Ordering Quantity Decisions Considering Uncertainty in Supply-Chain Logistics Operations

Hyoungtae Kim, Jye-Chyi Lu, Paul H. Kvam
School of Industrial and Systems Engineering
Georgia Institute of Technology, Atlanta, GA 30332

Abstract

This research seeks to determine the optimal order amount for the retailer given uncertainty in a supply-chain's logistics network due to unforseeable disruption or various types of defects (e.g., shipping damage, missing parts and misplacing products). Mixture distribution models characterize problems from solitary failures and contingent events causing network to function ineffectively. The uncertainty in the number of good products successfully reaching the distribution center (DC) and retailer poses a challenge in deciding product-order amounts. Because the commonly used ordering plan developed for maximizing expected profits does not allow retailers to address concerns about contingencies, this research proposes two improved procedures with risk-averse characteristics towards low probability and high impact events. Several examples illustrate the impact of DC's operation policies and model assumptions on retailer's product-ordering plan and resulting sales profit.

Key Words: contingency, logistics network, mixture distribution, stochastic optimization, supply chain.

1 Introduction

In this era of global sourcing, to reduce purchase costs and attract a larger base of customers, retailers such as Wal-Mart, Home Depot and Dollar General are constantly seeking suppliers with lower prices and finding them at greater and greater distances from their distribution centers (DCs) and stores. Consequently, a significant proportion of shipped products from overseas suppliers is susceptible to defects. Reasons for defects include missing parts, misplaced products (at DCs, stores) or mistakes in orders and shipments. Sometimes, products are damaged from mishandling in transportation or are affected by the low probability and high impact contingency such as extreme weather, labor dispute and terrorist attack. When there are logistics delays due to security inspections at U.S. borders and seaports, or simply traffic problems, orders, packagings and shipments do not arrive at the DCs or stores on time. Regardless of the problems contributed from the supply sources or logistics operations, this article considers all of them as *supply and logistics defects*. Two case studies with a major retailing chain indicate that the proportion of the "defects" could reach 20%. This creates significant challenge in product-ordering and shelf-space management.

If the defect rate is not accounted for in the purchase order, the resulting product shortages serve as precursors to several consequences, including inconveniencing their customers, compromising the retailer's reputation for service quality, and then having to trace, sell, repair or return the defective products. Based on our interaction experience with retailers, the stock-out problem can cause more than billion dollars in a large retail chain. On the other hand, use of excessive inventory to handle the uncertain supply and logistics defects is also costly. This article models the defect process in a three-level supply chain network with many suppliers, one DC and one store, and link it to DCs operation policy for developing an optimal product-ordering scheme.

The literature on supply-chain contract decisions feature methods that utilize a high-level general model to describe supply uncertainties without getting into any degree of logistics details. See examples in Sculli and Wu [8], Ramasech [7], Lau and Lau [2], Parlar and Perry [6], Parlar [5], Weng [11], and Mohebbi [3] for diverse implications of random production lead-time on inventory policies. Gulyani [1] studied the effects of poor transportation on the supply chain (i.e., highly ineffective freight transportation systems) and showed how it increases the probability of incurring damage in transit and total inventories, while also increasing overhead costs. Silver [10] used the Economic Order Quantity (EOQ) formulation to model the situation that the order quantity received from the supplier does not necessarily match the quantity requisitioned. He showed that the optimal order quantity depends only on the mean and standard deviation of the amount received. Shih [9] studied the optimal ordering schemes in the case where the proportion of defective products (PDP) in the accepted lots has a known probability distribution. Noori and Keller [4] extended Silver's model to obtain an optimal production quantity when the amount of products received at stores assumes probability distributions such as uniform, normal and gamma.

Papers concerned with logistics or probabilistic networks are more common in the area of transportation, especially in the hazardous material routing problem. In the transportation of hazardous material, the event of an accident with a truck fully loaded with hazardous material can be treated as contingency, but in most past models, the main objective has been to find the optimal route to minimize the expected total system cost, without concern for supply chain contract decisions such as the optimal ordering quantity.

In general, these models do not link the supply chain and logistics decision processes together. Without understanding the details of the logistics networks (e.g., how defects occurred in the supply network, how different operation policies in the DC affect the defect process), the supply-chain contract decisions are prone to inaccuracies, especially in dealing with stochastic optimizations due to supply uncertainties. For example, suppose the typical defect rate is θ_n and a contingent event occurs with probability p (e.g., 0.00001), and upon occurrence, $\theta_c \times 100$ percentage of the total shipment is damaged. Then, the overall defect rate is $(1-I)\times\theta_n + I\times\theta_c = \theta + I\times(\theta_c - \theta_n)$, where $I = 1$ under the contingency and $I = 0$ otherwise. Even though θ_c (e.g., 90%) product damage under the contingent situation can cause enormous stockout costs to the retailer, the average defect rate $\theta_n + p\times(\theta_c - \theta_n)$ is nearly the same as θ_n without the contingency due to the very small probability p. Consequently, orders based on the average defect rate (as seen in most of supply-chain contracts) do not prepare the retailer for potentially severe losses that accompany contingencies. Thus, it is important to know how defects will impact the uncertainty in the amount of good products arriving at stores and to develop optimization strategies to encounter these situations.

In this paper, we describe two procedures with which retailers can generate reasonable solutions that exhibit risk-averse characteristics toward extreme events. In Section 2 we first model processes for product defect rates between any two points in the network, which are directly linked to logistics operations. Next, we construct a random variable representing the total proportion of defective products (TPDP) by integrating models of defects at various stages of the supply-chain network. TPDP is based on a series of mixture distributions and characterizes the overall service levels (defect rates) of logistics operations in contingent and non-contingent circumstances. Moreover, we investigate the impact of two different policies of DCs operations on the resulting distributions of TPDP. Section 3 shows the ineffectiveness of using the expected profit in locating the optimal ordering quantity. In Section 4, a probability constrained optimization procedure is developed to handle the low probability and high impact events that lead to logistics uncertainties. Numerical examples are provided in Section 5. Section 6 summarizes the results of the article and outlines future research opportunities.

2 A Product-defect Model for Logistics Networks

We consider the problem of a retailer who is buying products from k identical suppliers. Each supplier provides the retailer with identical products at the same price. Products from the k suppliers are transported and stored in a single DC before being sent out to the retail outlet (see Figure 1).

A contingent event to products shipped from supplier j to the DC is assumed to have high impact, low probability and affects logistics operations (e.g., product defects or delivery delays) between suppliers and the DC. We assume that given a contingency, the damage proportion, denoted by X_{jC}, is independent of the size of actual shipment, with distribution function G_C. More generally, we define

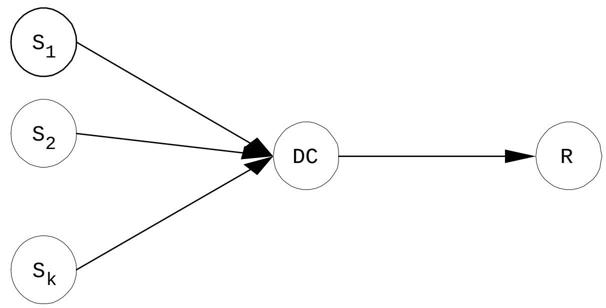

Figure 1: Supply chain network with k suppliers and one DC

X_{jC}	=	Proportion of defective products (PDP) due to contingency between supplier j and DC $X_{jC} \sim G_C$ where $E(X_{jC}) = \mu_C$, $Var(X_{jC}) = \sigma_C^2$,
X_{jN}	=	PDP due to non-contingency between supplier j and DC $X_{jN} \sim G_N$ where $E(X_{jN}) = \mu_N$, $Var(X_{jN}) = \sigma_N^2$,
I_j	=	1 if a contingent event occurs between supplier and DC, = 0 otherwise and $\{I_1, \dots, I_k\}$ are independent with $P(I_j = 1) = p_j$, and
P_{jW}	=	$(1 - I_j)X_{jN} + I_j X_{jC}$, PDP from supplier j to DC.

Note that P_{jW} is a simple mixture distribution, where I_j serves as the Bernoulli mixing distribution. We consider two scenarios to model how products from the supplier reach the retailer (see Figure 2).

2.1 Separate Supply Lines Between Suppliers and Retailer

In the first scenario, illustrated in Figure 2A (with $k = 2$), different trucks (or other methods of transport) are used for each supplier, so products from different suppliers are not mixed together in transport. We define

${X^*}_{jC}$	=	PDP of supplier j due to contingency between DC and retailer ${X^*}_{jC} \sim G_C$,
${X^*}_{jN}$	=	PDP of supplier j due to non-contingency between DC and retailer ${X^*}_{jN} \sim G_N$,
${I^*}_j$	=	1 if a contingent event occurs to supplier j between DC and retailer, = 0 otherwise and $\{{I^*}_1, \dots, {I^*}_k\}$ are independent with $P({I^*}_j = 1) = {p^*}_j$, and
P_{jR}	=	PDP of supplier j from DC to retailer = $(1 - {I^*}_j)X^*_{jN} + {I^*}_j X^*_{jC} \quad j = 1, 2, \cdots, k.$

To derive the distribution of the total proportion of defective products at the retail level, we first need to derive distributions of the PDP from each supplier that precedes it. For notational convenience, we use random variable Y to represent the TPDP. For the non-mixed case we denote this by Y_{NM}. If we let P_j be the proportion of defective products among all shipments from supplier j, then we have

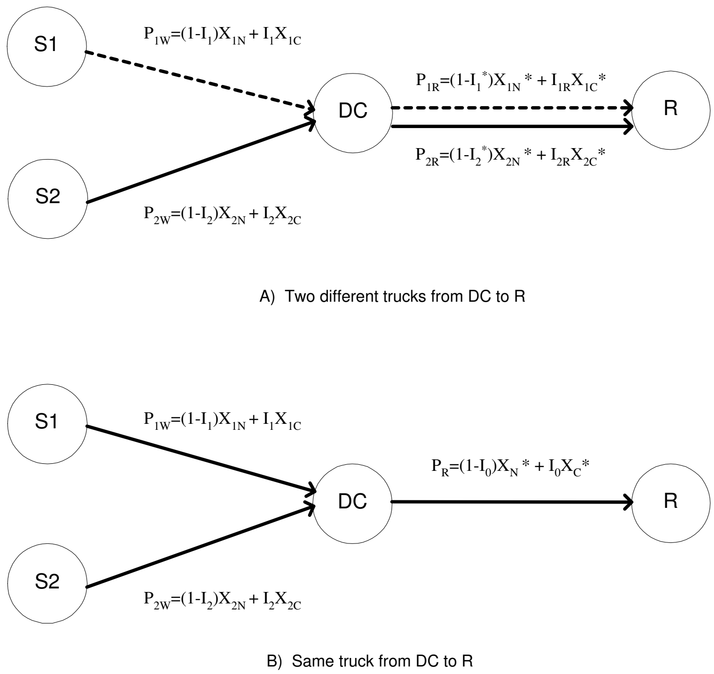

Figure 2: Different truck-load vs. same truck-load

$$P_j = P_{jW} + P_{jR}(1 - P_{jW}), \quad \text{and}$$
$$Y_{NM} = \frac{1}{k}\sum_{j=1}^{k} P_j, \quad j = 1, 2, \cdots, k.$$

2.2 Mixed Supply Lines Between Suppliers and Retailer

In the "mixed supply lines" scenario, products from different suppliers are delivered from the DC to the retail store using the same transport units (e.g., trucks), and only one indicator variable is required to model the logistics defects under contingency along the route; products loaded in the same truck are exposed to the same risk. Similar to the definitions in Section 2.1, let P_R be the PDP of the remaining good products that are shipped from the DC to the retail store, and define

$$\begin{aligned}
X^*{}_C &= \text{Total PDP due to contingency between DC and retailer} \\
&\quad\; X^*{}_C \sim G_C, \\
X^*{}_N &= \text{Total PDP due to non-contingency between DC and retailer} \\
&\quad\; X^*{}_N \sim G_N, \\
I_0 &= 1 \text{ if a contingent event occurs between DC and retailer, } = 0 \text{ otherwise} \\
&\quad\; \text{with } P(I_0 = 1) = p_0, \\
P_R &= \text{PDP from DC to retailer} = (1 - I_0)X_N^* + I_0 X_C^*, \text{ and} \\
P'_j &= \text{Total PDP from supplier } j.
\end{aligned}$$

Table 1: Structure of TPDP variables

	Separate (P_j)		Mixed (P'_j)	
	normal	contingency	normal	contingency
Supplier → DC	X_{jN}, $I_j = 0$	X_{jC}, $I_j = 1$	X_{jN}, $I_j = 0$	X_{jC}, $I_j = 1$
DC → Retailer	X^*_{jN}, $I^*{}_j = 0$	X^*_{jC}, $I^*{}_j = 1$	X^*_N, $I_0 = 0$	X^*_C, $I_0 = 1$

For this scenario, Y_M represents the aggregate PDP and

$$P'_j = P_{jW} + P_R(1 - P_{jW}),$$
$$Y_M = \frac{1}{k}\sum_{j=1}^{k} P_j{}', j = 1, 2, \cdots, k.$$

The total PDPs $(P'_1, \cdots, P'_k)$s are clearly not independent due to the shared risks. Construction of TPDP variables for the two scenarios is summarized in Table 1.

