (which is estimable) are significant. Another factor that is suspected to interact with k_I is C, and we explore the possibility of incorporating the $C \times k_I$ interaction into our analysis by replacing some insignificant effects in the preliminary model. Using the table of interactions for the L_{32} OA, we find that the $C \times k_I$ interaction can be estimated from columns 8 and 9 of the OA. Out of these two columns, 8 corresponds to factor M which is seen to be insignificant and 9 is a free column to which no other main effect or significant interaction is assigned. Thus we re-perform the analysis by including the $C \times k_I$ interaction (with two degrees of freedom) instead of M.

The half-normal plot (Figure 6) now identifies 6 effects standing above the rest. Apart from the four that were already seen to be significant, the interaction effects Ck_{I_l} and Ck_{I_q} turn out to be significant at 1% level.

From the plots of significant main effects (Figure 7) and the $G \times J$ interaction (Figure 8), we choose the optimal settings of k_0, J, G and Q as $k_0 = -1$,$J = 1$,$G = 1$ and $Q = 2$. Note that all of these factors (or interactions involving them) were found significant in the original analysis and the same levels (although different notations were used) were selected as the optimum ones.

The interaction plot of $C \times k_I$ (Figure 8) suggests that at neither of the two levels of C, the optimum k_I could be reached. Corresponding to $C = 1$, $k_I^*(C) \leq 0$ while corresponding to $C = -1$, $k_I^*(C) \geq 2$. We also note that curve corresponding to $C = 1$ is slightly convex as expected, whereas that corresponding to $C = -1$ exhibits a slight concavity, which can be attributed to sampling error or effect of higher order interactions. Since this difference in convexity results in significance of the quadratic component of the interaction, we may only consider the linear component of the interaction while modelling $\ln P\hat{M}$. From the interaction plot, we choose $C = -1$ and $k_I = 2$ as the optimal settings.

The following model is thus obtained:

$$\ln \widehat{PM} = -5.744 + 1.241x_{k_0} - 0.832x_G + 0.334x_G^2 + 1.073x_Q - 0.586x_Q^2 \\ +0.236x_C x_{k_I} - 0.124x_J x_G^2. \tag{17}$$

Substituting the optimal settings of the control factors in the above model, we get the optimal value of the performance measure as $\ln \widehat{PM}^* = -8.279$, which corresponds to a standard deviation of 0.016. The original experiment was able to reduce the output standard deviation drastically to 0.031 from the existing value of 0.121. We find from this re-analysis that with the newly recommended settings, it might have been possible to reduce the variation to almost 50% of what had been achieved.

This analysis also points out the importance of adopting a sequential approach for the performance modelling experiment. It is clear that with wider choices of levels of k_I, it might have been possible to reduce the output variance even further.

7 Concluding remarks

In this article we have developed a framework for robust parameter design of systems with feedback control. The suggested approach can be used to obtain the optimal control law and robust parameter design solution in a single stage. Appropriate performance measures are developed and the design and analysis of experiments for estimation and optimization of these performance measures are discussed. The benefits of using the proposed method are demonstrated using an example from a packing plant.

Although we have considered the pure-gain dynamic model and primarily the discrete PI control scheme, it should be possible to extend the proposed methodology to a much more generic class of models and other control schemes without much difficulty. However, systems that have on-line measurable noise factors along with provision for feedback control would be of interest and future research may consider robust parameter design with a combination of feedback and feedforward control.

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Table 1: Factors and Levels, Packing Plant Experiment

	Control Factor	Level 1	Level 2	Level 3
A.	Sample frequency	10	20	–
B.	Sample number	3	5	–
C.	Sample frequency timer	240	300	–
E.	Main feed blanking timer (sec)	0.5	0.9	–
F.	Dribble feed blanking timer (sec)	0.5	0.9	–
G.	Discharge timer (sec)	0.3	0.6	0.9
H.	Dribble feed time correction constant (sec)	0.1	0.8	–
I.	Gate allowance timer (sec)	0.4	2.0	–
J.	Feed delay timer (sec)	0.3	0.7	–
L.	Dribble feed quantity (start) (Kg)	8	12	–
M.	Discharge cut-off value (Kg)	20	30	–
N.	Overweight tolerance (Kg)	0.15	0.20	–
P.	Underweight tolerance (Kg)	0.10	0.15	–
Q.	Dribble feed time (sec)	1.2	1.4	1.6
	PI control scheme parameters	**Level 1**	**Level 2**	**Level 3**
k_I.	Auto compensation proportional constant	0.2	0.3	0.4
k_0.	In-flight material compensation (start)	2.0	3.0	–
	Noise factors	**Level 1**	**Level 2**	**Level 3**
Z.	Composition of material	Z_1	Z_2	Z_3

