Reliability Estimation Based on System Data with an Unknown Load Share Rule *

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Abstract

We consider a multi-component load-sharing system in which the failure rate of a given component depends on the set of working components at any given time. Such systems can arise in software reliability models and in multivariate failure-time models in biostatistics, for example. A load-share *rule* dictates how stress or load is redistributed to the surviving components after a component fails within the system. In this paper, we assume the load share rule is unknown and derive methods for statistical inference on load-share parameters. We consider components with (individual) constant failure rates in two environments: (1) the system load is distributed evenly among the working components, and (2) only assume the load for each working component increases when other components in the system fail. Tests for these special load-share models are investigated.

Key Words: Maximum Likelihood, Software Reliability, Order Restricted Inference, System dependence.

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1 Introduction

Consider a system of k components in parallel, for which component failure rates change only at the failure time of the other components within the system. For example, if the components have identical distributions with initial (constant) failure rate θ, then after the first system component fails, the failure rate of the remaining $k-1$ components changes to $\gamma_1\theta$, for some $\gamma_1 > 0$. After the next component failure, the failure rates of the other $k-2$ components change to $\gamma_2\theta$, and so on.

This is an example of a *load share* model, where component failure rates depend on the working state of the other components in the system. Early applications of the load-share system models were investigated by researchers in the textile industry [5] for studying the reliability of composite materials. Yarns and cables fail after the last fiber (or wire) in the bundle breaks, thus a bundle of fibers can be considered a parallel system subject to a constant tensile load. An individual fiber fails in time with an individual rate that depends on how the unbroken fibers within the bundle share the load of this stress. Depending on the physical properties of the fiber composite, this load sharing has different meanings in the failure model. Yarn bundles or untwisted cables tend to spread the stress load uniformly after individual failures (i.e., broken fibers). This leads to an *equal load share rule*, which implies the existence of a constant system load that is distributed equally among the working components. For the exponential model described above, if a constant load is distributed uniformly among the surviving components, then $\gamma_i = k/(k-i)$, for $i = 1, ..., k-1$. It is an interesting bi-product of the exponential distribution memoryless property that the sample component lifetimes in the equal load share model equate to an i.i.d. exponential distributed sample.

Load sharing models have been studied by Daniels [5], Rosen [14], Coleman [2], Birnbaum & Saunders [1], and by Phoenix [13]. Daniels, Rosen, and Coleman each used this model to study the strength behavior of fiber bundles. Birnbaum and Saunders adopted the load-share model to derive a more general lifetime distribution of materials. Phoenix showed that the system failure time is asymptotically normally distributed as the number of components grows large. This extended Coleman's research on the calculation of the asymptotic mean time to failure.

In some complex settings, a bonding matrix joins the individual fibers as a composite material, and an individual fiber failure affects the load of certain surviving fibers (e.g., neighbors) more than others. This characterizes a *local load sharing rule*, where a failed component's load is transferred to adjacent components; the proportion of the load the surviving components inherit depends on their distance to the failed component. A more general *monotone load sharing rule* assumes only that the load on any individual component is nondecreasing as other items fail. Harlow and Phoenix [8] first adopted this model to consider bundles with fibers in a circular arrangement. Lee, Durham, and Lynch [16] introduced the loading diagram to explicitly compute the bundle strength survival distribution under this local load share rule.

This kind of model dependence is not limited to materials testing. The load-sharing framework applies to more general problems of detecting members of a finite population. Suppose the resources allocated toward finding a finite set of items are defined globally, rather than assigned individually. That is, once items are detected, resources can be redistributed for the problem of detecting the remaining items, and this action gives rise to a load sharing model. In most cases, the items are identical to the observer, and an *equal load share rule* is appropriate for characterizing the system dependence.

From this framework, potential applications for the load share model extend far beyond the study of textile strength. In software debugging, the detection time for existing bugs in the software can depend on the number of other bugs in the software that have already been found. The discovery of a critical fault in the software might help reveal or conceal other yet undetected bugs. In manufacturing, as another example, a part can be considered failed after the failure of the entire set of welded joints that holds the part together. The failure of one or two welded joints can cause the increase of stress on the remaining joints, inducing a load-share model.