2.3 Mixed Case vs. Non-Mixed Case

For the non-mixed case, the mean and variance of Y_{NM} can be derived as

$$\begin{aligned} E[Y_{NM}] &= E\left(\frac{\sum_{j=1}^{k} P_j}{k}\right) = \frac{1}{k}\sum_{j=1}^{k} E[P_j], \\ Var[Y_{NM}] &= \frac{\sum_{j=1}^{k} Var[P_j]}{k^2}. \end{aligned}$$

where P_js are independent. Similarly, for the mixed case, the mean and variance of Y_M can be derived as

$$\begin{aligned} E[Y_M] &= E\left(\frac{\sum_{j=1}^{k} P'_j}{k}\right) = \frac{1}{k}\sum_{j=1}^{k} E[P'_j] \\ Var[Y_M] &= Var\left(\frac{\sum_{j=1}^{k} P'_j}{k}\right) = \frac{1}{k^2}\sum_{j=1}^{k} Var[P'_j] + \frac{1}{k^2}\sum\sum\nolimits_{i \neq j} Cov(P'_i, P'_j) \end{aligned}$$

where

$$\begin{aligned} Cov[P'_i, P'_j] &= Cov[P_{iW} + P_R(1 - P_{iW}), P_{jW} + P_R(1 - P_{jW})] \\ &= Cov[P_R(1 - P_{iW}), P_R(1 - P_{jW})] \\ &= Var[P_R](1 - E[P_{iW}])(1 - E[P_{jW}]) \geq 0. \end{aligned}$$

If we assume I_j's, $I^*{}_j$'s, and I_0 to be independent and identically distributed with $P[I_0 = 1] = p$ then we can further simplify the means and variances for both cases. For the non-mixed case,

$$
\begin{aligned}
E[Y_{NM}] &= \frac{1}{k}\sum_{j=1}^{k} E[P_j] = \frac{1}{k}\sum_{j=1}^{k} E[P_{jW} + P_{jR}(1 - P_{jW})] \\
&= [(1-p)\mu_N + p\mu_C]\Big(2 - [(1-p)\mu_N + p\mu_C]\Big), \qquad (2.1) \\
Var[Y_{NM}] &= \frac{\sum_{j=1}^{k} Var[P_j]}{k^2} = \frac{\sum_{j=1}^{k} Var[P_{jW} + P_{jR}(1 - P_{jW})]}{k^2} \\
&= \frac{1}{k^2}\sum_{j=1}^{k}[(1-p)\sigma_N^2 + p\sigma_C^2]\Big(2\Big[1 - ((1-p)\mu_N + p\mu_C)\Big] + (1-p)\sigma_N^2 + p\sigma_C^2\Big) \\
&= \frac{1}{k}[(1-p)\sigma_N^2 + p\sigma_C^2]\Big(2\Big[1 - ((1-p)\mu_N + p\mu_C)\Big] + (1-p)\sigma_N^2 + p\sigma_C^2\Big). \qquad (2.2)
\end{aligned}
$$

Under the mixed case we have

$$
\begin{aligned}
E[Y_M] &= \frac{1}{k}\sum_{j=1}^{k} E[P'_j] = \frac{1}{k}\sum_{j=1}^{k} E[P_{jW} + P_R(1 - P_{jW})] \\
&= [(1-p)\mu_N + p\mu_C]\Big(2 - [(1-p)\mu_N + p\mu_C]\Big) \qquad (2.3) \\
&= E[Y_{NM}], \\
Var[Y_M] &= Var\Big(\frac{\sum_{j=1}^{k} P'_j}{k}\Big) = \frac{1}{k^2}\sum_{j=1}^{k} Var[P'_j] + \frac{1}{k^2}\sum\sum\nolimits_{i\neq j} Cov(P'_i, P'_j) \\
&= \frac{1}{k}\Big\{[(1-p)\sigma_N^2 + p\sigma_C^2]\Big(2[1 - ((1-p)\mu_N + p\mu_C)] + (1-p)\sigma_N^2 + p\sigma_C^2\Big)\Big\} \\
&\quad + \frac{1}{k^2}\binom{k}{2}\Big((1-p)\sigma_N^2 + p\sigma_C^2\Big)\Big(1 - [(1-p)\mu_N + p\mu_C]\Big)^2. \qquad (2.4)
\end{aligned}
$$

These results are used in later sections to investigate the effect of contingency on the optimal order quantity and the retailer's profit function.

2.4 Risk-Pooling Effects of Mixed Supply Lines

In this section, we consider a two-supplier example to illustrate the risk pooling effects when the decision maker adopts the mixed supply line strategy. We focus on the implications of mixed versus non-mixed (or separated) supply lines and disregard all logistics operations between suppliers and DC.

The following notation aids in formulating of the example:

$$
\begin{array}{lcl}
Q & = & \text{total order quantity requested by retailer,} \\
c_1 & = & \text{unit wholesale price from supplier 1,} \\
c_2 & = & \text{unit wholesale price from supplier 2, assume } c_1 < c_2, \\
r & = & \text{unit retail price,} \\
m_1 & = & \text{profit margin of unit product purchased from supplier 1, } m_1 = r - c_1, \\
m_2 & = & \text{profit margin of unit product purchased from supplier 2, } m_2 = r - c_2, \\
 & & \text{(note that } m_1 > m_2), \\
h & = & \text{unit holding cost per period for unsold products,} \\
\pi & = & \text{unit shortage cost,} \\
\xi & = & \text{fixed total demand per period,} \\
P_1 & = & r.v. \text{ representing total proportion of defects among products from supplier 1,} \\
P_2 & = & r.v. \text{ representing total proportion of defects among products from supplier 2,} \\
Y & = & r.v. \text{ representing total proportion of defects among } Q \text{ in transit, } Y = \frac{1}{2}(P_1 + P_2).
\end{array}
$$

Following assumptions are made on risk pooling:

- $A1)$ Contingency dictates the magnitude of PDP from each truck
- $A2)$ Under mixed, the damage amount is evenly distributed among products from different suppliers
- $A3)$ Under non-mixed, we exclude the situation when both trucks experience a contingency simultaneously

In addition to these assumptions, we use the following discrete distributions for P_1 and P_2 and Y:

$$
P_1 = \begin{cases} 1, & \text{w/p } p; \\ 0, & \text{w/p } 1-p, \end{cases} \qquad P_2 = \begin{cases} 1, & \text{w/p } p; \\ 0, & \text{w/p } 1-p, \end{cases} \qquad Y = \begin{cases} 0.5, & \text{w/p } p; \\ 0, & \text{w/p } 1-p. \end{cases}
$$

We use distributions P_1 and P_2 for the non-mixed case and Y for the mixed case. According to the assumption $A2)$, $Y = 0.5$ represents $(P_1, P_2) = (0.5, 0.5)$.

Given Q, the retailer's profit is a function of P_1 and P_2:

$$
\begin{aligned}
\Pi(P_1, P_2) \equiv\; & rMin[\xi, (1-Y)Q] - c_1(1-P_1)Q/2 - c_2(1-P_2)Q/2 \\
& -h[(1-Y)Q - \xi]^+ - \pi[\xi - (1-Y)Q]^+ \\
=\; & (r - c_1)(1-P_1)Q/2 + (r - c_2)(1-P_2)Q/2 \\
& -(h+r)[(1-Y)Q - \xi]^+ - \pi[\xi - (1-Y)Q]^+ \\
=\; & m_1(1-P_1)Q/2 + m_2(1-P_2)Q/2 - (h+r)[(1-Y)Q - \xi]^+ - \pi[\xi - (1-Y)Q]^+.
\end{aligned}
$$

Under the mixed case, the possible profits are:

$$
\begin{aligned}
\Pi(0,0) &= (m_1 + m_2)Q/2 - (h+r)(Q - \xi)^+ - \pi(\xi - Q)^+ \\
\Pi(0.5, 0.5) &= (m_1 + m_2)Q/4 - (h+r)(0.5Q - \xi)^+ - \pi(\xi - 0.5Q)^+.
\end{aligned}
$$

Under the non-mixed case, retailer's profit can take one of following forms:

$$\begin{aligned}\Pi(0,0) &= (m_1+m_2)Q/2-(h+r)(Q-\xi)^+-\pi(\xi-Q)^+\\ \Pi(1,0) &= m_2Q/2-(h+r)(0.5Q-\xi)^+-\pi(\xi-0.5Q)^+\\ \Pi(0,1) &= m_1Q/2-(h+r)(0.5Q-\xi)^+-\pi(\xi-0.5Q)^+\\ \Pi(1,1) &= -\pi\xi.\end{aligned}$$

The expected profit based on the mixing case is:

$$E[\Pi(P_1,P_2)] = p\times\Pi(0.5,0.5)+(1-p)\times\Pi(0,0),$$

and the expected profit based on the non-mixing case is:

$$\begin{aligned}E[\Pi(P_1,P_2)] &= p^2\times\Pi(1,1)+p(1-p)\times\Pi(1,0)\\ &\quad +(1-p)p\times\Pi(0,1)+(1-p)^2\times\Pi(0,0)\\ &\simeq p\times\Pi(1,0)+p\times\Pi(0,1)+(1-2p)\times\Pi(0,0).\end{aligned}$$

Here, $E[\Pi(P_1,P_2)]$ is simplified by assuming $p^2\approx 0$. When there is no contingency (i.e., $P_1=0, P_2=0$), the two different strategies do not differ. Upon contingency, the mixed case generates equal amount of product damages among mixed products from two suppliers while the non-mixed case can result in the entire loss of products from each supplier. Because the profit margins of products from two suppliers are different ($m_1>m_2$), with $m_2<(m_1+m_2)/2<m_1$, profit using the non-mixed case can be larger and smaller compared to the profit using the mixed case depending on the source of the damaged products under contingency. But if the decision maker is a risk averse person who wants to avoid the worst case situation under which a contingency happens to the high margin products only, the mixed case strategy is the better choice than the non-mixed strategy because the half of damaged products among total fixed number of damages are from the low margin products when the mixed case is used.

2.5 Formulation of K-Supplier Product-Ordering Problems

The following notation aids the formulation of the uncertain supply problem in the logistics network and is slightly different from notation used in the previous section. We assume same wholesale prices for all the available suppliers and relax the fixed demand assumption:

Q	=	total order quantity requested by retailer
c	=	unit wholesale price
r	=	unit retail price
h	=	unit holding cost per period for unsold products
π	=	unit shortage cost
ξ	=	*r.v.* representing demand per period
$F(\xi)$	=	distribution fuction of ξ (with p.d.f. f(x))
Y	=	*r.v.* representing total proportion of defects among Q in transit

In the k-supplier model, we assume that the total order quantity Q is split equally between k suppliers. The retail price is fixed and strictly greater than wholesale cost ($r > c$) regardless of the terms of trade. The holding cost per period at the retail store level is h for each unsold product. In the event of a stock out, unmet demand is lost, resulting in the margin being lost (to the retailer). The related stock-out penalty cost is π. All cost parameters are assumed to be known.

