Positive recurrence of reflecting Brownian motion in three dimensions

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Abstract

Consider a semimartingale reflecting Brownian motion $Z$ whose state space is the $d$-dimensional non-negative orthant. The data for such a process are a drift vector $\theta$, a non-singular covariance matrix, and a $d \times d$ reflection matrix $R$ that specifies the boundary behavior of $Z$. We say that $Z$ is positive recurrent, or stable, if the expected time to hit an arbitrary open neighborhood of the origin is finite for every starting state.

In dimension $d = 2$, necessary and sufficient conditions for stability are known, but fundamentally new phenomena arise in higher dimensions. Building on prior work by El Kharroubi et al. (2000, 2002), we provide necessary and sufficient conditions for stability of SRBMs in three dimensions; to verify or refute these conditions is a simple computational task. As a by-product, we find that the fluid-based criterion of Dupuis and Williams (1994) is not only necessary but also sufficient for stability of SRBMs in three dimensions. That is, an SRBM in three dimensions is positive recurrent if and only if every path of the associated fluid model is attracted to the origin. The problem of recurrence classification for SRBMs in four and higher dimensions remains open.

Keywords: reflected Brownian motion, transience, Skorohod problem, fluid model, queueing networks, heavy traffic, diffusion approximation, strong Markov process

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1 Introduction

This paper is concerned with the class of $d$-dimensional diffusion processes called semi-martingale reflecting Brownian motions (SRBMs), which arise as approximations for open $d$-station queueing networks of various kinds, cf. Harrison and Nguyen (1993) and Williams (1995, 1996). The state space for a process $Z = \{Z(t), t \geq 0\}$ in this class is $S = \mathbb{R}^d_+$ (the non-negative orthant). The data of the process are a drift vector $\theta$, a non-singular covariance matrix $\Gamma$, and a $d \times d$ “reflection matrix” $R$ that specifies boundary behavior. In the interior of the orthant, $Z$ behaves as an ordinary Brownian motion with parameters $\theta$ and $\Gamma$, and roughly speaking, $Z$ is pushed in direction $R^j$ whenever the boundary surface $\{z \in S : z_j = 0\}$ is hit, where $R^j$ is the $j$th column of $R$ ($j = 1, \ldots, d$). To make this description more precise, one represents $Z$ in the form

$$Z(t) = X(t) + RY(t), \quad t \geq 0,$$

where $X$ is an unconstrained Brownian motion with drift vector $\theta$, covariance matrix $\Gamma$, and $Z(0) = X(0) \in S$, and $Y$ is a $d$-dimensional process with components $Y_1, \ldots, Y_d$ such that

1. $Y$ is continuous and non-decreasing with $Y(0) = 0$,
2. $Y_j$ only increases at times $t$ for which $Z_j(t) = 0$, $j = 1, \ldots, d$, and
3. $Z(t) \in S, \quad t \geq 0$.

The complete definition and essential properties of the diffusion process $Z$ will be reviewed in Appendix A, where we also discuss the notion of positive recurrence. As usual in Markov process theory, the complete definition involves a family of probability measures $\{\mathbb{P}_x, x \in S\}$ that specify the distribution of $Z$ for different starting states; informally, one can think of $\mathbb{P}_x(\cdot)$ as a conditional probability given that $Z(0) = x$. Denoting by $\mathbb{E}_x$ the expectation operator associated with $\mathbb{P}_x$ and by $\tau_A$ the first time $t \geq 0$ such that $Z(t) \in A$, we say that $Z$ is positive recurrent if $\mathbb{E}_x(\tau_A) < \infty$ for any $x \in S$ and any open neighborhood $A$ of the origin (see Appendix A for elaboration). For ease of expression we use the terms “stable” and “stability” as synonyms for “positive recurrent” and “positive recurrence,” respectively.

In the foundational theory for SRBMs, the following classes of matrices are of interest. First, a $d \times d$ matrix $R$ is said to be an $S$-matrix if there exists a $d$-vector $w \geq 0$ such that $Rw > 0$ (or equivalently, if there exists $w > 0$ such that $Rw > 0$), and $R$ is said to be completely-$S$ if each of its principal sub-matrices is an $S$-matrix. Second, a square matrix is said to be a $P$-matrix if all of its principal minors are positive (that is, each principal sub-matrix of $R$ has a positive determinant). $P$-matrices are a sub-class of completely-$S$ matrices, and the still more restrictive class of $M$-matrices is defined as in Chapter 6 of Berman and Plemmons (1979). References for the following key results can be found in the survey paper by Williams (1995): there exists a diffusion process $Z$ of the form described...
above if and only if $R$ is a completely-$S$ matrix; and moreover, $Z$ is unique in distribution whenever it exists.

Hereafter we assume that $R$ is completely-$S$. Its diagonal elements must then be strictly positive, so we can (and do) assume without loss of generality that

$$R_{ii} = 1 \text{ for all } i = 1, \ldots, d. \tag{1.5}$$

This convention is standard in the SRBM literature, and in Sections 5 through 7 of this paper (where our main results are proved) another convenient normalization of problem data will be used; Appendix B explains the scaling procedures that justify both (1.5) and the normalized problem format assumed in Sections 5 through 7.

We are concerned in this paper with conditions that assure the stability of $Z$. An important condition in that regard is

$$R^{-1} \theta < 0, \tag{1.6}$$

which includes the requirement that $R$ be non-singular. If $R$ is an $\mathcal{M}$-matrix, then (1.6) is known to be necessary and sufficient for stability of $Z$; Harrison and Williams (1987) prove that result and explain how the $\mathcal{M}$-matrix structure arises naturally in queueing network applications.