Until now, research involving load share models have emphasized the characterization of system reliability under a *known* load share rule [7, 10, 15, 16]. For the exponential model described above, this equates to assuming $(\gamma_1, ..., \gamma_{k-1})$ are known constants. Methods for reliability analysis based on *unknown* load share rules have not been fully developed. In this paper, we construct statistical methods for estimating the system and component lifetime distributions, and in addition, we seek to estimate the load share rule that dictates the system interdependency. For simplicity, we limit our discussion to a simple parallel system of identical components, and we focus on a load share rule where (working) component failure rates change uniformly after each failure within the system, but the magnitude of the change is unknown. This elementary model serves as an initial step toward drawing inference on more general load share rules. Extensions to more general rules and more complicated system configurations can be considered in subsequent research.

In Section 2, statistical methods for the load share model are derived based on the assumptions mentioned above. In Section 3, we consider systems with monotone load sharing, which comprise a significant number of load-sharing examples of interest. In Section 4, we derive a practical test of the load share rule. Specifically, we test to see if the load share parameters are monotonic. The test is based on order restricted inference and utilizes isotonic regression techniques.

2 Inference for the Load Share Rule

We assume the component lifetimes for n parallel systems are observable, and the individual component failure rates are constant and identical. Upon the first failures of the system, the initial (nominal) failure rate θ of the surviving components changes to $\gamma_1\theta, \gamma_2\theta, \cdots, \gamma_{k-1}\theta$ after 1^{st} failure, 2^{nd} failure, $\cdots$, and $(k-1)^{th}$ failures, respectively.

We seek maximum likelihood estimators (MLEs) of the k unknown parameters: θ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_{k-1})$. Suppose the random variable X_{ij} represents the lifetime of the j^{th} component in the i^{th} parallel system and that the random spacing T_{ij} is the time between j^{th} failure and $(j-1)^{th}$ failure for the i^{th} system. Here $i = 1, \cdots, n$ and $j = 1, \cdots, k$. The likelihood function for the i^{th} system is

$$L_i(\theta, \boldsymbol{\gamma} | t_{i1}, t_{i2}, \cdots, t_{ik}) = k!\theta^k \prod_{j=1}^{k-1} \gamma_j exp\left(-\theta \sum_{j=1}^{k} (k-j+1)\gamma_{j-1} t_{ij}\right)$$

where $\gamma_0 \equiv 1$ and the likelihood function based on n samples is

$$L(\theta, \gamma | \boldsymbol{T}) = (k!)^n (\theta)^{nk} (\prod_{j=1}^{k-1} \gamma_j)^n exp\left(-\theta \sum_{i=1}^{n} \sum_{j=1}^{k} (k-j+1)\gamma_{j-1} t_{ij}\right) \tag{2.1}$$

where $\boldsymbol{T} = \{t_{ij}; 1 \leq i \leq n, 1 \leq j \leq k\}$, $\theta > 0$ and $\boldsymbol{\gamma} > \mathbf{0}$. The corresponding k log-likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{nk}{\theta} - \sum_{i=1}^{n} \sum_{j=1}^{k} (k-j+1)\gamma_{j-1} t_{ij} = 0 \tag{2.2}$$

and

$$\frac{\partial \log L}{\partial \gamma_{j-1}} = \frac{n}{\gamma_{j-1}} - \theta \sum_{i=1}^{n} (k-j+1) t_{ij} = 0 \qquad j = 2, 3, \ldots, k \tag{2.3}$$

provide no general closed form solutions for the MLEs $(\hat{\theta}, \hat{\boldsymbol{\gamma}})$. However, from (2.2) we obtain

$$\theta = \frac{nk}{\sum_{i=1}^{n} \sum_{j=1}^{k} (k-j+1)\gamma_{j-1} t_{ij}}, \tag{2.4}$$

which, on substitution in (2.3), yields

$$\gamma_{j-1} - \frac{\sum_{i=1}^{n} \sum_{j=1}^{k} (k-j+1)\gamma_{j-1} t_{ij}}{k \sum_{i=1}^{n} (k-j+1) t_{ij}} = 0, \qquad j = 2, 3, \ldots, k. \tag{2.5}$$