The retailer's profit consists of three components: Sales revenue (SR) , procurement costs (PC) from suppliers, and the total system inventory cost (TSIC). After the completion of the logistics operations, the total amount of products received may or may not be enough to meet the demand amount, ξ. TSIC has two components: overstock inventory cost and total stock-out penalty cost. The shortage amount is primarily due to the unknown actual demand, but partly due to the potential logistics defects.

We can construct the retailer's profit function using the results from the previous section. For a given Q and TPDP (either case), the retailer's profit is

$$\begin{aligned} \Pi(Q,Y) \quad \equiv \quad & rMin[\xi,(1-Y)Q] - c(1-Y)Q \\ & -h[(1-Y)Q-\xi]^{+} - \pi[\xi-(1-Y)Q]^{+}, \end{aligned} \tag{2.5}$$

where $(x-y)^{+}$ represents $max[(x-y),0]$.

White [12] considers the problem of deciding the optimum batch production quantity when the probability of producing a good-for-sale item is p, so the total number of good items is distributed Binomial(Q,p). He shows the expected profit function is strictly concave and derives the optimum batch production quantity. In the next section, we address the concavity of the retailer's expected profit function and derive the optimal order quantity that maximizes the expected profit. We also explore the behavior of the optimal solution when parameters of the distribution for logistics defects changed.

3 Solution Strategies

3.1 Risk Neutral Solutions

This section shows how the standard expected value approach fails in the case of a low-probability-high-consequence contingency event. Below, we derive the optimal order quantity as a function of the logistics defect model parameters. From equation (2.5), the retailer's expected profit can be expressed as

$$\begin{aligned} E[\Pi(Q,Y)] \quad = \quad & rE[Min(\xi,(1-Y)Q)] - c(1-E[Y])Q \\ & -hE[((1-Y)Q-\xi)^{+}] - \pi E[(\xi-(1-Y)Q)^{+}] \\ & \equiv ESR - ETIC, \end{aligned}$$

where ESR (Expected Sales Revenue) and $ETIC$ (Expected Total Inventory Cost) are

$$\begin{aligned} ESR &= rE[Min(\xi,(1-Y)Q)] - c(1-E[Y])Q \\ &= rE[\xi] - c(1-E[Y])Q - r\int_0^1\int_{(1-y)Q}^{\infty}[\xi-(1-y)Q]f(\xi)g(y)d\xi dy, \\ ETIC &= h\int_0^1\int_0^{(1-y)Q}[(1-y)Q-\xi]f(\xi)g(y)d\xi dy \\ &+ \pi\int_0^1\int_{(1-y)Q}^{\infty}[\xi-(1-y)Q]f(\xi)g(y)d\xi dy. \end{aligned}$$

Shih [9] proved the convexity of $ETIC$; to prove the concavity of $E[\Pi(Q,Y)]$ it suffices to show the concavity of ESR, which is guranteed because its second derivative is

$$\frac{\partial^2 ESR}{\partial Q^2} = -r\int_0^1 (1-y)^2 f((1-y)Q)g(y)dy < 0, \qquad \text{for all} \quad Q. \tag{3.6}$$

The proof of equation 3.6 is listed in Appendix.

For illustration, we consider the simple case in which demand is uniformly distributed with parameters a and b:

$$f(\xi) = \begin{cases} (b-a)^{-1}, & \text{if } a \le \xi \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Under uniformly distributed demand, we have:

$$\int_0^{(1-y)Q} f(\xi)d\xi = \frac{(1-y)Q-a}{b-a} \quad \text{and} \quad \int_0^{(1-y)Q} \xi f(\xi)d\xi = \frac{(1-y)^2Q^2}{2(b-a)}$$

so that expected profit simplifies to:

$$\begin{aligned} E[\Pi(Q,Y)] &= r\frac{a+b}{2} - c(1-\mu)Q - \frac{1}{2(b-a)}\Big[(h+r+\pi)(\sigma^2+(1-\mu)^2)Q^2 \\ &\quad -2(1-\mu)(ah+b(r+\pi))Q + a^2h + b^2(r+\pi)\Big] \\ &= -\frac{(h+r+\pi)(\sigma^2+(1-\mu)^2)}{2(b-a)}\left(Q - \frac{(1-\mu)}{(1-\mu)^2+\sigma^2}\left[\frac{b(r+\pi-c)+a(h+c)}{r+\pi+h}\right]\right)^2 \\ &\quad + \frac{(a+b)}{2} + a^2h + b^2(r+\pi) + \frac{(1-\mu)^2}{(1-\mu)^2+\sigma^2}\frac{(b(r+\pi-c)+a(h+c))^2}{2(b-a)(h+r+\pi))}, \end{aligned} \tag{3.7}$$

where $\mu = E[Y]$, and $\sigma^2 = Var[Y]$. The optimal order quantity Q^* represents the boundary value between where an increased order provides cost or benefit.

Proposition 3.1 *Let $Q_0 = \{b(r+\pi-c)+a(h+c)\}/\{r+\pi+h\}$, which is the optimal order quantity in the conventional "news vendor" problem assuming no product defects in the order process (i.e.,*

$\mu = \sigma = 0$). In terms of Q_0, the order quantity which maximizes $E[\Pi(Q)]$ is

$$\begin{aligned} Q^* &= \frac{(1-\mu)}{(1-\mu)^2+\sigma^2}\left(\frac{b(r+\pi-c)+a(h+c)}{r+\pi+h}\right) \\ &= \frac{(1-\mu)}{(1-\mu)^2+\sigma^2} \times Q_0. \end{aligned}$$

Proposition 3.1 shows that the optimal order quantity which maximizes the retailer's expected profits depends only on the mean and the standard deviation of Y. Then, the amount received has the form of $Z = (1-Y)Q$, with $E[Z] = (1-\mu)Q$ and $Var[Z] = \sigma^2 Q^2$. This result coincides with results of Noori and Keller [4] where Q^* is proportional to μ and is reduced by an increase in the variability of Y.

Proposition 3.2 *The order quantities which maximize ESR and $ETIC$ are, respectively,*

$$Q_A^* = \frac{(1-\mu)}{(1-\mu)^2+\sigma^2}\left(b-(b-a)\frac{c}{r}\right),$$

$$Q_B^* = \frac{(1-\mu)}{(1-\mu)^2+\sigma^2}\left(a+(b-a)\frac{\pi}{\pi+h}\right).$$

Furthermore, Q^* is a convex combination of Q_A^* and Q_B^*:

$$Q^* = \lambda Q_A^* + (1-\lambda)Q_B^*, \quad \text{where} \quad \lambda = \frac{r}{r+\pi+h}.$$

Proposition 3.2 shows that the optimal order quantity is a weighted average of the separate order quantities that maximize ESR (Q_A^*) and ETIC (Q_B^*); if $r > \pi + h$, more weight is assigned to the quantity which maximizes ESR. Because the optimal order quantity depends only on the mean and the variance of Y, we focus on those parameters. If contingency probability is small (e.g., $p \leq 0.001$), the equations 2.1 through 2.4 show that

$$\begin{aligned} E[Y_{NM}] &= E[Y_M] \cong \mu_N(2-\mu_N), \\ Var[Y_{NM}] &\cong \frac{1}{k}\sigma_N^2[2(1-\mu_N)+\sigma_N^2] \quad \text{and} \\ Var[Y_M] &\cong Var[Y_{NM}] + \frac{k-1}{2k}\sigma_N^2(1-\mu_N)^2. \end{aligned} \tag{3.8}$$

The equations in 3.8 suggest that expected profit does not significantly change under contingency if p is small enough. A coherent solution for the decision making process must provide a means of protection against the severe effects of contingency; the expected value approach fails to do this. The following section introduces two procedures that allow the retailer to generate reasonable solutions reflecting natural risk-averse characteristics toward extreme events that have low probability.

3.2 Risk Averse Solutions

This section discusses two alternative solutions to the expected value approach for optimal ordering. The first method limits the solution space to the set of order quantities which guarantees an expected profit level under contingency. The second method features a constraint based on the quantile function of the profit distribution. While both methods restrict the solution space to control the consequence of the contingency, they differ in important ways; the first method considers only the measured contingency and not its probability while the second method is based directly on the contingency distribution. The following result (see Appendix for proof) is useful to understand the behavior of the retailer's profit function.

Proposition 3.3 *Whenever* $(c+h) > (r+\pi-c)$*, the variability of retailer's profit is increasing in order quantity* Q*.*

Proposition 3.3 states that whenever the profit margin loss from the unit surplus is greater than that from the unit short, the variance of retailer's profit is an increasing function of Q.

3.2.1 Constrained Optimization I - Profit constraint

Given a contingent event, we consider only solutions that lead to (conditionally) expected profit of at least Π_0. Let I denote the indicator function for a contingency. The problem becomes

$$\begin{aligned} \max_{Q\geq 0} \quad & E_G[\Pi(Q,Y)] \\ s.t. \quad & E_{G_C}[\Pi(Q,Y)] \equiv E_G[\Pi(Q,Y)|I=1] \geq \Pi_0. \end{aligned} \tag{3.9}$$

The retailer's strong risk-aversion can be reflected by increasing the value of Π_0. The restricted solution space is based on the following sets of order quantities:

$$\begin{aligned} S_{\Pi_0} &= \text{Set of possible order quantities which lead to unconditional expected profit} \geq \Pi_0, \\ &= \{Q \mid E_G[\Pi(Q,Y)] \geq \Pi_0\} \\ S_{\Pi_0,C} &= \text{Set of possible order quantities which lead to contingency expected profit} \geq \Pi_0, \\ &= \{Q \mid E_{G_C}[\Pi(Q,Y)] \geq \Pi_0\} \\ S^1_{\Pi_0} &= S_{\Pi_0} \cap S_{\Pi_0,C}. \end{aligned}$$

With uniformly distributed demand, the expected profit function in (3.7) simplifies:

$$E[\Pi(Q,Y)] = -A(\mu,\sigma^2)\left[Q - B(\mu,\sigma^2)\right]^2 + C(\mu,\sigma^2), \tag{3.10}$$

where

$$\begin{aligned} A(\mu,\sigma^2) &= \tfrac{(h+r+\pi)(\sigma^2+(1-\mu)^2)}{2(b-a)} \text{ determines the spread of the profit function,} \\ B(\mu,\sigma^2) &= \tfrac{(1-\mu)}{(1-\mu)^2+\sigma^2}\left[\tfrac{b(r+\pi-c)+a(h+c)}{r+\pi+h}\right] \text{ determines the optimal order quantity , and} \\ C(\mu,\sigma^2) &= \tfrac{(a+b)}{2} + a^2h + b^2(r+\pi) + \tfrac{(1-\mu)^2}{(1-\mu)^2+\sigma^2}\left[\tfrac{(b(r+\pi-c)+a(h+c))^2}{2(b-a)(h+r+\pi))}\right] \\ & \quad \text{determines the maximum expected profit.} \end{aligned}$$

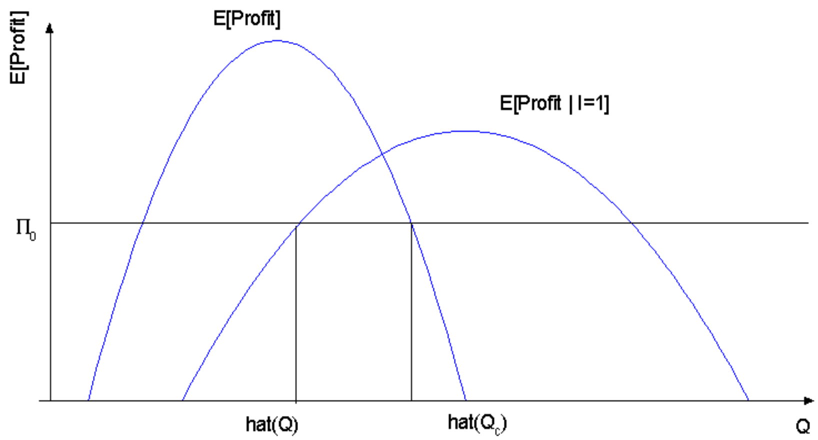

Figure 3: Expected profits based on increased mean in defect distribution after contingency

From the expression in (3.10), it is clearly seen that the expected profit only depends on the the mean and the variance of Y. By partitioning $E[\Pi(Q,Y)]$ into three parts, its behavior is more clearly manifest when the mean and the variance of the defect distribution vary. Specifically, A and C are decreasing functions of μ while B is an increasing function of μ. A is increasing in σ^2, while B and C decrease in σ^2. When the mean increases, the expected profit curve broadens ($\partial A/\partial\mu < 0$), shifts to the right (e.g. the optimal order quantity increases) ($\partial B/\partial\mu > 0$), and the corresponding maximum expected profit decreases ($\partial C/\partial\mu < 0$). When the variance increases, the curve shrinks ($\partial A/\partial\sigma^2 > 0$), shifts to the left (e.g. the optimal order quantity decreases) ($\partial B/\partial\sigma^2 < 0$), and the maximum expected profit decreases ($\partial C/\partial\sigma^2 < 0$).