El Kharroubi et al. (2000) further prove the following three results: first, the condition $R^{-1} \theta < 0$ is necessary for stability in general; second, when $d = 2$ one has stability if and only if $R^{-1} \theta < 0$ and $R$ is a $P$-matrix; and third, the condition $R^{-1} \theta < 0$ is not sufficient for stability in three and higher dimensions, even if $R$ is a $P$-matrix. In Appendix C of this paper we provide an alternative proof for the first of those results, one that is much simpler than the original proof by El Kharroubi et al. (2000). Also, Appendix A of Harrison and Hasenbein (2008) contains an alternative proof of the second result, and Section 3 of this paper reviews the ingenious example by Bernard and El Kharroubi (1991) that serves to establish the third result.

A later paper by El Kharroubi et al. (2002) established sufficient conditions for stability of SRBMs in three dimensions, relying heavily on the foundational theory developed by Dupuis and Williams (1994). In this paper we show that the conditions identified by El Kharroubi et al. (2002) are both necessary and sufficient for stability when $d = 3$; the relevant conditions are easy to verify or refute via simple computations.

The remainder of the paper is structured as follows. First, to allow precise statements of the main results, we introduce in Section 2 the “fluid paths” associated with an SRBM, and the linear complementarity problem that arises in conjunction with linear fluid paths. That section, like the paper’s three appendices, considers a general dimension $d$, whereas all other sections in the body of the paper consider $d = 3$ specifically. Section 3 identifies conditions under which fluid paths spiral on the boundary of the state space $S$, and then Section 4 states our main conclusions, which are achieved by combining the positive results of El Kharroubi et al. (2002) with negative results that are new; Figure 2 in Section 4 summarizes
succinctly the necessary and sufficient conditions for stability when \( d = 3 \), with notations as to which components of the overall argument are old and which are new. In Sections 5 through 7 we prove the new “negative results” referred to above, dealing first with the case where fluid paths spiral on the boundary, and then with the case where they do not. As stated above, Appendix A reviews the precise definition of SRBM, Appendix B explains the scaling procedures that give rise to normalized problem formats, and Appendix C contains a relatively simple proof that (1.6) is necessary for stability. Finally, Appendix D contains several technical lemmas that are used in the probabilistic arguments of Section 7.

2 Fluid paths and the linear complementarity problem

Definition 1. A fluid path associated with the data \((\theta, R)\) is a pair of continuous functions \(y, z : [0, \infty) \to \mathbb{R}^d\) that satisfy the following conditions:

\[(2.1) \quad z(t) = z(0) + \theta t + Ry(t) \text{ for all } t \geq 0,\]
\[(2.2) \quad z(t) \in S \text{ for all } t \geq 0,\]
\[(2.3) \quad y(\cdot) \text{ is continuous and nondecreasing with } y(0) = 0,\]
\[(2.4) \quad y_j(\cdot) \text{ only increases when } z_j(\cdot) = 0, \text{ i.e., } \int_0^\infty z_j(t) \, dy_j(t) = 0, \quad (j = 1, \ldots, d).\]

Definition 2. We say that a fluid path \((y, z)\) is attracted to the origin if \(z(t) \to 0\) as \(t \to \infty\).

Definition 3. A fluid path \((y, z)\) is said to be divergent if \(|z(t)| \to \infty\) as \(t \to \infty\), where, for a vector \(u = (u_i) \in \mathbb{R}^d\), \(|u| = \sum_i |u_i|\).

Theorem 1 (Dupuis and Williams 1994). Let \(Z\) be a \(d\)-dimensional SRBM with data \((\theta, \Sigma, R)\). If every fluid path associated with \((\theta, R)\) is attracted to the origin, then \(Z\) is positive recurrent.

Definition 4. A fluid path \((y, z)\) is said to be linear if it has the form \(y(t) = ut\) and \(z(t) = vt, \quad t \geq 0\), where \(u, v \geq 0\). Thus linear fluid paths are in one-to-one correspondence with solutions of the following linear complementarity problem (LCP):

\[(2.5) \quad \text{Find } u, v \geq 0 \text{ such that } v = \theta + Ru \text{ and } u \cdot v = 0,\]

where, for vectors \(u = (u_i)\) and \(v = (v_i) \in \mathbb{R}^d\), \(u \cdot v = \sum_i u_i v_i\) is the inner product of \(u\) and \(v\).

Definition 5. A solution \((u, v)\) of the LCP is said to be stable if \(v = 0\) and to be divergent otherwise. It is said to be non-degenerate if \(u\) and \(v\) together have exactly \(d\) positive components, and to be degenerate otherwise. A stable, non-degenerate solution of the LCP is called proper.
If (1.6) holds, then \((u^*, 0)\) is a proper solution of the LCP, where
\[
u^* = -R^{-1}\theta.
\]

Furthermore, remembering that (1.6) includes the requirement that \(R\) be non-singular, readers can easily verify the following: If (1.6) holds, then any solution of the LCP other than \((u^*, 0)\) must be divergent.

3 Fluid paths that spiral on the boundary

Bernard and El Kharroubi (1991) devised the following ingenious example with \(d = 3\), referred to hereafter as the B&EK example: let
\[
\theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}.
\]

This reflection matrix \(R\) is completely-\(S\) (moreover, it is a \(P\)-matrix), so \(Z\) is a well defined SRBM. (The covariance matrix \(\Sigma\) is immaterial to the discussion that follows, provided only that it is non-singular.) As Bernard and El Kharroubi (1991) observe, the unique fluid path with these process data, starting from \(z(0) = (0, 0, \kappa)\) with \(\kappa > 0\), is the one pictured in Figure 1; it travels in a counter-clockwise and piecewise linear fashion on the boundary, with the first linear segment ending at \((2\kappa, 0, 0)\), the second one ending at \((0, 4\kappa, 0)\), and so forth. El Kharroubi et al. (2000) prove that an SRBM with these data is not stable.
showing that if $\kappa$ is large then $|Z(t) - z(t)|$ remains forever small (in a certain sense) with high probability.