Any solution $(\theta, \boldsymbol{\gamma})$ of these equations in the space $[0, \infty)^k$ must be in the $k-1$ dimensional subspace $\{(\theta, \boldsymbol{\gamma}) \in [0, \infty)^k | \theta = nk/\Psi(\boldsymbol{\gamma})\}$ where

$$\Psi(\boldsymbol{\gamma}) = \sum_{i=1}^{n} \sum_{j=1}^{k} (k-j+1)\gamma_{j-1} t_{ij}.$$

Because the $k \times k$ Hessian matrix $\{\partial^2/\partial\gamma_i\partial\gamma_j \log L\}$ is negative definite (see appendix), for any fixed value of θ, there exists a vector $\boldsymbol{\gamma}$ in this subspace that yields a global maximum for (2.1), viewed as a function of $\boldsymbol{\gamma}$ alone. The induced profile likelihood function L_p, displayed below in (2.6), is obtained by replacing the parameter θ with $nk/\Psi(\boldsymbol{\gamma})$ in the likelihood function $L(\theta, \boldsymbol{\gamma})$:

$$L_p(\boldsymbol{\gamma}) = k!^n \left(\frac{nk}{\Psi(\boldsymbol{\gamma})}\right)^{nk} \left(\prod_{j=1}^{k} \gamma_{j-1}\right)^n exp(-nk). \tag{2.6}$$

Any point which maximizes the likelihood function $L(\theta(\boldsymbol{\gamma}), \boldsymbol{\gamma})$ must necessarily maximize $L_p(\boldsymbol{\gamma})$. The $k-1$ estimating equations corresponding to (2.6) are

$$\frac{\partial \log L_p}{\partial \gamma_{j-1}} = -\frac{nkT_j}{\Psi(\boldsymbol{\gamma})} + \frac{n}{\gamma_{j-1}} = 0, \qquad j = 2, 3, \ldots, k, \tag{2.7}$$

where $T_j = \sum_{i=1}^{n}(k-j+1)t_{ij}$. The MLE for θ is then deduced from (2.2). This leads to the following theorem, with the proof listed in the appendix.

Theorem 1: Let $(\hat{\theta}, \hat{\boldsymbol{\gamma}})$ be the maximum likelihood estimator of $(\theta, \boldsymbol{\gamma})$ from (2.4) and (2.7) in the exponential load-share model. The MLE exists and is unique. Furthermore, as $n \to \infty$, for $(\theta, \boldsymbol{\gamma}) > \mathbf{0}_k$, we have that $\sqrt{n}\{(\hat{\theta}, \hat{\boldsymbol{\gamma}})\prime - (\theta, \boldsymbol{\gamma})\prime)\}$ converges to a k-parameter Gaussian distribution with mean $\mathbf{0}_k$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} \theta^2 & -\theta\boldsymbol{\gamma}\prime \\ -\theta\boldsymbol{\gamma}\prime & \mathrm{D}(\boldsymbol{\gamma}^2) + \boldsymbol{\gamma}\boldsymbol{\gamma}\prime \end{pmatrix}$$

and $\mathrm{D}(\boldsymbol{\gamma}^2)$ is defined as the diagonal matrix with diagonal elements equal to $({\gamma_1}^2, ..., {\gamma^2}_{k-1})$.

Compared to an ordinary sample of n i.i.d. exponential random variables, the asymptotic variance for $\hat{\theta}$ in Theorem 1 is equal to that of the MLE based on the i.i.d. sample that is k times smaller in size. Clearly, in a parallel system, an unknown load-share condition is detrimental to any analysis of the system's component lifetime distributions. On the other hand, for systems that fail as a series system if not for the ability to transfer load (e.g., mechanical systems), load sharing actually boosts reliability, but the unknown load-share condition still hinders the statistical inference.