Based on these properties, we consider two ways contingency affects the expected profit through the distribution of the total defect proportion: (1) contingency increases the mean of Y, and (2) contingency increases the variance of Y. Let $\hat{Q}$ = the optimal ordering quantity of the unconditional problem, $\hat{Q}_C$ = the optimal ordering quantity under contingency, and Q^* = the optimal solution to the constrained optimization problem in (3.9). As Π_0 increases, the solution Q^* increases toward $\hat{Q}_C$; as Π_0 decreases, the constraint eventually disappears. Figure 3 illustrates the approach with the conditional and unconditional profit functions. This problem formulation provides flexibility to the decision maker, with Π_0 serving as a utility function that shrinks the unconstrained solution towards $\hat{Q}_C$ in the case of contingency.

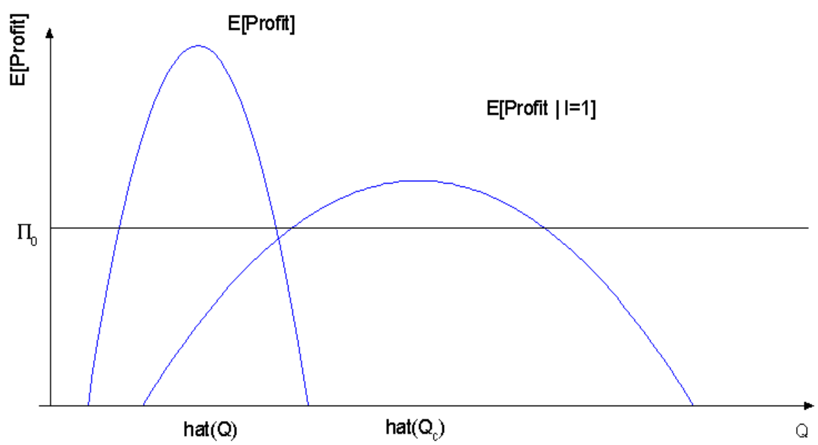

Figure 4: Increased mean causes infeasible solution

We first examine the case where the contingency causes a location shift (to the right) in defect distribution, so G_C is increased from G by a constant. The location shift (to the right) in G causes both the location shift (to the right) and increased variability in the expected profit (see Figure 3). In the case $S^1_{\Pi_0} = S_{\Pi_0} \cap S_{\Pi_0,C} \neq \emptyset$, if $\hat{Q} \in S^1_{\Pi_0}$ then $Q^* = \hat{Q}$, otherwise $Q^* = \min_{Q \in S^1_{\Pi_0}} \{Q\}$. If $S^1_{\Pi_0} = \emptyset$, it is not possible to keep conditional expected profit above Π_0 (in case of contingency) without allowing overall expected profit to go below Π_0 (see Figure 4). To rectify this problem, Π_0 must be reduced until there is an overlap between S_{Π_0} and $S_{\Pi_0,C}$. In general, the order quantity increasing in the level of risk-aversion (e.g. when Π_0 increases, Q^* increases) under the location shift case.

From Proposition 3.1, the optimal order quantity which maximizes expected profit decreases as the variance increases. Accordingly, increased variability in Y shifts the expected profit to the left as illustrated in Figure 5. Furthermore, the increase in variance results in a decrease in expected profit. The vertical distance between the two local maxima in the expected profit curves represents the decrease in maximum possible profits due to the increase in variance. When the variance under contingency is $\sigma^2 + \delta$, this distance is

$$\begin{aligned}\Delta &= C(\mu, \sigma^2) - C(\mu, \sigma^2 + \delta) \\ &= \left[\frac{(1-\mu)^2}{(1-\mu)^2 + \sigma^2} - \frac{(1-\mu)^2}{(1-\mu)^2 + \sigma^2 + \delta}\right]\left[\frac{(b(r+\pi-c) + a(h+c))^2}{2(b-a)(h+r+\pi))}\right].\end{aligned}$$

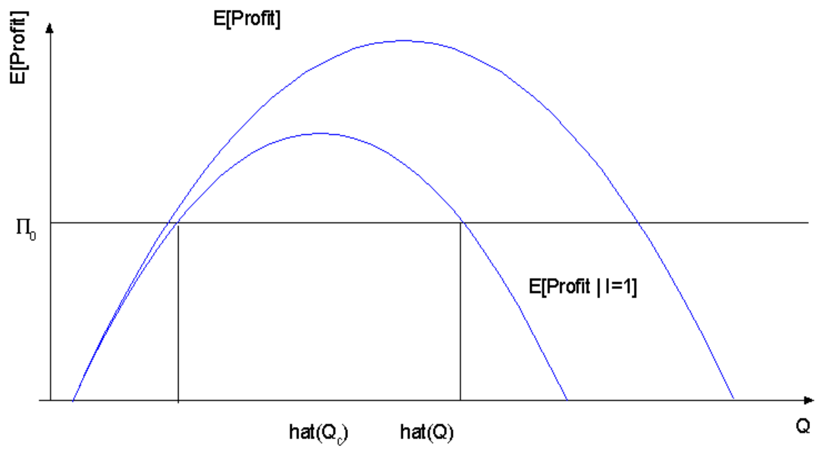

Figure 5: Increased variance shifts expected profits to the left

If $S^1_{\Pi_0} = S_{\Pi_0} \cap S_{\Pi_0,C} \neq \emptyset$, then a unique solutions exists; if $\hat{Q} \in S^1_{\Pi_0}$ then $Q^* = \hat{Q}$, otherwise $Q^* = \max_{Q \in S^1_{\Pi_0}} \{Q\}$. Again, if no overlap between S_{Π_0} and $S_{\Pi_0,C}$ exists, the constraint Π_0 must be reduced. Not like the location shift case, strong risk-aversion (bigger value of Π_0) reduces the order quantity due to the increased variance. The section that follows illustrates these solution procedures with a numerical example.

3.2.2 Constrained Optimization II - Probability constraint

As an alternative to controlling the profit function by conditioning on the occurrence of a contingency, here we restrict the solutions space by restricting the probability space of the profit distribution. That is, we bound from below (with γ) the probability that the profit is less than an amount Π_1. If we assume, for simplicity, that the demand level ξ is fixed, the problem becomes

$$\begin{aligned} \max_{Q \geq 0} \quad & E_g[\Pi(Q, Y)] \\ s.t. \quad & P_g(\Pi(Q, Y) \leq \Pi_1) \leq \gamma, \end{aligned} \tag{3.11}$$

where

$$\Pi(Q, Y) = \begin{cases} r\xi - c(1-Y)Q - h((1-Y)Q - \xi), & (1-Y)Q \geq \xi, \\ r\xi - c(1-Y)Q - (r+\pi)(\xi - (1-Y)Q), & (1-Y)Q \leq \xi. \end{cases}$$

In this case, the retailer's strong risk-aversion can be reflected by either increasing the value of Π_1 or decreasing the value of γ. Denote by $\hat{Q}$ the optimal order quantity for the unconstrained maximization problem that satisfies the following expression (see Shih [9]):

$$\int_0^{1-\xi/\hat{Q}} (1-y)g(y)dy = \frac{(1-\mu)(r-c+\pi)}{r+h+\pi}. \tag{3.12}$$

If we tacitly assume $Q \geq \xi$, the probability constraint becomes

$$\begin{aligned} P[\Pi(Q,Y) \leq \Pi_1] &= P\Big(Y \geq 1 - \frac{\pi\xi + \Pi_1}{(r+\pi-c)Q}\Big| Y \geq 1 - \frac{\xi}{Q}\Big) P\Big(Y \geq 1 - \frac{\xi}{Q}\Big) \\ &+ P\Big(Y \leq 1 - \frac{r\xi + h\xi - \Pi_1}{(h+c)Q}\Big| Y \leq 1 - \frac{\xi}{Q}\Big) P\Big(Y \leq 1 - \frac{\xi}{Q}\Big). \end{aligned}$$

The following propositions characterize the solution to the stochastic constraint placed on the profit distribution.

Proposition 3.4 *For any fixed target profit level Π_1, there exists a critical order size $Q_1 = \Pi_1/(r-c)$ such that for any demand*

$$\begin{aligned} Q \leq Q_1 \quad &\rightarrow \quad P[\Pi(Q,Y) \leq \Pi_1] = 1, \\ Q > Q_1 \quad &\rightarrow \quad P[\Pi(Q,Y) \leq \Pi_1] = 1 - G(1 - \frac{\pi\xi + \Pi_1}{(r+\pi-c)Q}) + G(1 - \frac{r\xi + h\xi - \Pi_1}{(h+c)Q}). \end{aligned}$$

Proposition 3.5 *For any given target profit level Π_1 and probability γ, there exists a feasible set for (3.11) of the form $S(A,B)$=$\{Q|Q_L \leq Q \leq Q_U\}$ with*

$$\begin{aligned} Q_L &= \frac{\pi\xi + \Pi_1}{(r+\pi-c)(1-G^{-1}(1-\gamma))}, \\ Q_U &= \Big\{Q : \gamma = 1 - G\big(1 - \frac{\pi\xi + \Pi_1}{(r+\pi-c)Q}\big) + G\big(1 - \frac{r\xi + h\xi - \Pi_1}{(h+c)Q}\big)\Big\}, \end{aligned}$$

where G^{-1} represents the inverse cumulative distribution function of Y.

As Π_1 increases, the feasible set S shrinks; the lower boundary Q_L decreases in γ while the upper boundary Q_U increases so S widens as γ increases.