Table 2: Data, Packing Experiment

Control Factor														PI		Variation over 3 noise levels	
A	B	C	E	F	G	H	I	J	L	M	N	P	Q	k_I	k_0	s.d.	$\ln(s^2)$
+	−	+	+	+	0	−	+	−	+	+	−	+	0	1	−	0.0584	-5.6821
−	−	−	+	−	0	−	−	−	−	+	+	−	1	0	+	0.1167	-4.2966
+	+	−	−	+	0	+	+	−	+	−	+	−	0	1	+	0.1320	-4.0499
−	+	+	−	−	0	+	−	−	−	−	−	+	1	0	−	0.0979	-4.6484
+	+	+	−	+	0	+	−	+	−	+	−	−	1	1	−	0.0287	-7.1007
−	+	−	−	−	0	+	+	+	+	+	+	+	0	0	+	0.0347	-6.7234
+	−	−	+	+	0	−	−	+	−	−	+	+	1	1	+	0.1100	-4.4138
−	−	+	+	−	0	−	+	+	+	−	−	−	0	0	−	0.0289	-7.0905
−	+	+	+	−	2	−	−	−	−	+	+	+	0	1	−	0.0235	-7.4974
+	+	−	+	+	2	−	+	−	+	+	−	−	1	0	+	0.1066	-4.4772
−	−	−	−	−	2	+	−	−	−	−	−	−	0	1	+	0.1065	-4.4792
+	−	+	−	+	2	+	+	−	+	−	+	+	1	0	−	0.0421	-6.3351
−	−	+	−	−	2	+	+	+	+	+	+	−	1	1	−	0.0232	-7.5304
+	−	−	−	+	2	+	−	+	−	+	−	+	0	0	+	0.1465	-3.8414
−	+	−	+	−	2	−	+	+	+	−	−	+	1	1	+	0.1021	-4.5632
+	+	+	+	+	2	−	−	+	−	−	+	−	0	0	−	0.0206	-7.7693
+	−	−	−	−	1	−	+	−	−	+	−	−	0	1	−	0.0165	-8.2079
−	−	+	−	+	1	−	−	−	+	+	+	+	2	2	+	0.1134	-4.3529
+	+	+	+	−	1	+	+	−	−	−	+	+	0	1	+	0.0955	-4.6972
−	+	−	+	+	1	+	−	−	+	−	−	−	2	2	−	0.0130	-8.6886
+	+	−	+	−	1	+	−	+	+	+	−	−	2	1	−	0.0187	-7.9619
−	+	+	+	+	1	+	+	+	−	+	+	+	0	2	+	0.0644	-5.4851
+	−	+	−	−	1	−	−	+	+	−	+	+	2	1	+	0.1531	–3.7536
−	−	−	−	+	1	−	+	+	−	−	−	−	0	2	−	0.0255	-7.5840
−	+	−	−	+	2	−	−	−	+	+	+	+	0	1	−	0.0259	-7.3038
+	+	+	−	−	2	−	+	−	−	+	−	−	2	2	+	0.1737	-3.5010
−	−	+	+	+	2	+	−	−	+	−	−	−	0	1	+	0.1190	-4.2572
+	−	−	+	−	2	+	+	−	−	−	+	+	2	2	−	0.0222	-7.6150
−	−	−	+	+	2	+	+	+	−	+	+	−	2	1	−	0.0180	-8.0317
+	−	+	+	−	2	+	−	+	+	+	−	+	0	2	+	0.1510	-3.7810
−	+	+	−	+	2	−	+	+	−	−	−	+	2	1	+	0.0311	-6.9418
+	+	−	−	−	2	−	−	+	+	−	+	−	0	2	−	0.0178	-8.0519