There are a variety of iterative methods designed for solving systems of nonlinear equations in (2.7). The Gauss-Seidel method [12], is especially well suited for the log-likelihood equations in this problem. The Gauss-Seidel iterations solve the $k-1$ nonlinear equations

$$Q_{j-1}(\gamma_1, \gamma_2, \cdots, \gamma_{k-1}) = \frac{n}{\gamma_{j-1}} - \frac{nk\sum(k-j+1)t_{ij}}{\sum_{p=1}^{n}\sum_{q=1}^{k}(k-q+1)\gamma_{q-1}t_{pq}} = 0, \qquad j = 2, \cdots, k.$$

The MLE can be solved using the following four steps:

1 Choose initial solutions $\gamma_1^{(0)}, \cdots, \gamma_{k-2}^{(0)}$ and solve $Q_{k-1}(\gamma_1^{(0)}, \cdots, \gamma_{k-2}^{(0)}, \gamma_{k-1}) = 0$ for γ_{k-1}. Denote the solution as $\gamma_{k-1}^{(1)}$.

2 Solve $Q_{k-2}(\gamma_1^{(0)}, \cdots, \gamma_{k-3}^{(0)}, \gamma_{k-2}, \gamma_{k-1}^{(1)}) = 0$ for γ_{k-2} denote the solution as $\gamma_{k-2}^{(1)}$.

3 Continue in this manner, solving for γ_{j-1} by fixing the other variables at their last solution, and finding $\gamma_{j-1}^{(1)}$ such $Q_{j-1} = 0$.

4 Repeat these steps in sufficient number of iterations until convergence to $\hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_{k-1}$ has been achieved. Then $\hat{\theta}$ is computed through (2.4).

Confidence statements and hypothesis tests, based on the likelihood ratio, can be constructed for any combination of the failure rate parameter θ and load share parameters $\boldsymbol{\gamma}$. The inverse of the observed Fisher information matrix $\boldsymbol{I_o}$ provides an estimate of the covariance in the large sample normal distribution of $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}$, where $\boldsymbol{\beta} \equiv (\theta, \boldsymbol{\gamma})$ and $\hat{\boldsymbol{\beta}}$ is the MLE of $\boldsymbol{\beta}$. For large samples, the approximate $(1-\alpha)$ confidence ellipsoid for $(\theta, \boldsymbol{\gamma}) \in (0, \infty)^k$ is

$$(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' I_o^{-1} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \leq \chi^2_{k,\alpha}, \tag{2.8}$$

centered at the MLE $\hat{\boldsymbol{\beta}}$. Here, $\chi^2_{k,\alpha}$ is the upper α^{th} quantile of the chi-square distribution with k degrees of freedom. The computation of $\boldsymbol{I_o^{-1}}$ is included in the appendix.

Consider the following example to illustrate uncertainty estimation for the load-share parameters. Sample data for 10 identical load-share systems were generated by using $\theta = 0.1$, $\gamma_1 = 2$ and $\gamma_2 = 4$. For illustration, we simplify (2.8) to obtain a confidence region for $\boldsymbol{\gamma}$ rather than $\boldsymbol{\beta}$. The numerical method described above provides MLEs $\hat{\gamma}_1 = 2.212$, $\hat{\gamma}_2 = 4.148$, and a $(1-\alpha)$ confidence region for (γ_1, γ_2), based on (2.8) is

$$\frac{1}{n}(\hat{\gamma}_1 - \gamma_1, \hat{\gamma}_2 - \gamma_2)\begin{pmatrix} 2\hat{\gamma}_1^2 & \hat{\gamma}_1\hat{\gamma}_2 \\ \hat{\gamma}_2\hat{\gamma}_1 & 2\hat{\gamma}_2^2 \end{pmatrix}\begin{pmatrix} \hat{\gamma}_1 - \gamma_1 \\ \hat{\gamma}_2 - \gamma_2 \end{pmatrix} \leq \chi_2^2(0.95)$$

or $81.825 + 0.979\gamma_1^2 - 32.846\gamma_2 + 3.441\gamma_2^2 - 12.389\gamma_1 + 1.943\gamma_1\gamma_2 \leq 5.99$. Contour lines of the confidence regions for (γ_1, γ_2) at α equal 0.80, 0.90, 0.95 and 0.99 are displayed in Figure 1. The figure also conveys the strong negative correlation between $\hat{\gamma_1}$ and $\hat{\gamma_2}$.