To generalize the B&EK example, let $C_1$ be the set of $(\theta, R)$ pairs that satisfy the following system of inequalities (here $R_{ij}$ denotes the $(i, j)^{th}$ element of $R$, or equivalently, the $i^{th}$ element of the column vector $R^j$):

\begin{align*}
\theta &< 0, \\
\theta_1 &> \theta_2 R_{12} \quad \text{and} \quad \theta_3 < \theta_2 R_{32}, \\
\theta_2 &> \theta_3 R_{23} \quad \text{and} \quad \theta_1 < \theta_3 R_{13}, \\
\theta_3 &> \theta_1 R_{31} \quad \text{and} \quad \theta_2 < \theta_1 R_{21}.
\end{align*}

(Notation used in this section agrees with that of El Kharroubi et al. (2000, 2002) in all essential respects, but is different in a few minor respects.) To explain the meaning of these inequalities, we consider a fluid path associated with $(\theta, R)$ that starts from $z(0) = (0, 0, \kappa)$, where $\kappa > 0$; it is the unique fluid path starting from that state, but that fact will not be used in our formal results. Over an initial time interval $[0, \tau_1]$ the fluid path is linear and adheres to the boundary $\{z_2 = 0\}$, as in Figure 1. During that interval one has $\dot{y}(t) = (0, -\theta_2, 0)'$ and hence

$$\dot{z}(t) = \theta + R\dot{y}(t) = \theta - \theta_2 R^2 = \begin{pmatrix} \theta_1 - \theta_2 R_{12} \\ 0 \\ \theta_3 - \theta_2 R_{32} \end{pmatrix}.$$ 

Thus (3.1) and (3.2) together give the following: as in Figure 1, a fluid path starting from state $(0, 0, \kappa)$ has an initial linear segment in which $z_3$ decreases, $z_1$ increases, and $z_2$ remains at zero; that initial linear segment terminates at the point $z(\tau_1)$ on the $z_1$ axis that has

$$z_1(\tau_1) = \left( \frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3} \right) \kappa > 0.$$ 

Similarly, from (3.1) and (3.3), the fluid path is linear over an ensuing time interval $[\tau_1, \tau_2]$, with $z_1$ decreasing, $z_2$ increasing, and $z_3$ remaining at zero; that second linear segment terminates at the point $z(\tau_2)$ on the $z_2$ axis that has

$$z_2(\tau_2) = \left( \frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3} \right) \left( \frac{\theta_2 - \theta_3 R_{23}}{\theta_3 R_{13} - \theta_1} \right) \kappa > 0.$$ 

Finally, from (3.1) and (3.4), the fluid path is linear over a next time interval $[\tau_2, \tau_3]$, with $z_2$ decreasing, $z_3$ increasing, and $z_1$ remaining at zero; that third linear segment terminates at the point $z(\tau_3)$ on the $z_3$ axis that has $z_3(\tau_3) = \beta_1(\theta, R) \kappa$, where

$$\beta_1(\theta, R) = \left( \frac{\theta_1 - \theta_2 R_{12}}{\theta_2 R_{32} - \theta_3} \right) \left( \frac{\theta_2 - \theta_3 R_{23}}{\theta_3 R_{13} - \theta_1} \right) \left( \frac{\theta_3 - \theta_1 R_{31}}{\theta_1 R_{21} - \theta_2} \right) > 0.$$ 

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Thereafter, the piecewise linear fluid path continues its counter-clockwise spiral on the boundary in a self-similar fashion, like the path pictured in Figure 1, except that in the general case defined by (3.1) through (3.4) the spiral may be either inward or outward, depending on whether \( \beta_1(\theta, R) < 1 \) or \( \beta_1(\theta, R) > 1 \); if \( \beta_1(\theta, R) = 1 \) then the three linear segments described above are simply repeated in subsequent cycles, and we say that the fluid path follows a **bounded orbit**.

To repeat, \( C_1 \) consists of all \((\theta, R)\) pairs that satisfy (3.1) through (3.4), and the single-cycle gain \( \beta_1(\theta, R) \) for such a pair is defined by (3.5). As we have seen, fluid paths associated with problem data in \( C_1 \) spiral counter-clockwise on the boundary of \( S \). Now let \( C_2 \) consist of all \((\theta, R)\) pairs that satisfy (3.1) and further satisfy (3.2) through (3.4) with all six of the strict inequalities reversed. It is more or less obvious that \((\theta, R)\) pairs in \( C_2 \) are those giving rise to clockwise spirals on the boundary, and the appropriate analog of (3.5) is

\[
\beta_2(\theta, R) = \frac{1}{\beta_1(\theta, R)} = \left( \begin{array}{c} \theta_3 - \theta_2 R_{32} \\ \theta_2 R_{12} - \theta_1 \\ \theta_3 R_{23} - \theta_2 \\ \theta_2 R_{31} - \theta_3 \end{array} \right) > 0.
\]

Hereafter we define \( C = C_1 \cup C_2, \beta(\theta, R) = \beta_1(\theta, R) \) for \((\theta, R) \in C_1\), and \( \beta(\theta, R) = \beta_2(\theta, R) \) for \((\theta, R) \in C_2\). Thus \( C \) consists of all \((\theta, R)\) pairs whose associated fluid paths spiral on the boundary, and \( \beta(\theta, R) \) is the single-cycle gain for such a pair.

### 4 Summary of results in three dimensions

The following is a slightly weakened version of Theorem 1 by El Kharroubi et al. (2002), which the original authors express in a more elaborate notation; we have deleted one part of their result that is irrelevant for current purposes. The corollary is immediate from Theorem 1 above (the Dupuis-Williams fluid stability criterion).

**Theorem 2** (El Kharroubi et al. (2002)). Suppose that (1.6) holds and that either of the following additional hypotheses is satisfied: (a) \((\theta, R) \in C \) and \( \beta(\theta, R) < 1 \); or (b) \((\theta, R) \not\in C \) and the unique solution of the linear complementarity problem (2.5) is the proper solution \((u^*, 0)\) defined in Section 2. Then all fluid paths associated \((\theta, R)\) are attracted to the origin.