Figure 1: Feedback control with control & noise factors

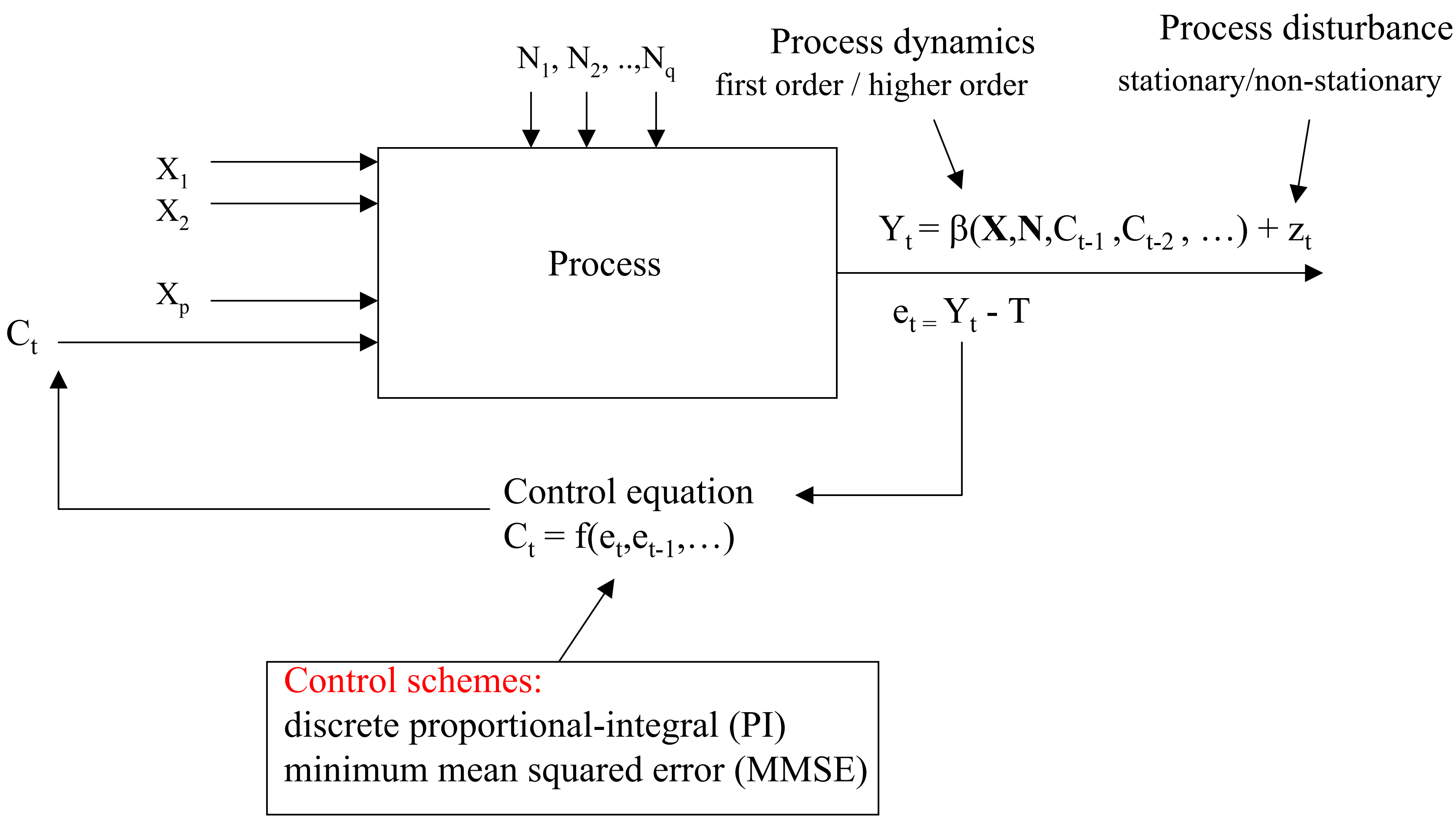


Figure 2: Surface plot of PM(X,k_I)