Proposition 3.6 *If $\gamma < \gamma_1 = 1 - G(1-\zeta(\Pi_1))$, where $\zeta(t) = \big(\frac{\pi\xi+t}{r+\pi-c}\big)\big(\frac{r\xi+h\xi-t}{h+c}\big)^{-1}$, then $S(\Pi_1, \gamma) = \emptyset$, e.g., there is no feasible solution in (3.11).*

Proof. We need to show that $P[\Pi(Q,Y) \leq \Pi_1]$ has only one minimum point at $(r\xi + h\xi - \Pi_1)/(h+c)$. Because $1 - G\big(1 - (\pi\xi + \Pi_1)/((r+\pi-c)Q)\big)$ is a decreasing function of Q and $G\big(1 - (r\xi + h\xi - \Pi_1(/((h+c)Q_U)\big)$ is increasing in Q, $P[\Pi(Q,Y) \leq \Pi_1]$ has its minimum at $((r\xi + h\xi - \Pi_1)/(h+c)$. □

Proposition 3.7 *If* $Q \geq Q_1$*, then* γ_1 *is increasing in* Π_1 *and* $\gamma_1 = 1$ *when* Π_1 *is set at the maximum profit level, e.g.* $\Pi_1 = (r-c)\xi$*, which can be achieved only when there are no product shortages nor unsold products.*

Proof. It is easy to see that $\zeta(t)$ is an increasing function of t. If $Q \geq Q_1$, it can be shown that $\zeta(\Pi_1) \leq 1$. When $\Pi_1 = (r-c)\xi$, then $\zeta(\Pi_1) = 1$, hence $\gamma_1 = 1$. □

Proposition 3.8 *For any given profit level* Π_1 *and probability* γ*,* $\gamma \geq \gamma_1$*, the optimal order quantity for the constrained optimization problem (3.11) is*

$$Q^* = \begin{cases} \hat{Q}, & \textit{if } \hat{Q} \in S(\Pi_1, \gamma), \\ Q_L, & \textit{if } \hat{Q} \leq Q_L, \\ Q_U, & \textit{if } \hat{Q} \geq Q_U \end{cases}$$

where $\hat{Q}$ *is determined by (3.12).*

Whenever the order quantity $\hat{Q}$ satisfies the probability constraint (e.g. $\hat{Q} \in S(\Pi_1, \gamma)$), then the retailer orders $\hat{Q}$. Otherwise, the retailer should order either Q_L or Q_U according to whether $\hat{Q} \leq Q_L$ or not. Figure 6 shows the result in Proposition 3.8.

In the previous section, only the mean and the variance of Y are needed to derive the optimal solution, but in this case, the distribution of Y must be known to apply probability constraints and derive an optimal solution. If the distribution is known along with appropriate values of Π_1 and γ, profit loss can be avoided in the case of contingency. However, because of the small probability of contingency, the value of γ must be selected carefully.

Remarks: It is interesting to note that traditional Mean-Variance and Max-Min procedures are not appropriate to generate risk-averse solutions under possible contingency. First, we consider the Mean-Variance procedure. The objective function can be written as:

$$\max_{Q \geq 0} \quad E_G[\Pi(Q,Y)] - \alpha Var_G[\Pi(Q,Y)]$$

$$\textit{or}$$

$$\begin{aligned} \max_{Q \geq 0} \quad & (E_{G_N}[\Pi(Q,Y)] - \alpha Var_{G_N}[\Pi(Q,Y)])P[I=0] \\ & +(E_{G_C}[\Pi(Q,Y)] - \alpha Var_{G_C}[\Pi(Q,Y)])P[I=1]. \end{aligned}$$

Assuming $p = P[I=1]$ is vary small, the solution to the above maximization problem will maximize the objective function under the non-contingency case. Due to its small probability, the contingency does not effect the derivation of risk-averse solutions.

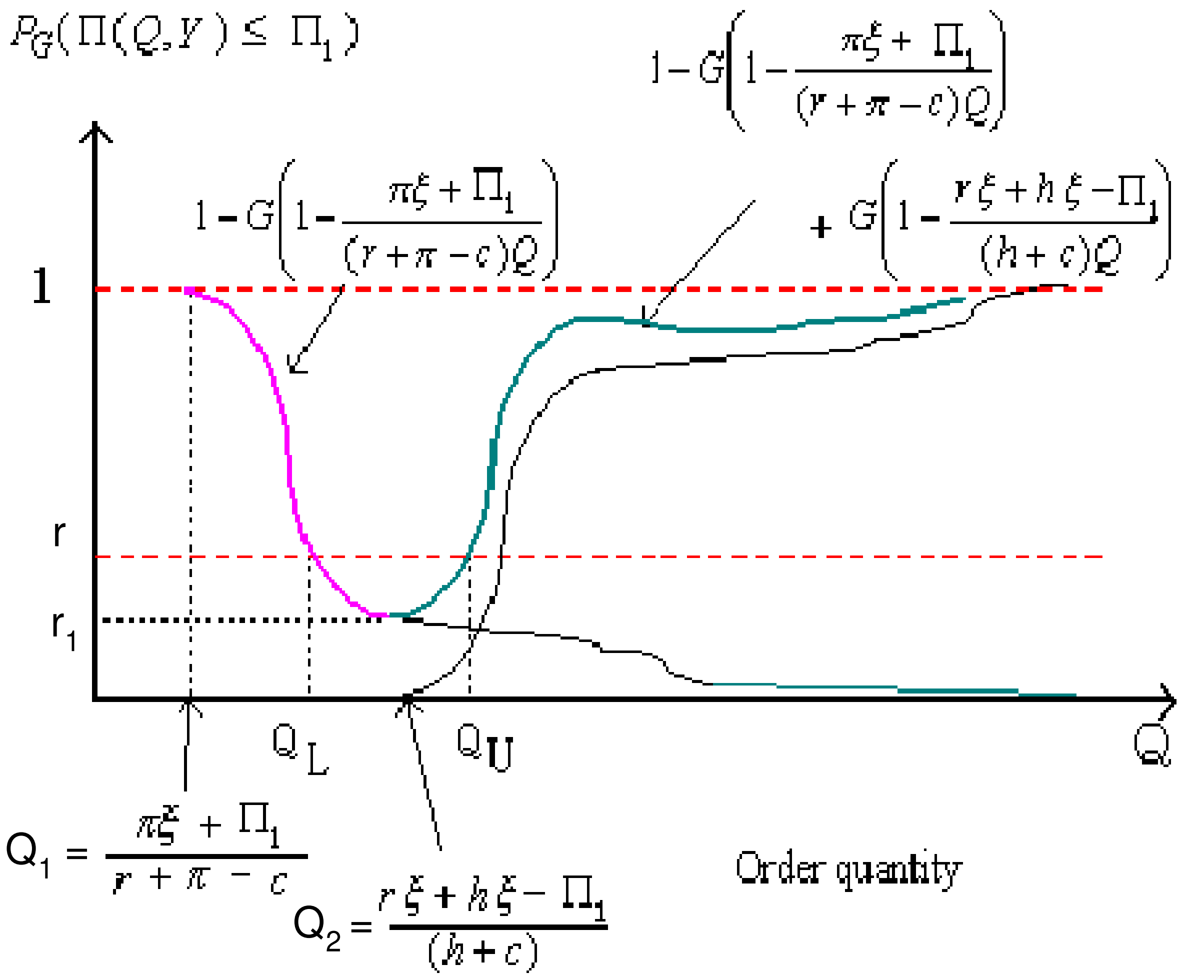

Figure 6: Shape of the probability constraint

Table 2: Parameter values in the case of contingency

(μ_c,σ^2)	(0.05, 0.01)	(0.1, 0.01)	(0.2, 0.01)	(0.3, 0.01)
	(0.4, 0.01)	(0.5, 0.01)	(0.6, 0.01)	(0.7, 0.01)
(μ, σ_c^2)	(0.01, 0.05)	(0.01, 0.1)	(0.01, 0.2)	(0.01, 0.3)
	(0.01, 0.4)	(0.01, 0.5)	(0.01, 0.6)	(0.01, 0.7)

Now, the formal objective function under the max-min procedure is:

$$\max_Q \min_G E_G[\Pi(Q, Y)].$$

The max-min procedure provides a solution which maximizes the expected profit under the worst case scenario. Figures in Section 3.2.1 can be used to illustrate Max-Min solutions. Max-Min solutions of Figure 3 and Figure 4 correspond to ordering quantities where two curves intersect. In Figure 5, the Max-Min solution coincides with the ordering quantity which maximizes the expected profit under contingency. In this way, the Max-Min solution does not provide any flexibility in terms of the resulting ordering quantity. Contrary to these two traditional methods for risk-averse solutions, the two proposed methods in this section provide not only a way to handle the low probability events but also offers great flexibility in deriving ordering quantity decisions in accordance with decisionmaker's various degrees of risk preferences by adjusting parameter values.

4 Numerical Examples

To further understand the implication of results from the previous section, we consider numerical examples employing a variety of model parameters.

4.1 Constrained Optimization I - Profit constraint

For the following example, we assign parameter values:

$$(r, c, \pi, h) = (\$50, \$10, \$30, \$2)$$

with the demand distribution of $U(a = 100, b = 150)$. We assume the (unconditional) mean and the variance of Y are $(\mu, \sigma^2) = (0.01, 0.01)$. In the case of contingency, we use the parameter values listed in Table 2. We let μ_c and σ_c^2 respectively represent the mean and the variance of Y under contingency.

Table 3 lists optimal ordering quantities which maximize expected profits under various combinations of mean and variance of Y. As stated previously, $\hat{Q}$ increases as the mean of Y increases and decreases as the variance of Y increases. For the unconditional case, the optimal ordering

Table 3: Example: case 1 vs. case 2

Mean	Variance	$\hat{Q}$	E(Profit)	%	Mean	Variance	$\hat{Q}$	E(Profit)	%
0.01	0.01	143	$4,575	100.0	0.01	0.01	143	$4,575	100.0
0.05	0.01	149	$4,561	0.3	0.01	0.05	137	$3,934	14.0
0.1	0.01	157	$4,540	0.8	0.01	0.1	131	$3,198	30.1
0.2	0.01	176	$4,487	1.9	0.01	0.2	120	$1,915	58.1
0.3	0.01	200	$4,410	3.6	0.01	0.3	110	$831	81.8
0.4	0.01	231	$4,293	6.2	0.01	0.4	102	-$95	102.1
0.5	0.01	274	$4,102	10.3	0.01	0.5	95	-$896	119.6
0.6	0.01	336	$3,762	17.8	0.01	0.6	89	-$1,595	134.9
0.7	0.01	428	$3,075	32.8	0.01	0.7	84	-$2,211	148.3

Table 4: Solutions when Π_0= $4,000 and $\sigma^2 = 0.01$

μ_c	$S_{4000,C}$	S_{4000}	$\bar{S}_{4000}$	Q^*	$E[\Pi(Q^*)]$
0.05	122, 175	117, 169	122, 169	143	4575
0.1	129, 184	117, 169	129, 169	143	4575
0.2	146, 205	117, 169	146, 169	146	4566
0.3	169, 231	117, 169	169, 169	169	4012
0.4	201, 262	117, 169	$\emptyset$	201	1813
0.5	253, 296	117, 169	$\emptyset$	253	-5308
0.6	$\emptyset$	117, 169	$\emptyset$	infeasible	NA
0.7	$\emptyset$	117, 169	$\emptyset$	infeasible	NA

quantity $\hat{Q} = 143$ with $E[\Pi(\hat{Q})] = \$4,575$. If the expected profit under contingency is restricted to be at or above Π_0=$4000, the solution satisfies

$$\begin{aligned} \max_{Q \geq 0} \quad & E_g[\Pi(Q, Y)] \\ s.t. \quad & E_{g_C}[\Pi(Q, Y)] \geq 4000. \end{aligned}$$

When contingency increases the mean to $\mu_c = 0.05$, we have $S^1_{4000} = S_{4000} \cap S_{4000,C} = \{117 \leq Q \leq 169\} \cap \{122 \leq Q \leq 175\} = \{122 \leq Q \leq 169\}$. The solution $Q^* = \hat{Q} = 143$ because $143 \in S^1_{4000}$. Table 4 summarizes results using different values of μ_c. When $\mu_c \geq 0.4$, S^1_{4000} is the empty set, so there is no solution which guarantees a minimum expected profit of $4,000 regardless of contingency. If we choose a constraint value of Π_0 smaller that $4,000, say $3,000, we have nonempty set of S^1_{3000} when $\mu_c = 0.4$ as shown in Table 5. Figure 7 shows plots of expected profits versus ordering quantity at different levels of μ given $\sigma^2 = 0.01$.

If a contingency leads to the increase in variance of Y, the optimal ordering quantity decreases.