**Corollary.** Suppose that (1.6) holds and, in addition, either (a) or (b) holds. Then \( Z \) is positive recurrent.

The new results of this paper are Theorems 3 and 4 below, which will be proved in Sections 5 through 7. Figure 2 summarizes the logic by which these new results combine with previously known results to provide necessary and sufficient conditions for stability (that is, positive recurrence) of \( Z \).

**Theorem 3.** If \((\theta, R) \in C \) and \( \beta(\theta, R) \geq 1 \), then \( Z \) is not positive recurrent.

**Theorem 4.** If there exists a divergent solution for the linear complementarity problem (2.5), then \( Z \) is not positive recurrent.
5 Proof of Theorem 3

Throughout this section and the next, we assume without loss of generality that our problem data satisfy not only (1.5) but also

\begin{equation}
\theta_i \in \{-1, 0, 1\} \quad \text{for } i = 1, 2, 3.
\end{equation}

Appendix B explains the scaling procedures that yield this normalized form. To prove Theorem 3 we will assume that \((\theta, R) \in C_1\) and \(\beta_1(\theta, R) \geq 1\), then show that \(Z\) is not stable; the proof of instability when \((\theta, R) \in C_2\) and \(\beta_2(\theta, R) \geq 1\) is identical. Given the normalizations (1.5) and (5.1), the conditions (3.1) through (3.4) that define \(C_1\) can be restated as follows:

\begin{align}
\theta &= (-1, -1, -1)', \\
R_{12}, R_{23}, R_{31} &> 1, \quad \text{and} \quad R_{13}, R_{21}, R_{32} < 1.
\end{align}

Let us now define a 3 \times 3 matrix \(V\) by setting \(V_{ij} = R_{ij} - 1\) for \(i, j = 1, 2, 3\). Then \(V^j\) (the \(j^{th}\) column of \(V\)) is the vector \(\theta - \theta_j R^j\) that was identified in Section 3 as the velocity vector on the face \(\{Z_j = 0\}\) for a fluid path corresponding to \((\theta, R)\).

**Lemma 1.** Because \(\beta_1(\theta, R) \geq 1\) by assumption, there exists a vector \(u > 0\) such that \(u'V \geq 0\), or equivalently, \(u'V^j \geq 0\) for each \(j = 1, 2, 3\).
Proof. From (1.5) and (5.3) we have that

\[
V = \begin{pmatrix}
0 & a_2 & -b_3 \\
-b_1 & 0 & a_3 \\
a_1 & -b_2 & 0
\end{pmatrix},
\]

where \(a_i, b_i > 0\) for \(i = 1, 2, 3\). In this notation the definition (3.5) is as follows:

\[
\beta_1(\theta, R) = \frac{a_1 a_2 a_3}{b_1 b_2 b_3},
\]

Setting \(u_1 = 1, u_2 = \frac{a_1 a_2}{b_1 b_2}\) and \(u_3 = \frac{a_2}{b_2}\), it is easy to verify that \(u'V^1 = u'V^2 = 0\), and \(u'V^3 = b_3(\frac{a_1 a_2 a_3}{b_1 b_2 b_3} - 1)\). The definition (5.5) and our assumption that \(\beta_1(\theta, R) \geq 1\) then give \(u'V^3 \geq 0\). \(\square\)

For the remainder of the proof let \(e\) denote the three-vector of ones, so (5.2) is equivalently expressed as \(\theta = -e\), and we can represent \(X\) as

\[
X(t) = X(0) + B(t) - et, \quad t \geq 0,
\]

where \(B\) is a driftless Brownian motion with non-singular covariance matrix and \(B(0) = 0\). Also, we choose a starting state \(x = X(0) = Z(0)\) that satisfies

\[
Z_1(0) \geq 0, \quad Z_2(0) = 0 \quad \text{and} \quad Z_3(0) > 0.
\]

In this section, because the initial state is fixed, we write \(E(\cdot)\) rather than \(E_x(\cdot)\) to signify the expectation operator associated with the probability measure \(P_x\) (see Appendix A). Also, when we speak of stopping times and martingales, the relevant filtration is the one specified in Appendix A.

Let \(u > 0\) be chosen to satisfy \(u'V \geq 0\) (see Lemma 1), and further normalized so that \(u'e = 1\). It is immediate from the definition of \(V\) that \(u'V = u'R - e'\), and thus one has the following:

\[
u'R \geq e'.
\]

Now define \(\xi(t) = u'Z(t), \ t \geq 0\). From (1.1), (5.6) and (5.8) one has

\[
\xi(t) - \xi(0) = u'B(t) - u'et + u'RY(t) \geq u'B(t) - t + e'Y(t) \quad \text{for} \quad t \geq 0.
\]

Next, let

\[
\tau_1 = \inf\{t > 0 : Z_3(t) = 0\}, \quad \tau_2 = \inf\{t > \tau_1 : Z_1(t) = 0\}, \quad \tau_3 = \inf\{t > \tau_2 : Z_2(t) = 0\},
\]