3 Monotone Load Sharing

In many practical applications, $1 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{k-1}$ (or *monotone* load sharing) might be a reasonable assumption; a component failure can cause the increase in the work-load of the other components, which can equate to an increase of failure rate. We consider estimation of the load-share parameters under this order restriction as well as a corresponding test of hypothesis.

After the j^{th} failure in the system, the conditional failure rate of the $k-j$ remaining components is $\gamma_j\theta$, so the conditional likelihood between the $(j-1)^{th}$ and j^{th} failure can be computed as $L_j(\alpha_j) = (k-j+1)\alpha_j \exp(-(k-j+1)t_j\alpha_j)$ where $\alpha_j = \gamma_{j-1}\theta$, $j = 1, \cdots, k$. Then $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_k)$, where $\alpha_1 \equiv \theta$, is isotonic if and only if $\boldsymbol{\gamma}$ is. The full log-likelihood, in terms of $\boldsymbol{\alpha}$, is

$$\log L(\boldsymbol{\alpha}) = n\sum_{j=1}^{k}\log(k-j+1) + n\sum_{j=1}^{k}\log\alpha_j - \sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)t_{ij}\alpha_j. \tag{3.9}$$

The problem of maximizing (3.9) subject to $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ is equivalent to maximizing the log-likelihood

$$\begin{aligned}\log L &= n\sum_{j=1}^{k}\log\alpha_j - \sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)t_{ij}\alpha_j \\ &= \sum_{j=1}^{k}\left(\frac{n\log\alpha_j}{(k-j+1)\sum_{i=1}^{n}t_{ij}} - \alpha_j\right)(k-j+1)\sum_{i=1}^{n}t_{ij}\end{aligned}$$

subject to $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$.

Robertson, Wright and Dykstra (Chapter 1.5) [18] showed that the restricted least squares estimate coincides with the maximum likelihood estimate from this log-likelihood function. For brevity, further references to their book will denoted by RWD. By applying Theorem 1.4.4 of RWD, the order restricted MLE can be solved as an isotonic regression. Specifically, if we let $f(j) = \alpha_j$, $g(j) = n/\sum_{i=1}^{n}t_{ij}(k-j+1)$ and $w(j) = (k-j+1)\sum_{i=1}^{n}t_{ij}$, the solution $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\cdots$, $\hat{\alpha}_k$ is the isotonic regression g^* of g and is given by

$$\begin{aligned}g^*(j) &= max_{s\leq j}min_{t\geq j}\frac{\sum_{u=s}^{t}g(u)w(u)}{\sum_{u=s}^{t}w(u)} \\ &= max_{s\leq j}min_{t\geq j}\frac{n(t-s+1)}{\sum_{u=s}^{t}(k-u+1)\sum_{i=1}^{n}t_{iu}}.\end{aligned} \tag{3.10}$$

The order restricted MLE of γ_{j-1} is $\tilde{\gamma}_{j-1} = g^*(j)\tilde{\theta}^{-1}$ where $\tilde{\theta} = g^*(1)$.

To illustrate the order restricted estimation we generated $n = 20$ failure times from two systems comprised of three components. The first system is characterized by the parameters

($\theta = 0.1$, $\gamma_1 = 1.5$, $\gamma_2 = 3$) and the second system by ($\theta = 0.1$, $\gamma_1 = 3$, $\gamma_2 = 1.5$). The simulated data are listed in Table 1 and the load-share parameter estimators are listed in Table 2. For System 1, the unrestricted MLEs are already isotonic and thus have the same values as the order-restricted MLEs. For system 2, the corresponding unrestricted MLEs are not isotonic so they do not match the order restricted MLEs.

4 Hypothesis Testing for Unknown Load Share Rules

In this paper, three load sharing rules have been discussed: equal load-sharing, local load-sharing, and monotone load-sharing. Equal load sharing dictates that at any moment a constant total system load is distributed equally to each working component. As components fail, the total system load remains unchanged, so that the load increases for each of the remaining components according to the rule $\gamma_i = k/(k-i)$, $i = 1, 2, \cdots, k-1$. As reported in Section 1, for the exponential model, this generates another sample of system data that are also i.i.d. exponential. The memoryless property actually preserves the i.i.d. failure data distribution.