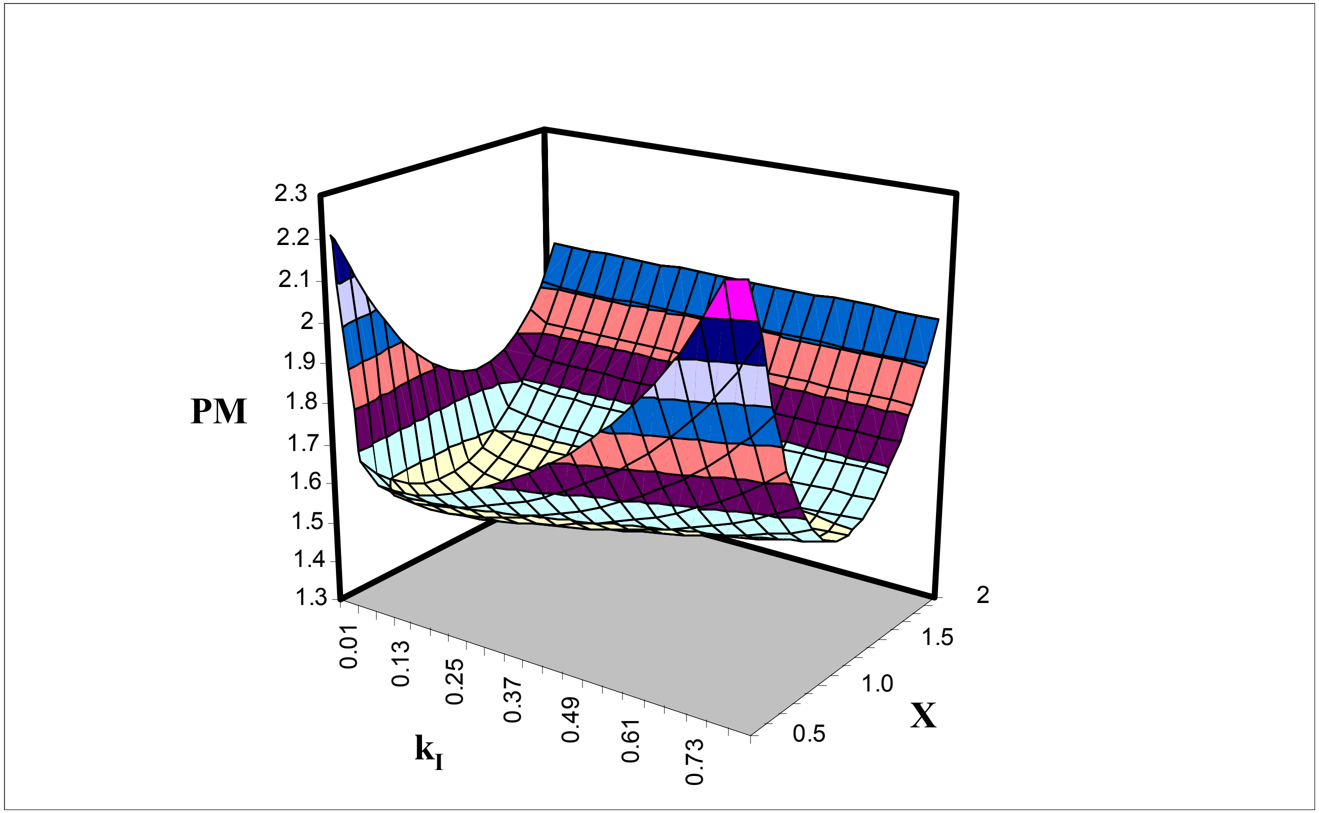


Figure 3: DOE for response modelling

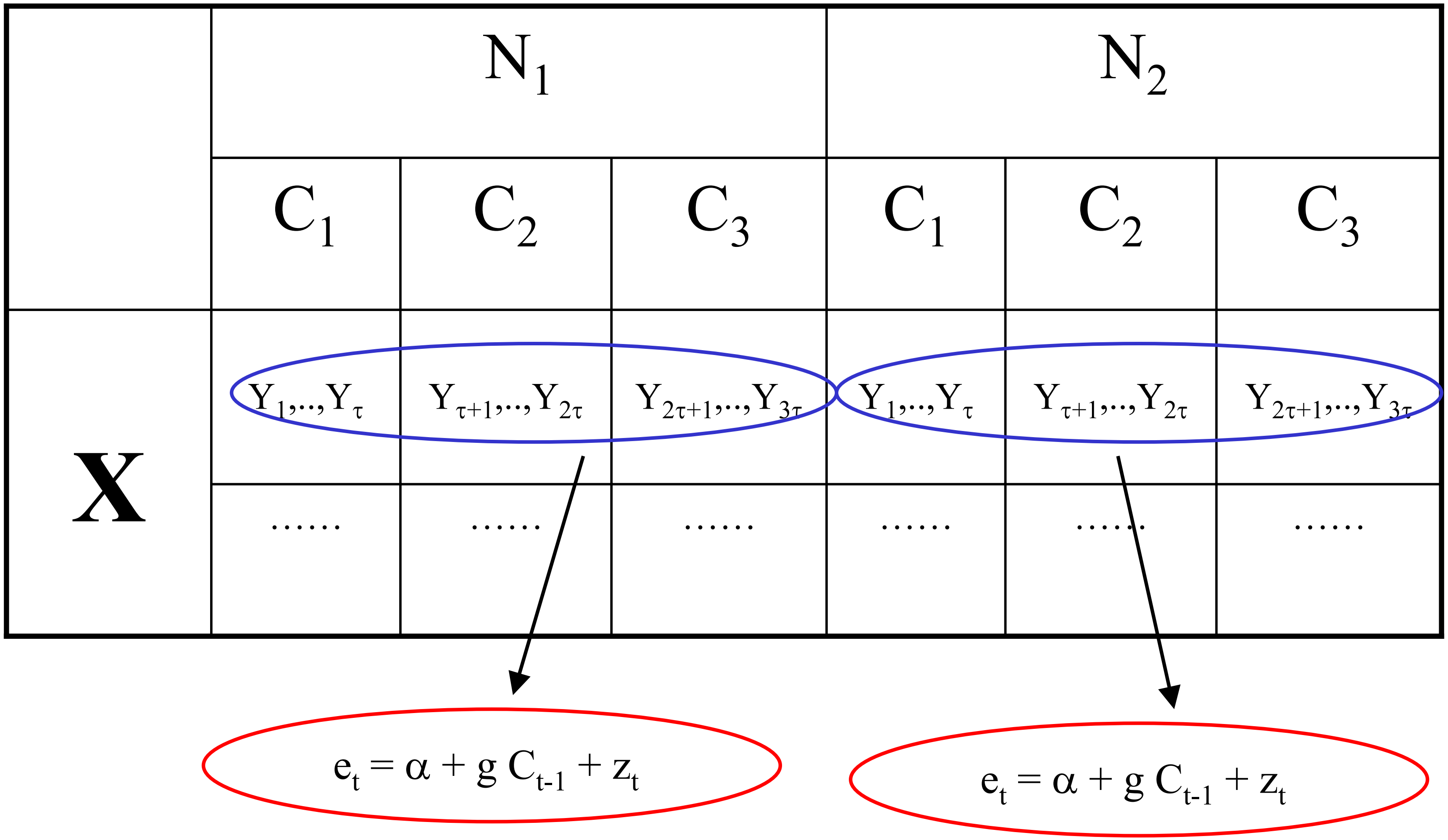