Table 5: Solutions when Π_0= \$3,000 and $\sigma^2 = 0.01$

μ_c	$S_{3000,C}$	S_{3000}	$\bar{S}_{3000}$	Q^*	$E[\Pi(Q^*)]$
0.05	103, 194	99, 186	103, 186	143	4575
0.1	109, 204	99, 186	109, 186	143	4575
0.2	123, 228	99, 186	123, 186	143	4575
0.3	142, 258	99, 186	142, 186	143	4575
0.4	167, 296	99, 186	167, 186	167	4095
0.5	203, 346	99, 186	$\emptyset$	203	1620
0.6	262, 409	99, 186	$\emptyset$	262	-6986
0.7	398, 411	99, 186	$\emptyset$	398	-48355

Table 6: Solutions when Π_0=\$3,000 and $\mu = 0.01$

σ_c^2	$S_{3000,C}$	S_{3000}	$\bar{S}_{3000}$	Q^*	$E[\Pi(Q^*)]$
0.05	104, 170	99, 186	104, 170	143	4575
0.1	116, 145	99, 186	116, 145	143	4575
0.2	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.3	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.4	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.5	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.6	$\emptyset$	99, 186	$\emptyset$	infeasible	NA
0.7	$\emptyset$	99, 186	$\emptyset$	infeasible	NA

For example, at Π_0 = \$3000, Table 6 shows solutions for eight different values of σ_c^2 between 0.05 and 0.70. In Figure 8, we fix the mean at a constant value, μ=0.01, and change the variance to see the effect on expected profit. As expected, the optimal order quantity increases as the mean of Y increases and decreases as the variance increases. It can be seen that the maximum expected profit is much more sensitive to changes in the variance compared to changes in the mean. The expected profit loss from the increase in the mean can be compensated by increasing the quantity of the order. The profit loss from an increase in variance can be much more dramatic, however, and the constrained maximization problem becomes infeasible quickly in this case. Profit loss becomes worse with larger order quantities because the larger order naturally creates more variability in the amount of total defects. The proposed approach provides a more robust solution to the optimization problem and is more appropriate when the contingency increases the mean of Y while variance remains stable. Otherwise, even with a reasonable constraint amount Π_0, the feasible set S_{Π_0} can be empty.

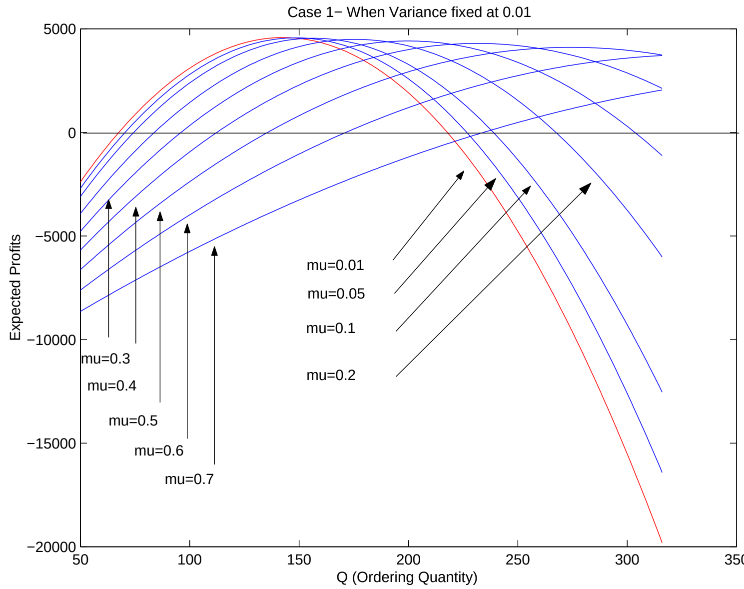

Figure 7: Profit functions under fixed variance

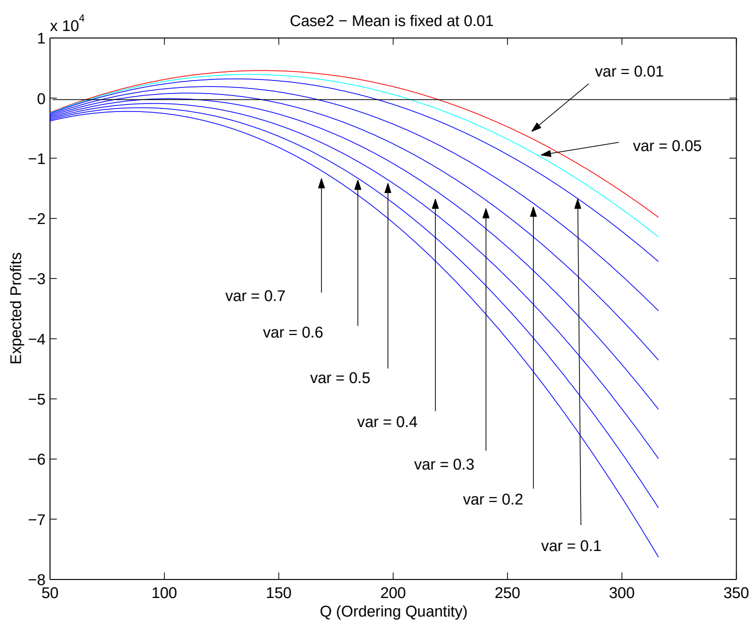

Figure 8: Profit functions under fixed mean

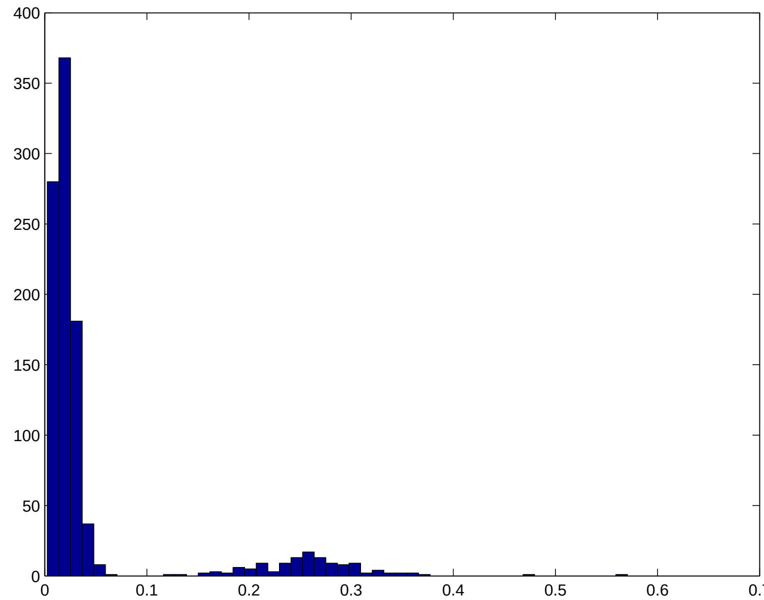

Figure 9: Simulated *pdf* of Y_{NM}

4.2 Constrained Optimization II - Probability constraint

This example uses the same parameter values as before, $(r, \pi, h) = (\$50, \$30, \$2)$, with a fixed demand level at 120 units per period and $k = 2$ suppliers, and the following distributional assumptions:

$$\begin{array}{lcl} X_{jC}, X^*{}_C & \sim & iid\ Beta(10, 10) \\ X_{jN}, X^*{}_N & \sim & iid\ Beta(1, 99) \\ I_j, I^*{}_j, I_0 & \sim & iid\ Bernoulli \text{ with } p = 0.01. \end{array}$$

Beta distributions are effective and flexible for modeling damage proportions. Parameters of beta distributions are chosen so that the distribution of damage proportion without contingency has a smaller mean and a smaller variance compared to the mean and variance of the distribution under contingency. Thus, in our example, we assume that the proportion of damaged products incurred during logistics operations under the normal operation without contingency follows a beta distribution with parameters $a = 1$ and $b = 99$ with the mean of $a/(a+b) = 0.01$ and the variance of $ab/(a+b+1)(a+b)^2 = (9.802) \times 10^{-5}$. On the other hand, we assume Beta(10,10) for the damage proportion under contingency, which has the mean of $a/(a+b) = 0.5$ and the variance of $ab/(a+b+1)(a+b)^2 = 0.0119$.

Figure 9 shows the shape of the simulated distribution of Y under the separated logistics operations assumption. Four different values of Π_1 and three different probabilities for γ are considered: $\{\Pi_1 \in 1000, 2000, 3000, 4000\}$ and $\{\gamma \in 0.1, 0.01, 0.001\}$. Table 7 summarizes values of considered parameters.

Table 7: Parameter values I

Parameter	Value
r	\$50
ξ	120 units per period
h	\$2
π	\$30
γ	0.1, 0.01, 0.001
Π_1	\$3000, \$4000
(c_1, c_2)	(\$10, \$10), (\$5, \$15), (\$1, \$19)
$X_{jC},\ X_C^*$	i.i.d. Beta(10, 10)
$X_{jN},\ X_N^*$	i.i.d. Beta(1, 99)
$I_j,\ I_j^*,\ I_0$	i.i.d. Bernoulli with p=0.01

The optimal order quantity ($\hat{Q}$) for the unconstrained maximization problem is 124 units independent of two different logistics operations. Resulting solutions under "Separated" and "Integrated" logistics operations (described in Section 2) are summarized in Table 8. Furthermore, Figure 10 gives detailed comparisons between solutions from two cases graphically. For each fixed value of Π_1, the optimal order quantity Q^* increases as γ decreases, reflecting strong risk-aversion of the decision maker. In Figure 10, expected profits at the optimal order quantities computed by adopting separated logistics operations are always higher compared to the optimal expected profits in case of integrated logistics operations. From this, we can conclude that when products' retail prices, holding costs, and shortage costs are identical, separated logistics channels for products from different suppliers always produce more profits than the integrated logistics channel as long as there are no additional costs for adopting separated logistics channels. This improved performance from the separated logistics channels is due to the smaller variability in the damage amount compared to that of the integrated logistics channel. But, despite the reduced variability of separated logistics channels, if shortage costs, which are due to product damages, are not same for products from different suppliers, the integrated logistics channel may perform better under certain situations. In the following section, we illustrate this using numerical examples.

4.3 Constrained Optimization II - Risk-Pooling using Integrated Operations

In this section, we show that the integrated logistics operation performs better than separated one under different product shortage costs. We first consider a simple example to convey the idea and consider a more general example later.

Suppose that proportions of damaged products from two suppliers under separate logistics

Table 8: Solutions under separated and integrated logistics operations

	$\pi_1 = \pi_2$=\$30		separated operations		integrated operations	
	Π_1	γ	Q^*	$E[\Pi(Q^*)]$	Q^*	$E[\Pi(Q^*)]$
c_1=\$10	\$3000	0.1	124	\$4,703	124	\$4,690
c_2=\$10	\$3000	0.01	132	\$4,661	134	\$4,630
	\$3000	0.001	139	\$4,598	154	\$4,432
	\$4000	0.1	124	\$4,703	124	\$4,690
	\$4000	0.01	152	\$4,477	154	\$4,432
	\$4000	0.001	160	\$4,398	177	\$4,190
c_1=\$5	\$3000	0.1	124	\$4,703	124	\$4,690
c_2=\$15	\$3000	0.01	124	\$4,703	130	\$4,645
	\$3000	0.001	129	\$4,663	161	\$4,332
	\$4000	0.1	124	\$4,703	124	\$4,690
	\$4000	0.01	140	\$4,561	150	\$4,445
	\$4000	0.001	149	\$4,476	186	\$4,057
c_1=\$1	\$3000	0.1	124	\$4,703	124	\$4,690
c_2=\$19	\$3000	0.01	124	\$4,703	124	\$4,690
	\$3000	0.001	137	\$4,589	158	\$4,363
	\$4000	0.1	124	\$4,703	124	\$4,690
	\$4000	0.01	143	\$4,533	142	\$4,525
	\$4000	0.001	158	\$4,387	182	\$4,102