A more practical test for reliability applications is for detecting an increasing load within the system: $H_0 : \gamma_1 = \gamma_2 = \cdots = \gamma_{k-1}$ versus $H_1 : \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_{k-1}$. Consistent with the likelihood approach used in estimation, we consider a test based on the likelihood ratio statistic $(\sup_{H_0} L(\boldsymbol{\beta}))/(\sup_{H_1} L(\boldsymbol{\beta})) = L(\hat{\boldsymbol{\beta}})/L(\tilde{\boldsymbol{\beta}})$, where $\tilde{\boldsymbol{\beta}} = (\tilde{\theta}, \tilde{\boldsymbol{\gamma}})$ are the order restricted MLEs computed in Section 3. In the likelihood function for the monotone load share model, it will be more convenient to work with the notation $\eta_j = (\theta\gamma_j)^{-1}$ and $\eta_0 = 1/\theta$, so an equivalent set of hypotheses is $H_0 : \eta_0 = \eta_1 = \eta_2 = \cdots = \eta_{k-1}$ versus $H_1 : \eta_0 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_{k-1}$.

Here H_0 indicates that there is no actual "load"; the component failure rates remain unchanged after failures within the system. Let T_{ij} to be the time between $(j-1)^{th}$ failure and j^{th} failure in i^{th} sample. Then $T_{ij} \sim$ exponential with failure rate $(k-j+1)/\eta_{j-1}$. Under H_0, the MLEs are $\hat{\eta_0} = (\sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)t_{ij})(nk)^{-1}$ and, for $j > 1$, $\hat{\eta}_{j-1} = n^{-1}\sum_{i=1}^{n}(k-j+1)t_{ij}$. The likelihood ratio statistic is computed by plugging

$$L_0 = \sup_{H_0} L(\boldsymbol{\eta}) = L(\hat{\boldsymbol{\eta}}) = (k!)^n \prod_{j=1}^{k} \left(\frac{1}{\hat{\eta_0}}\right)^n exp\left(-\frac{\sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)t_{ij}}{\hat{\eta_0}}\right)$$

into the numerator. If we define functions $p_1(\eta_j) = -1/\eta_j$, $p_2(\theta) = 1$, $K(t_j|\theta) = (k-j+1)t_j$, $S(t_j|\theta) = \ln(k-j+1)$ and $q(\beta_j|\theta) = -\ln \eta_j$, then regularity conditions 1.5.7, 1.5.8, and 1.5.9 from RWD are satisfied for their Theorem 1.5.2, which proves that under H_1, the MLE $\tilde{\boldsymbol{\eta}}$ is solved as the isotonic regression in (3.10) with weights $w(x_i) = n$. For the denominator of the likelihood ratio, we have

$$L_1 = \sup_{H_1} L(\boldsymbol{\eta}) = L(\tilde{\boldsymbol{\eta}}) = (k!)^n \prod_{j=1}^{k} \left(\frac{1}{\tilde{\eta}_{j-1}} \right)^n exp\left(-\frac{\sum_{j=1}^{k} (k-j+1) t_{ij}}{\tilde{\eta}_{j-1}} \right).$$

Due to the order restrictions, we lack the regularity conditions to guarantee the likelihood ratio statistic will have a Chi-square distribution. However, for this particular order restriction, Theorem 4.1.1 of RWD holds and we can approximate the distribution of the test statistic as a mixture of Chi-square distributions. Specifically, if $T_{01} = -2(\log L_0 - \log L_1)$, then

$$T_{01} = 2 \sum_{j=1}^{k} n(-\log \tilde{\eta}_{j-1} + \log \hat{\eta}_0) + 2 \sum_{j=1}^{k} n\hat{\eta}_{j-1} \left(-\frac{1}{\tilde{\eta}_{j-1}} + \frac{1}{\hat{\eta}_0} \right).$$

Under H_0, the asymptotic distribution function of T_{01} is

$$P(T_{01} \le c) = 1 - \sum_{l=2}^{k} \mathrm{P}(l,k) \mathrm{P}(\chi^2_{l-1} > \mathrm{c}). \tag{4.11}$$

The level probability $P(l,k)$ denotes the probability that given k groups under H_0 the isotonic regression will result in l level sets. Level sets are sets of constancy of isotonic functions, and $\sum_{l=1}^{k} \mathrm{P}(l,k) = 1$.