Figure 4: DOE for performance measure modeling

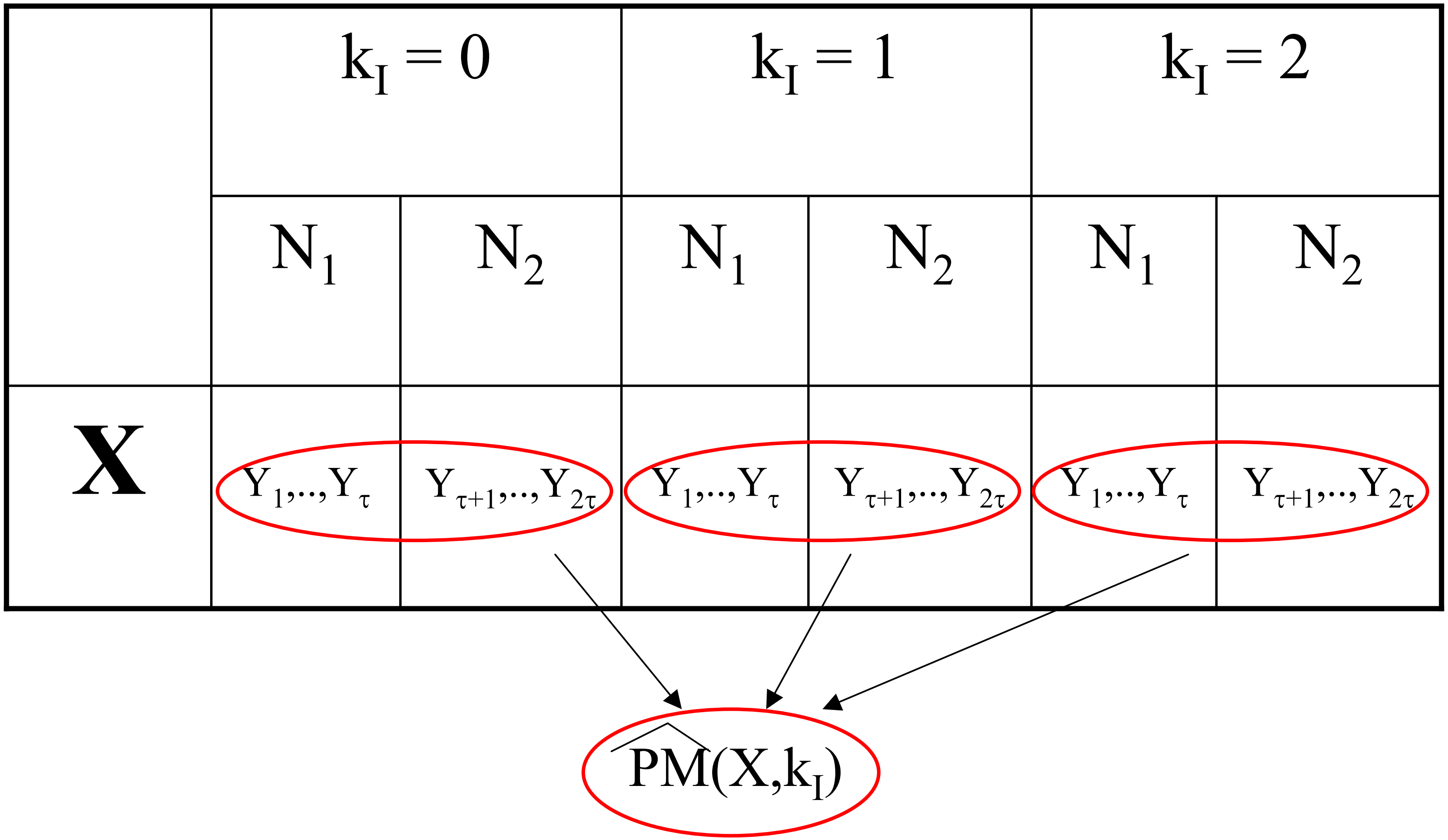


Figure 5: Half-normal plot