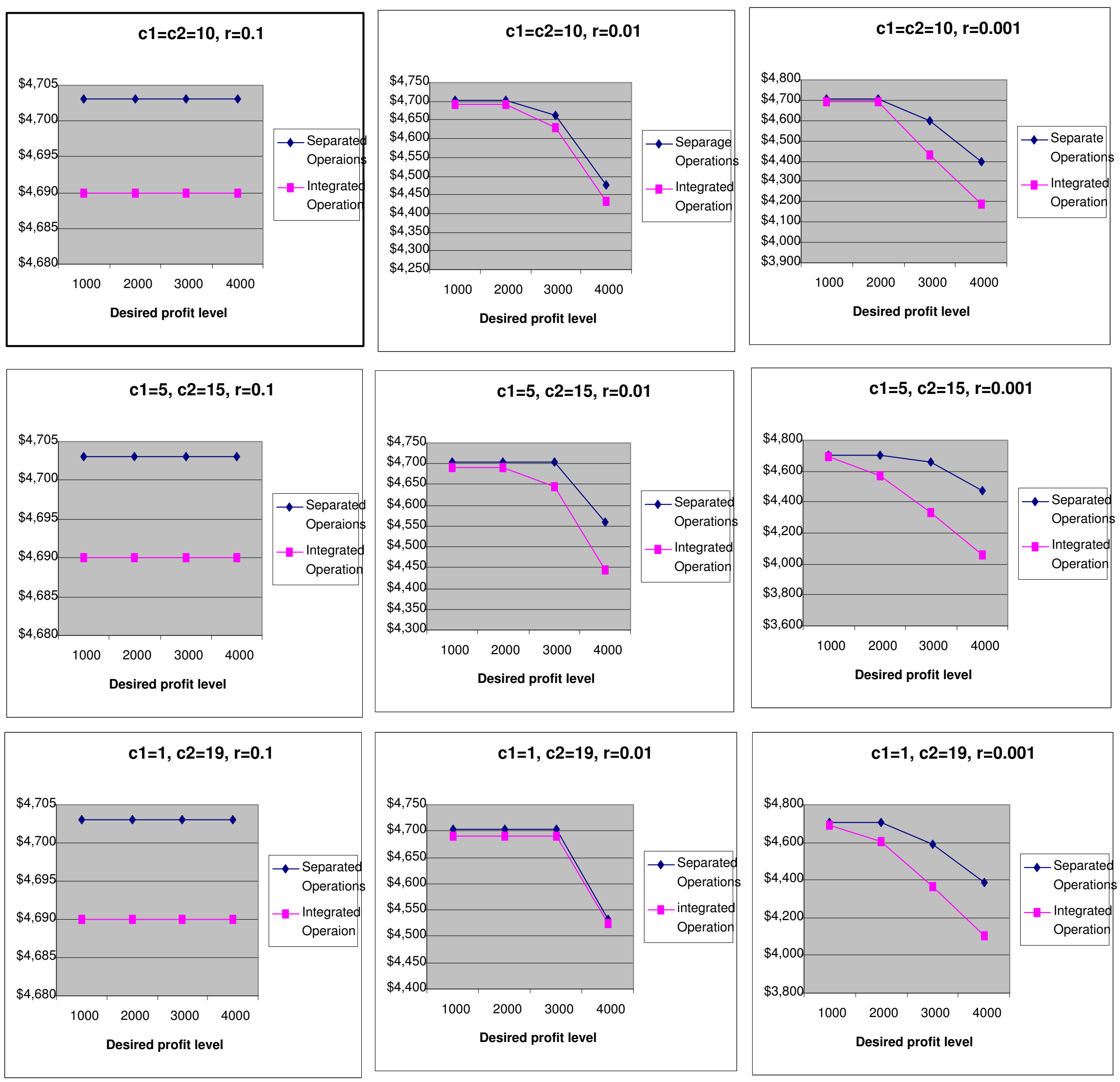

Figure 10: Comparison between separated logistics operation and integrated logistics operation

operations and integrated logistics operation are distributed as follows:

$$P_1(and\ P_2) = \begin{cases} 0, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.5. \end{cases} \quad P_1'(and\ P_2') = \begin{cases} 0, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.5. \end{cases}$$

Separated logistics operations (e.g. two trucks to separately ship products from two suppliers) result in independence of P_1 and P_2 while integrated logistics operation renders P_1' and P_2' dependent. We adopt a shock model to explain the dependency between proportions of product damages from two suppliers in case of integrated operation. When only one truck is used to transport products from two suppliers from DC to the retailer, only two contingency cases are possible. If a contingency occurs, products from both suppliers will jointly experience an increased rate of product damages. If a contingency does not occur, proportions of damaged products will be smaller. Based on this, we construct following distributions for total proportion of damaged products:

$$Y_{NM} = \begin{cases} 0, & \text{w/p } 0.25; \\ 0.1, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.25. \end{cases} \quad Y_M = \begin{cases} 0, & \text{w/p } 0.5; \\ 0.2, & \text{w/p } 0.5. \end{cases}$$

In our first example, we use different shortage costs for different suppliers ($\pi_1 = \$1, \pi_2 = \$29$), γ is fixed at 0.5 and $\Pi_1 = (\$4200, \$4250, \$4300)$. All other parameter values are kept same as in 4.2. Figure 11 shows how the integrated logistics operations perform better than the separated operations. This second example illustrates the risk-pooling effects of the integrated operation with more general distributional assumptions. Instead of using simple two-point mass discrete distributions as in the first example, we use set of beta distributions shown in Table 13. In this example, we assume shortage costs of $\pi_1 = \$5$, $\pi_2 = \$25$. As explained in Section 6.1, beta distribution parameters are chosen so that the distribution of damage proportion without contingency has a smaller mean and variance compared to mean and variance under contingency. The results are summarized in Table 10, Figure 12 and Figure 13. Again, the integrated logistics operations perform better in the cases where parameters are $\Pi_1 = (\$3800,\ \$4000,\ \$4200)$ and $\gamma = (0.02, 0.03, 0.04, 0.05, 0.1)$. In both examples, by adopting integrated logistics operation, the retailer will produce more profits whenever the logistics operations are exposed to possible contingencies and inventory costs such as unit shortage cost are different for products from different suppliers (e.g. $\pi_1 = \$5$, $\pi_2 = \$25$). Contingency to the logistics operation for products with higher unit shortage cost may significantly affect the retailer profit. But if the retailer employs an integrated operation then the total resulting damages from contingency may be evenly spread among pooled products from different suppliers. In this way, product-pooling, using the integrated logistics operation, dampens the possible risk of having a large portion of damaged products, which have higher shortage costs by introducing the risk of having evenly distributed damaged products among all products.

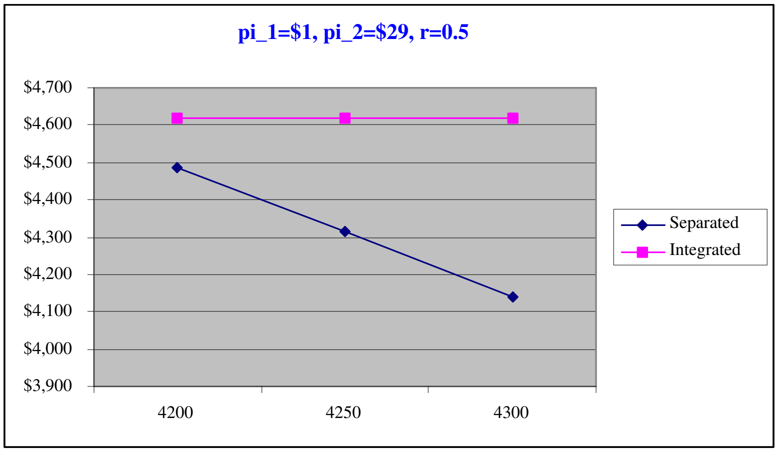

Figure 11: Risk-pool effects of integrated logistics operation

Table 9: Parameter values II

Parameter	Value
r	\$50
ξ	120 units per period
h	\$2
π	π_1=\$5, π_1=\$25
γ	0.1, 0.05, 0.04, 0.03, 0.02, 0.01
Π_1	\$3800, \$4000, \$4200
$c = c_1 = c_2$	\$10
$X_{jC},\ X_C^*$	i.i.d. Beta(10, 10)
$X_{jN},\ X_N^*$	i.i.d. Beta(1, 99)
$I_j,\ I_j^*,\ I_0$	i.i.d. Bernoulli with p=0.01

Table 10: Solutions under separated and integrated logistics operations with different shortage costs

π_1=5, π_2=25		separated operations		integrated operations	
Π_1	γ	Q^*	$E[\Pi(Q^*)]$	Q^*	$E[\Pi(Q^*)]$
\$3800	0.1	124	\$4,709	124	\$4,717
\$3800	0.05	124	\$4,709	124	\$4,717
\$3800	0.04	124	\$4,709	124	\$4,717
\$3800	0.03	133	\$4,627	124	\$4,717
\$3800	0.02	141	\$4,548	138	\$4,577
\$3800	0.01	151	\$4,447	161	\$4,331
\$4000	0.1	124	\$4,701	124	\$4,714
\$4000	0.05	124	\$4,701	124	\$4,714
\$4000	0.04	133	\$4,621	124	\$4,714
\$4000	0.03	143	\$4,524	139	\$4,565
\$4000	0.02	153	\$4,425	151	\$4,438
\$4000	0.01	166	\$4,290	171	\$4,221
\$4200	0.1	124	\$4,709	124	\$4,710
\$4200	0.05	124	\$4,709	124	\$4,710
\$4200	0.04	128	\$4,677	124	\$4,710
\$4200	0.03	143	\$4,529	133	\$4,625
\$4200	0.02	154	\$4,417	153	\$4,415
\$4200	0.01	164	\$4,311	n/a	n/a