For example, with Sample 1 in Table 1, $T_{01} = 14.33$ and the P-value $= P(T_{01} > 14.33) =$ 0.00023, which strongly suggests the ordering described by H_1 is present in the data. For the second sample, we have $T_{01} = 3.7089$ with P-value $= 0.053$. In this case the evidence of load-share parameter ordering is less convincing. For the cases of $k \in \{3, 4, 5\}$, Table 3 lists upper quantiles for the null-distribution for this test of hypothesis.

5 Discussion

In terms of model uncertainty, there is a slight disadvantage to estimating system or component lifetime distributions in a load share system when the load share rule is assumed to be known. An accelerated lifetime model with known acceleration levels is a suitable analog. The inference is more elaborate than inferences for regular lifetime models. However, if the load share rule cannot be assumed exactly, the load sharing property severely hinders statistical inference on the system. This was seen in the results of Theorem 1, where it was shown that the variance of the lifetime model parameter estimates was k times larger than the variances in a regular i.i.d. sample.

This fact can have important ramifications in practical examples in which the failure rates of system components can change after a failure event occurs within the system. If the load share model is appropriate for a software reliability problem (discussed in Section 1), the more traditional modeling and analyses are likely to lead to inference that grossly underestimates parameter uncertainty. For example, in problems where the number of remaining bugs in a piece of software is being estimated, upper bounds for this unknown number of bugs will be too small.

As mentioned in Section 2, load sharing can also serve to benefit system reliability. The ability to transfer a load after a key component failure can save a system that would otherwise fail, such as a system of support structures. The event of the World Trade Centers' collapse actually serves as an example. For primary support, the towers relied on interior columns as well as pinstripe columns running up each tower's facade, which were turned into additional load-bearing supports. As an afterthought, a system of supports put on the top of the building bound the exterior columns to the core. The structure, called a *hat truss*, was originally installed to hold up antennae, but after the impact of the speeding commercial jet, the hat truss served to spread the load of the damaged columns onto undamaged columns. This load-sharing, as reported in the *New York Times* [6], helped prevent the instantaneous collapse of the towers after the plane impact events.

This paper represents an important first step in drawing inference on load sharing properties for basic systems. Extending the load share model to more general lifetime distributions (e.g., Weibull, lognormal, normal) will be problematical in likelihood based inference, undoubtedly. On-going research includes a nonparametric lifetime model under unknown equal load sharing. In many applications, basic systems of identical components can be modeled adequately by the exponential load-share model if component failure rates remain approximately constant between component failures. In large systems with several components, this is sometimes a common assumption.

Table 1: Failure times for load-share samples

n	Sample 1			Sample 2		
	t_{i1}	t_{i12}	t_{i3}	t_{i1}	t_{i2}	t_{i3}
1	1.94	0.37	6.93	3.85	6.49	0.36
2	7.44	0.06	2.42	0.32	0.14	7.57
3	0.14	0.20	0.20	8.29	0.12	5.98
4	2.14	1.62	2.34	0.86	6.12	3.43
5	1.91	5.70	1.96	2.42	1.19	6.00
6	8.23	2.25	4.60	1.53	0.20	6.26
7	1.40	2.50	0.09	2.50	1.18	1.01
8	0.79	2.44	7.27	1.30	1.19	9.13
9	0.92	0.12	0.06	4.32	2.08	3.62
10	0.73	0.79	8.61	2.89	0.49	6.28
11	2.78	7.22	1.38	3.25	3.88	6.22
12	0.85	2.81	5.05	17.87	4.18	0.03
13	8.50	4.13	0.52	8.99	0.46	27.63
14	12.93	5.67	1.11	4.08	2.17	15.02
15	4.46	0.96	3.54	1.93	6.81	10.18
16	3.50	7.16	2.38	2.70	0.37	5.04
17	19.59	0.32	1.89	0.34	0.97	2.47
18	4.98	7.32	1.54	5.16	2.64	5.43
19	10.29	2.58	8.61	4.03	0.10	2.38
20	2.22	1.73	1.22	0.16	3.98	2.26