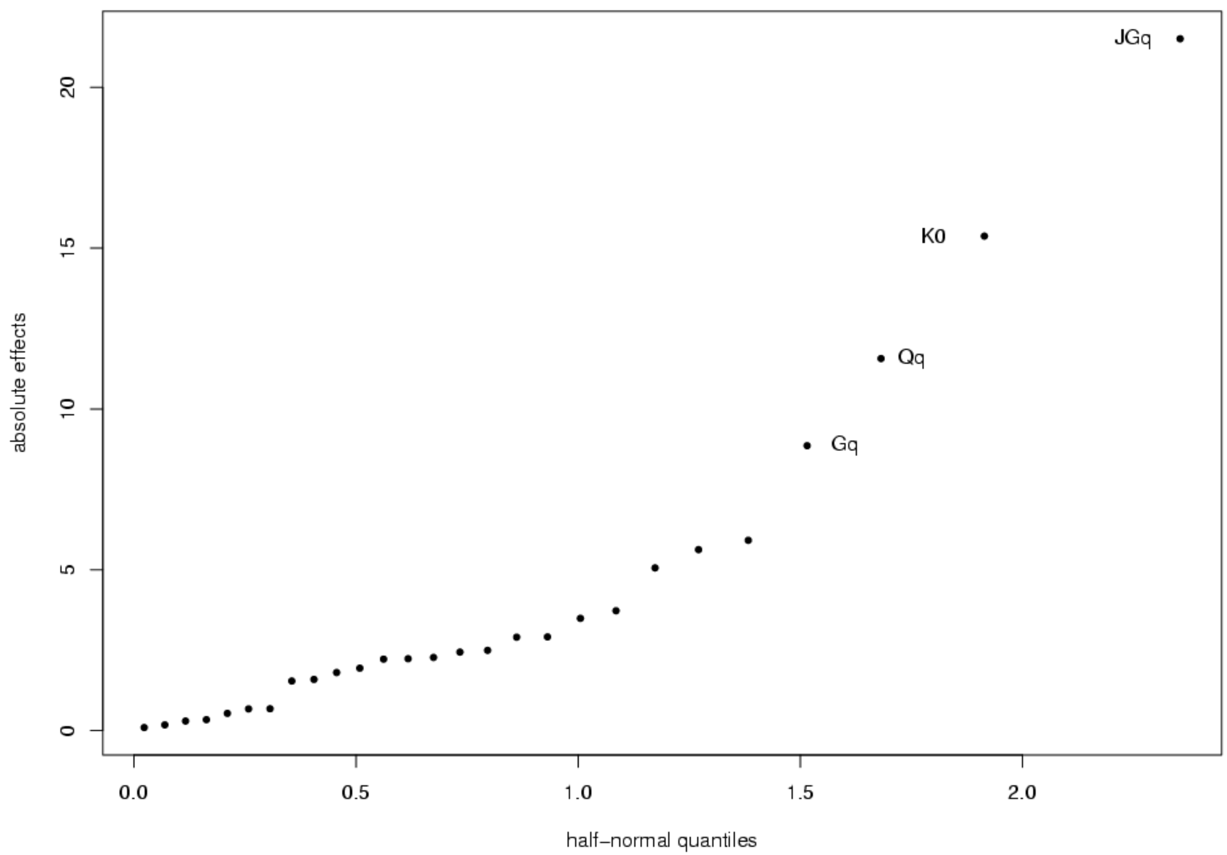


Figure 6 : Modified half-normal plot (with C×k_I interaction)