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and so forth. (These stopping times are analogous to the points in time at which the piecewise linear fluid path in Figure 1 changes direction.) The crucial observation is the following: $Z_3(\cdot) > 0$ over the interval $[0, \tau_1)$, $Z_1(\cdot) > 0$ over $[\tau_1, \tau_2)$, $Z_2(\cdot) > 0$ over $[\tau_2, \tau_3)$, and so forth; thus $Y_3(\cdot)$ does not increase over $[0, \tau_1)$, $Y_1(\cdot)$ does not increase over $[\tau_1, \tau_2)$, $Y_2(\cdot)$ does not increase over $[\tau_2, \tau_3)$, and so forth. From (1.1) and (5.6) we then have the following relationships:

\begin{align}
(5.10) \quad Z_2(t) &= B_2(t) - t + Y_2(t) \\
& \quad + R_{21}Y_1(t), \quad 0 \leq t \leq \tau_1, \\
(5.11) \quad Z_3(t) &= [B_3(t) - B_3(\tau_1)] - (t - \tau_1) + [Y_3(t) - Y_3(\tau_1)] \\
& \quad + R_{32}[Y_2(t) - Y_2(\tau_1)], \quad \tau_1 \leq t \leq \tau_2, \\
(5.12) \quad Z_1(t) &= [B_1(t) - B_1(\tau_2)] - (t - \tau_2) + [Y_1(t) - Y_1(\tau_1)] \\
& \quad + R_{13}[Y_3(t) - Y_3(\tau_2)], \quad \tau_2 \leq t \leq \tau_3,
\end{align}

and so forth. Now (5.10) gives

\begin{align}
(5.13) \quad Y_2(t) &= t + Z_2(t) - B_2(t) - R_{21}Y_1(t) \quad \text{for} \quad 0 \leq t \leq \tau_1.
\end{align}

Because $Y_3 \equiv 0$ on $[0, \tau_1)$ we can substitute (5.13) into (5.9) to obtain the following:

\begin{align}
(5.14) \quad \xi(t) - \xi(0) &\geq u'B(t) - t + Y_1(t) + [t + Z_2(t) - B_2(t) - R_{21}Y_1(t)] \quad \text{for} \quad 0 \leq t \leq \tau_1.
\end{align}

From the definition of $V$ and (5.4) we have $1 - R_{21} = b_1 > 0$, so (5.14) can be rewritten

\begin{align}
(5.15) \quad \xi(t) - \xi(0) &\geq M(t) + A(t) \quad \text{for} \quad 0 \leq t \leq \tau_1,
\end{align}

where

\begin{align}
(5.16) \quad M(t) &= u'B(t) - B_2(t) \quad \text{for} \quad 0 \leq t \leq \tau_1, \\
(5.17) \quad A(t) &= Z_2(t) + b_2Y_1(t) \quad \text{for} \quad 0 \leq t \leq \tau_1.
\end{align}

Defining $\tau = \lim \tau_n$, we now extend the definition (5.16) to all $t \in [0, \tau)$ as follows:

\begin{align}
(5.18) \quad M(t) &= M(\tau_1) + u'[B(t) - B(\tau_1)] - [B_3(t) - B_3(\tau_1)] \quad \text{for} \quad \tau_1 \leq t \leq \tau_2, \\
(5.19) \quad M(t) &= M(\tau_2) + u'[B(t) - B(\tau_2)] - [B_1(t) - B_1(\tau_2)] \quad \text{for} \quad \tau_2 \leq t \leq \tau_3,
\end{align}

and so forth. Finally, on $\{\tau < \infty\}$ we set $M(t) = M(\tau)$ for all $t \geq \tau$. Then $M = \{M(t), t \geq 0\}$ is a continuous martingale whose quadratic variation $\langle M, M \rangle(\cdot)$ satisfies

\begin{align}
(5.20) \quad \langle M, M \rangle(t) - \langle M, M \rangle(s) &\leq \gamma(t - s) \quad \text{for} \quad 0 < s < t < \infty, \, \text{where} \, 0 < \gamma < \infty.
\end{align}

Also, we extend (5.17) to all $t \in [0, \tau)$ via

\begin{align}
(5.21) \quad A(t) &= A(\tau_1) + Z_3(t) + b_3[Y_2(t) - Y_2(\tau_1)] \quad \text{for} \quad \tau_1 \leq t \leq \tau_2, \\
(5.22) \quad A(t) &= A(\tau_2) + Z_1(t) + b_3[Y_3(t) - Y_3(\tau_2)] \quad \text{for} \quad \tau_2 \leq t \leq \tau_3,
\end{align}

and so forth. Thus the process $A = \{A(t), 0 \leq t < \tau\}$ is non-negative and continuous.
Lemma 2. \( \xi(t) - \xi(0) \geq M(t) + A(t) \) for all \( t \in [0, \tau) \).

Proof. It has already been shown in (5.15) that this inequality is valid for \( 0 \leq t \leq \tau_1 \). In exactly the same way, but using (5.11) instead of (5.10), one obtains

\[
(5.23) \quad \xi(t) - \xi(\tau_1) = [M(t) - M(\tau_1)] + [A(t) - A(\tau_1)] \quad \text{for} \quad \tau_1 \leq t \leq \tau_2,
\]

so the desired inequality holds for \( 0 \leq t \leq \tau_2 \). Continuing in this way, the desired inequality is established for \( 0 \leq t < \tau \). \qed

To complete the proof of Theorem 3, let \( T = \inf\{t > 0 : \xi(t) = \varepsilon\} \) and let \( \sigma = \inf\{t > 0 : \xi(0) + M(t) = \varepsilon\} \), where \( 0 < \varepsilon < \xi(0) \). From Lemma 2, the non-negativity of \( A(\cdot) \), and the fact that \( \xi(\tau) = 0 \) on \( \{\tau < \infty\} \), we have the following inequalities: \( 0 < \sigma \leq T \leq \tau \). From the martingale property of \( M \) and (5.20) it follows easily that \( \mathbb{E}(\sigma) = \infty \), so \( \mathbb{E}(T) = \infty \) as well, which establishes that \( Z \) is not positive recurrent.

6 Categories of divergent LCP solutions

Our goal in the remainder of the paper is to prove Theorem 4. We continue to assume the canonical problem format in which \( R \) satisfies (1.5) and \( \theta \) satisfies (5.1). In the following lemma and later, the term “LCP solution” is used to mean a solution \((u,v)\) of the linear complementarity problem (2.5). For a vector \( v \) we write \( v > 0 \) to mean that each component of \( v \) is positive.