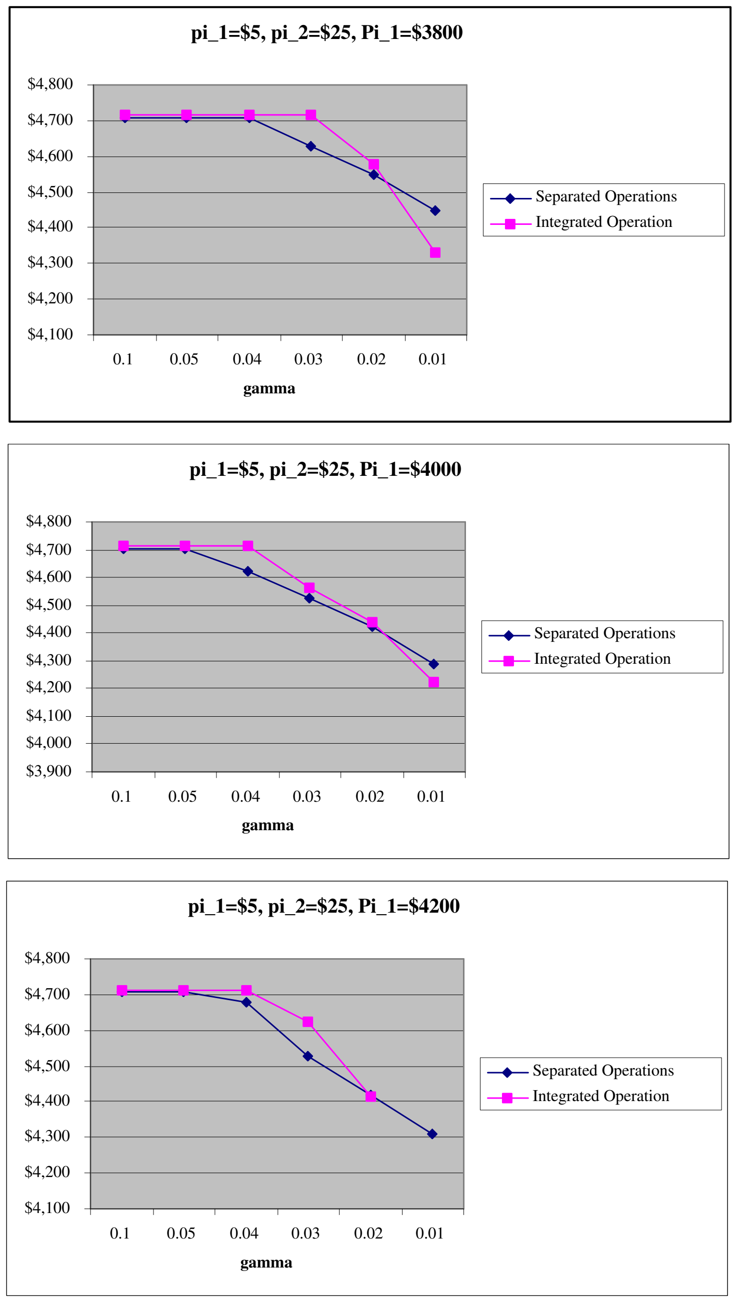

Figure 12: Risk-pooling effects of integrated logistics operation I

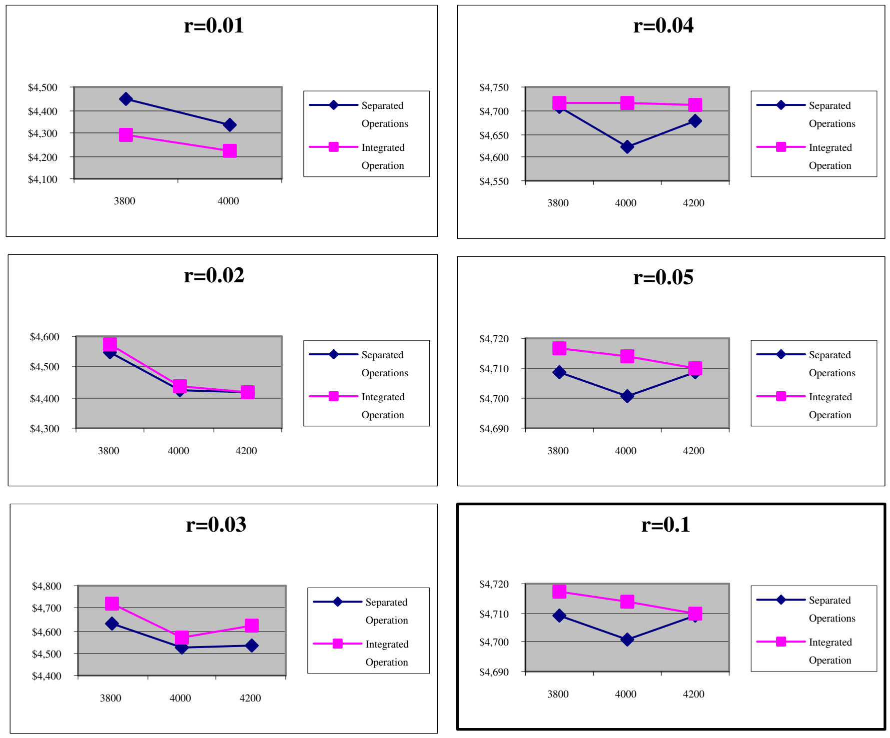

Figure 13: Risk-pooling effects of integrated logistics operation II

5 Summary and Conclusions

In brief, the main goal of this research is to build a bridge between the quantitative uncertain-supply problem and the problem of logistics network planning and vehicle routing, so that retailers can incorporate supply chain logistics uncertainties in product-ordering decisions. We adopt statistical concepts to characterize the underlying uncertainty. We also investigate the impact of two different logistics operational policies on the resulting solutions.

To achieve this goal, we examine the consequences of supply disruption on the retailer's profits. The supply disruptions take the form of high-impact and low-probability contingencies which can threaten large sections of the supply chain. The traditional expected-value approaches for product-ordering decisions fail to provide the retailer with sufficient means of protection against the effects of contingency. To provide the decision-maker facing possible contingency with a systematic way to handle this situation, we propose two procedures.

In the first procedure, by constraining the maximization problem with respect to conditional expected profit, a more stable and risk-averse solution can be found. We consider two cases where contingency either increases the mean of the proportion of damage distribution or increases the variance of the distribution. An increase in variance causes a rapid drop in expected profit, leaving no other alternatives to compensate the profit loss. On the other hand, the more robust methods introduced here compensate for a mean shift of the profit curve, resulting in an increased quantity of the order. In practice, it is recommended that the retailer investigate the characteristic of potential contingency to see how and if it affects the mean or variance of Y. If the model implies that the contingency changes only the mean, then the retailer can benefit from the constrained optimization solution in Section 5. However, if the contingency adversely affects the variance, the decision maker should find a way to reduce the variance, perhaps using multiple sourcing or purchasing options by which the damage distribution can be truncated.

In the other procedure, we utilize the probability constraint to restrict possible solutions. With this procedure, the decision-maker has more options to include risk preferences in the solution; one can change either the target profit level or the target probability level to derive risk-averse solutions. This procedure illustrates the risk-pooling effects of the integrated logistics operations under certain conditions. Whenever the inventory holding cost, the shortage cost, and retail prices of products from different suppliers are identical, separated logistics operations between the distribution center and the retailer generate solutions with higher expected profits compared to those of the integrated logistics operation case. But in our examples, we show that the resulting expected profits may be higher under the integrated logistics operation strategy than the expected profits under separated logistics operations when shortage costs are significantly different.

The investigation of the effects of non i.i.d. defect distributions associated with different routes can be considered for future research. To make our model more practical, we also need to introduce the logistics cost element in the retailer's profit function. In that case, it will be an interesting problem to study the systematic trade-off methods between the cost saving effects due to the reduced variance from separated logistics operations and the additional logistics costs required for separated logistics operations. Extending our problem to a multi-period setting is another challenging task.

APPENDIX

Proof of Proposition 5.2:

Proof. If $Q \le Q_1$, using the previous assumption, $Q \ge \xi$, we can show that

$$\frac{\pi\xi + \Pi_1}{(r+\pi-c)} \ge \frac{\pi\xi + (r-c)Q}{(r+\pi-c)} \ge \frac{\pi\xi + (r-c)\xi}{(r+\pi-c)} = \xi,$$

thus, $\{Y \ge 1 - \frac{\pi\xi+\Pi_1}{(r+\pi-c)Q}\} \cap \{Y \ge 1 - \frac{\xi}{Q}\} = \{Y \ge 1 - \frac{\xi}{Q}\}$ and similarly

$$\frac{r\xi + h\xi - \Pi_1}{(h+c)} \le \frac{r\xi + h\xi - (r-c)Q}{(h+c)} \le \frac{r\xi + h\xi - (r-c)\xi}{(h+c)} = \xi,$$

and $\{Y \le 1 - \frac{r\xi+h\xi-\Pi_1}{(h+c)Q}\} \cap \{Y \le 1 - \frac{\xi}{Q}\} = \{Y \le 1 - \frac{\xi}{Q}\}$. From these results, the conditional probabilities from the first and the second term in $P[\Pi(Q,Y) \le \Pi_1]$ become equal to 1 so that

$P[\Pi(Q,Y) \le \Pi_1] = P(Y \ge 1-\frac{\xi}{Q})+P(Y \le 1-\frac{\xi}{Q}) = 1$. If $Q > Q_1$, then $\{Y \ge 1-\frac{\pi\xi+\Pi_1}{(r+\pi-c)Q}\}\cap\{Y \ge 1-\frac{\xi}{Q}\} = \{Y \ge 1-\frac{\pi\xi+\Pi_1}{(r+\pi-c)Q}\}$ and $\{Y \le 1-\frac{r\xi+h\xi-\Pi_1}{(h+c)Q}\}\cap\{Y \le 1-\frac{\xi}{Q}\} = \{Y \le 1-\frac{r\xi+h\xi-\Pi_1}{(h+c)Q}\}$ to yield

$$P[\Pi(Q,Y) \le \Pi_1] = 1 - G\big(1-\frac{\pi\xi+\Pi_1}{(r+\pi-c)Q}\big)+G\big(1-\frac{r\xi+h\xi-\Pi_1}{(h+c)Q}\big).$$

□

Proof of Equation 3.6:

The first derivative is:

$$\begin{aligned}\frac{\partial ESR}{\partial Q} &= -c(1-\mu) - r\frac{\partial}{\partial Q}\Big(\int_0^1\int_{(1-y)Q}^{\infty}[\xi-(1-y)Q]f(\xi)g(y)d\xi dy\Big)\\ &= -c(1-\mu) - r\frac{\partial}{\partial Q}\Big(\int_0^1\int_{(1-y)Q}^{\infty}\xi f(\xi)d\xi g(y)dy - \int_0^1(1-y)Q\int_{(1-y)Q}^{\infty}f(\xi)d\xi g(y)dy\Big)\end{aligned}$$

where

$$\frac{\partial}{\partial Q}\Big(\int_0^1\int_{(1-y)Q}^{\infty}\xi f(\xi)d\xi g(y)dy\Big) = -\int_0^1(1-y)^2Qf((1-y)Q)g(y)dy,$$

$$\begin{aligned}\frac{\partial}{\partial Q}\Big(\int_0^1(1-y)Q\int_{(1-y)Q}^{\infty}f(\xi)d\xi g(y)dy\Big) &= \int_0^1(1-y)\bar{F}((1-y)Q)g(y)dy\\ &\quad -\int_0^1(1-y)^2Qf((1-y)Q)g(y)dy.\end{aligned}$$

If we simplify the above expression we have:

$$\frac{\partial ESR}{\partial Q} = -c(1-\mu) + r\int_0^1(1-y)\bar{F}((1-y)Q)g(y)dy.$$

And now it is easy to see the following result:

$$\frac{\partial^2 ESR}{\partial Q^2} = -r\int_0^1(1-y)^2f((1-y)Q)g(y)dy$$

which is negative for all possible values of Q. □

Proof of Proposition 3.3:

The variance of the profit is

$$
\begin{aligned}
Var[\Pi(Q,Y)] &= Var\Big(\Pi(Q,Y)|Y \leq 1-\frac{\xi}{Q}\Big)P\Big(Y \leq 1-\frac{\xi}{Q}\Big) \\
&\quad +Var\Big(\Pi(Q,Y)|Y > 1-\frac{\xi}{Q}\Big)P\Big(Y > 1-\frac{\xi}{Q}\Big) \\
&= Var\Big(c(1-Y)Q + h((1-Y)Q-\xi)\Big)P\Big(Y \leq 1-\frac{\xi}{Q}\Big) \\
&\quad +Var\Big(c(1-Y)Q + (r+\pi)(\xi-(1-Y)Q)\Big)P\Big(Y > 1-\frac{\xi}{Q}\Big) \\
&= (c+h)^2 Var[(1-Y)Q]G\Big(1-\frac{\xi}{Q}\Big) \\
&\quad +(r+\pi-c)^2 Var[(1-Y)Q]\Big(1-G\Big(1-\frac{\xi}{Q}\Big)\Big) \\
&= (c+h)^2 Q^2\sigma^2 G\Big(1-\frac{\xi}{Q}\Big)+(r+\pi-c)^2 Q^2\sigma^2\Big(1-G\Big(1-\frac{\xi}{Q}\Big)\Big) \\
&= \Big\{(c+h)^2 G\Big(1-\frac{\xi}{Q}\Big)+(r+\pi-c)^2\Big(1-G\Big(1-\frac{\xi}{Q}\Big)\Big)\Big\}Q^2\sigma^2.
\end{aligned}
$$

The first derivative of this variance with respect to the order quantity Q is

$$
\begin{aligned}
\frac{\partial Var[\Pi(Q,Y)]}{\partial Q} &= 2\sigma^2 Q\Big\{(c+h)^2 G\Big(1-\frac{\xi}{Q}\Big)+(r+\pi-c)^2\Big[1-G\Big(1-\frac{\xi}{Q}\Big)\Big]\Big\} \\
&+ \sigma^2\xi\Big((c+h)^2-(r+\pi-c)^2\Big)g\Big(1-\frac{\xi}{Q}\Big).
\end{aligned}
$$

The first term is always positive and the second term is positive whenever $(c+h) > (r+\pi-c)$.

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