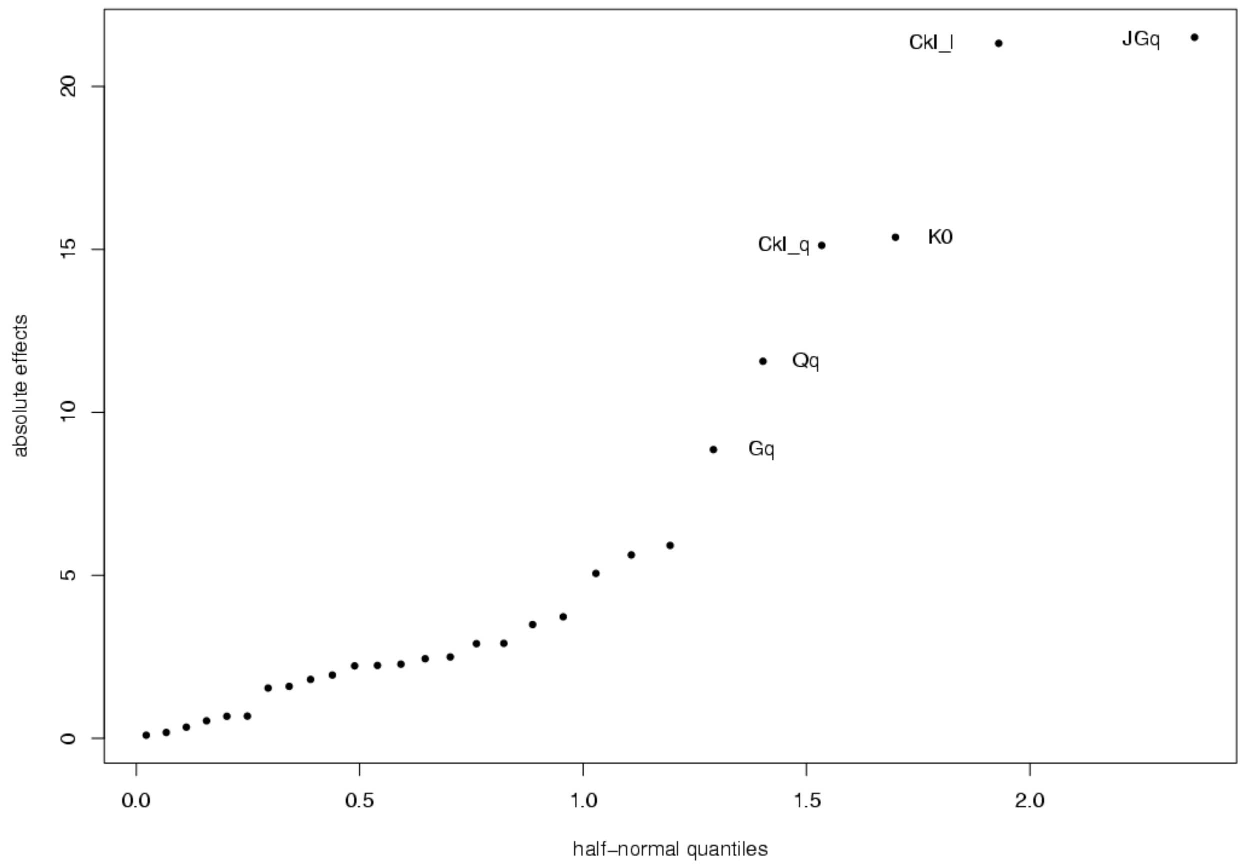
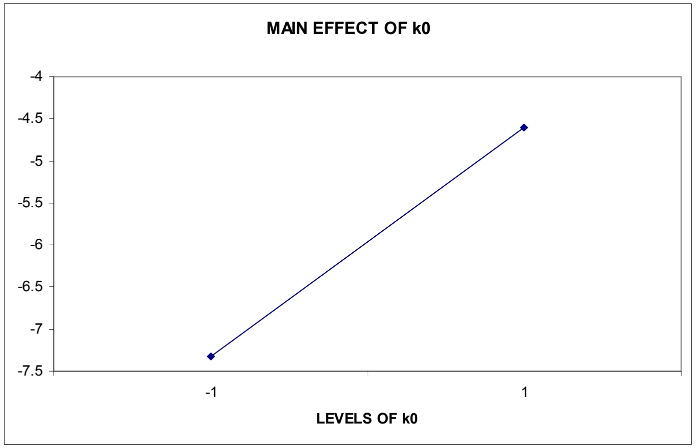
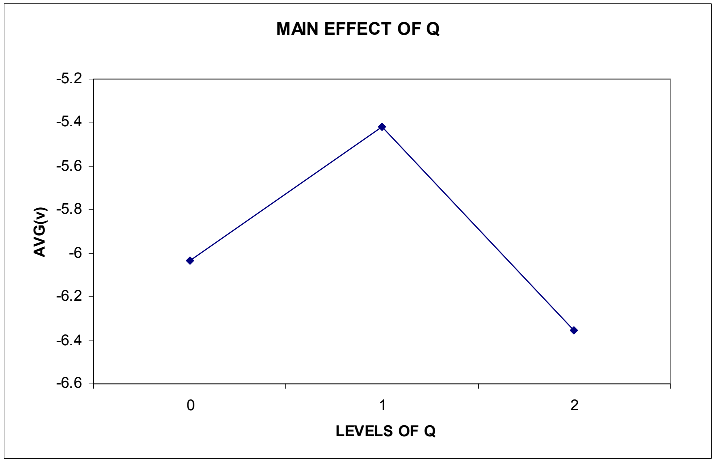


Figure 7: Main effects plots





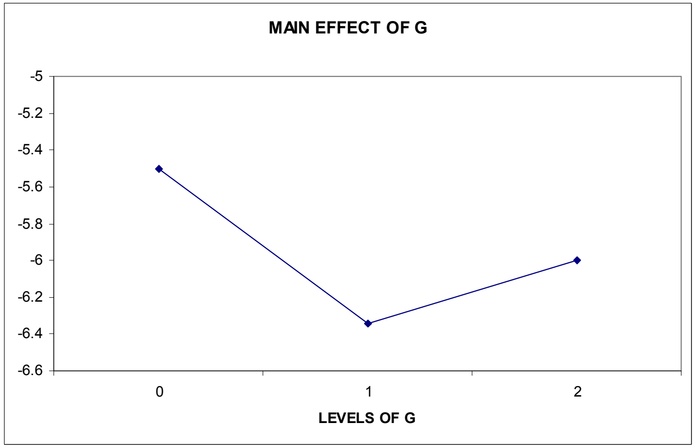


Figure 8: Interaction plots