Lemma 3. If (1.6) holds, then it cannot be that \( \theta \geq 0 \), and there cannot exist an LCP solution \((u,v)\) with \( v > 0 \).

Proof. Because \( R \) is completely-\( S \) by assumption, its transpose is also completely-\( S \), cf. Proposition 1.1 of Dai and Williams (1995). Thus there exists a vector \( a > 0 \) such that \( a'R > 0 \). Now (1.6) says that \( \theta + Ry = 0 \), where \( y > 0 \). Multiplying both sides of that equation by \( a' \) and rearranging terms, one has \( a'\theta = -a'Ry < 0 \), so it cannot be that \( \theta \geq 0 \). Also, if \((u,v)\) is an LCP solution with \( v > 0 \), one has from (2.5) that \( u = 0 \) and \( v = \theta \), which is a contradiction. \qed

Hereafter we assume that \( \theta \) and \( R \) jointly satisfy (1.6), because otherwise the conclusion of Theorem 4 is immediate (see Appendix C). We take as given a divergent LCP solution \((u,v)\), so \( v \neq 0 \) as a matter of definition. Also, by Lemma 3, it cannot be that \( u = 0 \), because then (2.5) would imply \( \theta = v \geq 0 \). We need to show that \( Z \) is not positive recurrent. Combining (2.5) and Lemma 3 with the restrictions \( u \neq 0 \) and \( v \neq 0 \), one sees that only LCP solutions in the following five categories need to be considered. Immediately after each category is defined, we shall exhibit a pair \((R,\theta)\) which admits an LCP solution \((u,v)\) in that category, or else direct the reader to a proposition that shows the category to be empty; readers may verify that the reflection matrix \( R \) appearing in each of our examples...
is completely-S. Also, we shall display the vector $-R^{-1}\theta$ for each example, confirming that assumption (1.6) is satisfied.

**Category I**  
Exactly two components of $v$ are positive, and the complementary component of $u$ is positive.

\[
R = \begin{pmatrix} 1 & 1/3 & 1/3 \\ 2 & 1 & -1/2 \\ 2 & -1/2 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad -R^{-1}\theta = \begin{pmatrix} 1/5 \\ 6/5 \\ 6/5 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]

**Category II**  
Exactly one component of $v$ is positive, $\det(\hat{R}) > 0$, and the two complementary components of $u$ are not both zero, where $\hat{R}$ is the $2 \times 2$ principal sub-matrix of $R$ corresponding to the two zero components of $v$.

\[
R = \begin{pmatrix} 1 & 1 & 1/2 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad -R^{-1}\theta = \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad u = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

In the example above, the two complementary components of $u$ are both positive; in the following example, which also falls in Category II, just one of them is positive.

\[
R = \begin{pmatrix} 1 & 1/2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad -R^{-1}\theta = \begin{pmatrix} 1/5 \\ 2/5 \\ 1/5 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

**Category III**  
Exactly one component of $v$ is positive, $\det(\hat{R}) = 0$, and the two complementary components of $u$ are not both zero.

It will be shown in Lemma 6 below that no such LCP solutions exist, given our other restrictions on $R$ and $\theta$.

**Category IV**  
Exactly one component of $v$ is positive, $\det(\hat{R}) < 0$, and the two complementary components of $u$ are both positive.

\[
R = \begin{pmatrix} 1 & 11/10 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad -R^{-1}\theta = \begin{pmatrix} 19/68 \\ 15/34 \\ 1/12 \end{pmatrix}, \quad u = \begin{pmatrix} 1/2 \\ 5/6 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \\ 2/3 \end{pmatrix}.
\]

It will be shown in Lemma 4 below that if there exists an LCP solution in Category IV, given our other restrictions on $R$ and $\theta$, there also exists a solution in Category I or Category II (or both). For the example above, a
second LCP solution is \((\hat{u}, \hat{v})\) where \(\hat{u} = (0, 1, 0)'\) and \(\hat{v} = (1/10, 0, 1)'\); this second solution lies in Category I. Despite considerable effort, we have not been able to generate an example where there exist LCP solutions in both Category IV and Category II, and where our other restrictions on \(R\) and \(\theta\) are also satisfied.

**Category V**

Exactly one component of \(v\) is positive, \(\det(\hat{R}) < 0\), and exactly one of the two complementary components of \(u\) is positive.

\[
R = \begin{pmatrix} 1 & 1 & -2/5 \\ 2 & 1 & -6/5 \\ -2 & -1/10 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad -R^{-1}\theta = \begin{pmatrix} 9/8 \\ 5/14 \\ 45/28 \end{pmatrix}, \quad u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

It will be shown in Section 7 that \(Z\) cannot be positive recurrent if there exists an LCP solution in either Category I or Category II or Category V. In the rest of this section and Section 7, when an LCP solution \((u, v)\) falls into Category II through Category V, we assume without loss of generality that the first two components of \(v\) are zero and the third component of \(v\) is positive; namely,

\[v_1 = v_2 = 0 \text{ and } v_3 > 0.\]

Because \(R\) satisfies (1.5), we can write

\[R = \begin{pmatrix} 1 & a' & c \\ a & 1 & c' \\ b & b' & 1 \end{pmatrix}\]

for some constants \(a, a', b, b', c\) and \(c'\). Under convention (6.1) the \(2 \times 2\) principal sub-matrix of \(R\) corresponding to \(v\) is

\[\hat{R} = \begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}.
\]

The following lemma reduces the analysis of Category IV solutions to analysis of two other cases. Finally, Lemma 6 at the end of this section will show that LCP solutions in Category III cannot occur when (1.6) holds.

**Lemma 4.** If there exists an LCP solution in Category IV, then there also exists a solution in Category I or Category II (or both).