Table 2: MLE vs order restricted MLE

	Sample 1		Sample 2	
	γ_1	γ_2	γ_1	γ_2
MLE	1.7875	3.2393	2.2337	1.5837
ORDERED MLE	1.7875	3.2393	1.8534	1.8534

Table 3: Upper α-quantiles for the mixture distribution in (4.11)

k	$\alpha = 0.20$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$
3	3.047	4.487	5.927	7.363	9.273
4	3.446	4.977	6.491	7.990	9.870
5	3.748	5.345	6.907	8.461	10.486

Figure 1: Confidence regions (80%, 90%, 95%, 99%) for (γ_1, γ_2)

APPENDIX

Proof of Theorem 1: For the load-share model, the computation of the covariance, based on the information matrix $I_\theta = \Sigma^{-1}$ is straightforward if we write

$$I_\theta = \Sigma^{-1} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where $I_{11} = nk/\theta^2$, $I_{12} = I_{21}\prime = n\theta^{-1}\boldsymbol{\gamma}\prime$, and $I_{22} = nD(1/\gamma_1, \cdots, 1/\gamma_{k-1})$. If we define

$$\Sigma = I_\theta^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

then the inverse solution is a common matrix identity and we have $\Sigma_{11} = n^{-1}\theta^2$, $\Sigma_{12} = \Sigma_{21}\prime = -n^{-1}\theta\boldsymbol{\gamma}\prime$, and $\Sigma_{22} = n^{-1}(D(\boldsymbol{\gamma}^2 + \boldsymbol{\gamma}\boldsymbol{\gamma}\prime))$.

Proof that matrix H is negative definite: The components of the Hessian matrix H in are given by

$$H_{j,j} = \frac{\partial^2}{\partial \gamma_{j-1}^2} \log L = \frac{nkT_{j+1}^2}{\Psi(\boldsymbol{\gamma})^2} - \frac{n}{\gamma_j^2}, \qquad j = 1, \dots, k-1, \quad \text{and}$$

$$H_{j,l} = H_{l,j} = \frac{\partial^2}{\partial \gamma_j \gamma_l} \log L = \frac{nkT_{j+1}T_{l+1}}{\Psi(\boldsymbol{\gamma})^2}, \qquad 1 \le j \ne l \le k-1.$$

Here we establish that H is negative definite, thus the MLE in (2.6) exists and is unique. To show that H is negative definite, we need $\mathbf{Z}'\mathbf{HZ} < 0$, where $\mathbf{Z}$ represents a $k-1$ vector, $T_j = \sum_{i=1}^n (k-j+1)t_{ij}$, $\Psi(\boldsymbol{\gamma}) = \sum_{j=2}^k \gamma_{j-1}T_j$, and

$$\mathbf{Z}'\mathbf{HZ} = \sum_{i=2}^k \sum_{j=2}^k \frac{nkT_iT_jZ_{i-1}Z_{j-1}}{\Psi(\boldsymbol{\gamma})^2} - n\sum_{j=2}^k \frac{Z_{j-1}^2}{\gamma_{j-1}^2}.$$

From (2.7), $1/\gamma_{j-1} = kT_j/\Psi(\boldsymbol{\gamma})$, and we can write

$$\begin{aligned} \mathbf{Z}'\mathbf{HZ} &= \frac{nk}{\Psi(\boldsymbol{\gamma})^2}\left(\Big(\sum_{j=2}^k T_jZ_{j-1}\Big)^2 - k\sum_{j=2}^k T_j^2 Z_{j-1}^2 \right) \\ &= -\frac{nk}{\Psi(\boldsymbol{\gamma})^2} \sum_{i\neq j}^k \big(T_jZ_{j-1} - T_iZ_{i-1}\big)^2 < 0, \end{aligned}$$

which establishes that H is negative definite.

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