**Proof.** Denoting by \((u, v)\) a solution in Category IV, we assume that no solution in Category I exists. It will then suffice to prove that a solution in Category II exists.
Because \( v \) is assumed to satisfy (6.1), one has \( u_1 > 0, u_2 > 0, \) and \( u_3 = 0 \). By Lemma 5 below, one then has \( a, a' > 1 \) and \( \theta = (-1,-1,1)' \). We shall assume that \( c' \geq c \), then construct an LCP solution \( (\tilde{u}, \tilde{v}) \) that falls into Category II and has \( \tilde{u}_1 > 0, \tilde{v}_2 > 0, \tilde{u}_3 > 0 \); in exactly the same way, if \( c > c' \) then one can construct an LCP solution \( (\tilde{u}, \tilde{v}) \) that falls into Category II and has \( \tilde{v}_1 > 0, \tilde{u}_2 > 0, \tilde{u}_3 > 0 \). Our first observation is the following: it must be that \( b, b' \leq -1 \), because otherwise, contrary to the assumption imposed in the first paragraph of this proof, there would exist an LCP solution that falls into Category I; for example, if \( b > -1 \) then one has a divergent LCP solution \( (\tilde{u}, \tilde{v}) \) with \( \tilde{u} = (1,0,0)' \), \( \tilde{v}_1 = 0, \tilde{v}_2 = a - 1 > 0 \), and \( \tilde{v}_3 = b + 1 > 0 \).

To recapitulate, our immediate objective is to construct an LCP solution \( (\tilde{u}, \tilde{v}) \) that falls into Category II and has \( \tilde{u}_1 > 0, \tilde{v}_2 > 0, \tilde{u}_3 \geq 0 \), given that \( a, a' > 1, \theta = (-1,-1,1)', \) \( c' \geq c \) and \( b, b' \leq -1 \). The 2×2 sub-matrix of \( \tilde{R} \) that is relevant for this construction is

\[
\tilde{R} = \begin{pmatrix} 1 & c \\ b & 1 \end{pmatrix}.
\]

Because \( \tilde{R} \) is an \( S \)-matrix and \( b < 0 \), we know that \( bc < 1 \) (that is, \( \det(\tilde{R}) > 0 \)), and because \( b \leq -1 \) we then know that \( c > -1 \). Let \( \gamma = (\gamma_1, \gamma_2)' \) be the two-vector satisfying \( \tilde{R}\gamma = (1,-1)' \), that is,

\[
\gamma = \frac{1}{1 - bc} \begin{pmatrix} 1 & -c \\ -b & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{1 - bc} \begin{pmatrix} 1 + c \\ -1 - b \end{pmatrix}.
\]

Defining \( \tilde{u} = (\gamma_1,0,\gamma_2)' \) and \( \tilde{v} = \theta + \tilde{R}\tilde{u} \), one has \( \tilde{u}_1 = \gamma_1 > 0, \tilde{u}_2 = 0, \tilde{u}_3 = \gamma_2 \geq 0 \), and \( \tilde{v}_1 = \tilde{v}_3 = 0 \) by construction. Comparing the first and second rows of \( R \) term by term, and noting that the first two components of \( \theta \) are identical, one sees that

\[
\tilde{v}_2 - \tilde{v}_1 = \frac{1}{1 - bc} \left[(a - 1)(1 + c) - (c' - c)(1 + b)\right].
\]

From the inequalities \( a > 1, c > -1, c' \geq c \) and \( b \leq -1 \), we see that the quantity inside the square brackets in (6.4) is positive. Thus \( \tilde{v}_2 > 0 \), and hence \( (\tilde{u}, \tilde{v}) \) is an LCP solution that falls into Category II.

\[\Box\]

**Lemma 5.** Assume that there does not exist an LCP solution that falls into Category I, and that there is a divergent LCP solution \((u, v)\) with \( u_1 > 0, u_2 > 0, u_3 = 0, v_1 = v_2 = 0 \) and \( v_3 > 0 \). Assume further that \( R \) is of the form (6.2) and the principal sub-matrix \( \tilde{R} \) defined by (6.3) satisfies \( \det(\tilde{R}) < 0 \). Then \( \theta = (-1,-1,1)' \) and \( a, a' > 1 \).

**Proof.** Because \((u, v)\) is a solution of the LCP (2.5), one has

\[
\begin{pmatrix} 1 & a' & c \\ a & 1 & c' \\ b & b' & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\theta_1 \\ -\theta_2 \\ -\theta_3 + v_3 \end{pmatrix}.
\]
Set $\hat{\theta} = (\theta_1, \theta_2)'$. We show that $\hat{\theta} = (-1, -1)'$. Because $\hat{R}$ is an $S$-matrix with negative determinant, it must be that

\begin{equation}
(6.6) \quad a, a' > 0 \text{ and } aa' > 1.
\end{equation}

Setting $\hat{u} = (u_1, u_2)'$, we have from (6.5) that

\begin{equation}
(6.7) \quad \hat{R} \hat{u} = -\hat{\theta}.
\end{equation}

Because $u_1 > 0$ and $u_2 > 0$ by hypothesis, it is immediate from (6.6) and (6.7) that both components of $\hat{\theta}$ are negative, so our canonical rescaling gives $\hat{\theta} = (-1, -1)'$. From (6.6), (6.7) and $\theta = (-1, -1)'$, one concludes that

\begin{equation}
(6.8) \quad a, a' > 1.
\end{equation}

Suppose that $\theta = (-1, -1, -1)'$. Then (6.5) becomes

\begin{equation}
(6.9) \quad \begin{pmatrix} 1 & a' & c \\ a & 1 & c' \\ b & b' & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 + v_3 \end{pmatrix}.
\end{equation}

It must be true that $b, b' \leq 1$, because otherwise there would be a solution of the LCP that falls into Category I; for example, if $b > 1$ then one has a divergent LCP solution $(\bar{u}, \bar{v})$ with $\bar{u} = (1, 0, 0)'$, $\bar{v}_1 = 0$, $\bar{v}_2 = a - 1 > 0$ and $\bar{v}_3 = b - 1 > 0$. However, one cannot have $a, a' \geq 1$, (6.9) and $b, b' \leq 1$ holding simultaneously, so we have arrived at a contradiction.

Next suppose that $\theta = (-1, -1, 0)'$. Now we must have $b, b' \leq 0$, because otherwise there would be a solution of the LCP that falls into Category I. Now (6.9) holds with $v_3$ in place of $1 + v_3$ on the right-hand side, but that is obviously inconsistent with $b, b' \leq 0$. Thus we have again arrived at a contradiction, and the only remaining possibility under our canonical rescaling is $\theta = (-1, -1, 1)'$.

**Lemma 6.** Given that (1.6) holds, there cannot exist an LCP solution in Category III.

**Proof.** Arguing as in the proof of Lemma 4, we assume the existence of an LCP solution $(u, v)$ in Category IV, and focus on the specific case where $u_1 > 0$, $u_2 \geq 0$, $u_3 = 0$, $v_1 = v_2 = 0$ and $v_3 > 0$; other cases can be treated in exactly the same way. Again we use the notation (6.2) and define $\hat{R}$ by (6.3). Then a minor variation of the first paragraph in the proof of Lemma 5 establishes the following conclusions for the current case: first, both $\theta_1$ and $\theta_2$ are negative, so $\theta_1 = \theta_2 = -1$ with our scaling convention; and then one has $a = a' = 1$ in (6.2), because now $\det(\hat{R}) = 0$ by assumption. But (6.2) demands that $\theta + Ru^* = 0$ where $u^* > 0$, which leads to the conclusion that $c = c'$ in (6.2). That is, we conclude that the first two rows of $R$ are identical, whereas (6.2) includes the requirement that $R$ be non-singular.  

\[\square\]
7 Proof of Theorem 4

Given the development in Section 6, the proofs of Lemmas 7, 8, and 9 below will complete the proof of Theorem 4. We continue to adopt the canonical setting stated at the beginning of Section 6. The SRBM $Z$ is said to be transient if there exists an open ball $C$ centered at the origin such that $\mathbb{P}\{\tau_C = \infty\} > 0$ for some initial state $Z(0) = x \in \mathbb{R}^3$ that is outside of the ball, where $\tau_C = \inf\{t \geq 0 : Z(t) \in C\}$. Clearly, when $Z$ is transient, it is not positive recurrent.

Lemma 7. If there is an LCP solution $(u, v)$ in Category I, then $Z$ is transient.

Proof. Without loss of generality, we assume that

$$u_1 > 0, \quad u_2 = 0, \quad u_3 = 0, \quad v_1 = 0, \quad v_2 > 0, \quad v_3 > 0. \tag{7.1}$$

Because $(u, v)$ satisfies (2.5), one has $\theta_1 < 0$. By our scaling convention $\theta_1 = -1$. It follows from (2.5) that $u_1 = 1, \ v_2 = \theta_2 + a > 0$ and $v_3 = \theta_3 + b > 0$.

Assume that $R$ is of the form (6.2). Then equation (1.1) for $Z$ becomes

$$Z_1(t) = Z_1(0) + \theta_1 t + B_1(t) + Y_1(t) + a'Y_2(t) + c'Y_3(t), \quad t \geq 0, \tag{7.2}$$
$$Z_2(t) = Z_2(0) + \theta_2 t + B_2(t) + Y_2(t) + aY_1(t) + cY_3(t), \quad t \geq 0, \tag{7.3}$$
$$Z_3(t) = Z_3(0) + \theta_3 t + B_3(t) + Y_3(t) + bY_1(t) + b'Y_2(t), \quad t \geq 0, \tag{7.4}$$

where $B = \{B(t), t \geq 0\}$ is the three-dimensional driftless Brownian motion with covariance matrix $\Sigma$. Let $Z(0) = (0, N + N_0, N + N_0)'$ for some constant $N_0 \geq 0$ and $N > 0$. Let $\tau = \inf\{t \geq 0 : Z_2(t) = N_0 \text{ or } Z_3(t) = N_0\}$. We will show that for each $N_0 \geq 0$, $\mathbb{P}\{\tau = \infty\} > 0$ for sufficiently large $N$, and thus $Z$ is transient. Because the proof for each $N_0 \geq 0$ is identical, in the rest of this proof, we set $N_0 = 0$. Because $\theta_1 = -1$ and $Y_2(t) = Y_3(t) = 0$ for $t \in [0, \tau)$, one has

$$Z_1(t) = -t + B_1(t) + Y_1(t), \quad t < \tau,$$
$$Z_2(t) = N + \theta_2 t + B_2(t) + aY_1(t), \quad t < \tau,$$
$$Z_3(t) = N + \theta_3 t + B_3(t) + bY_1(t), \quad t < \tau.$$

For $t \geq 0$, let $\tilde{Y}_1(t) = \sup_{0 \leq s \leq t}(-s + B_1(s))^-$. Define

$$\tilde{Z}_1(t) = -t + B_1(t) + \tilde{Y}_1(t), \quad t \geq 0, \tag{7.5}$$
$$\tilde{Z}_2(t) = N + \theta_2 t + B_2(t) + a\tilde{Y}_1(t), \quad t \geq 0, \tag{7.6}$$
$$\tilde{Z}_3(t) = N + \theta_3 t + B_3(t) + b\tilde{Y}_1(t), \quad t \geq 0. \tag{7.7}$$

Clearly, the process of $Z$ is equal to $\tilde{Z}$ up to time $\tau$ on each sample path. In particular, $\tau = \hat{\tau}$, where $\hat{\tau} = \inf\{t \geq 0 : \tilde{Z}_2(t) = 0 \text{ or } \tilde{Z}_3(t) = 0\}$. It suffices to prove that for sufficiently
large $N$, $\mathbb{P}(\hat{\tau} = \infty) > 0$. By the functional strong-law-of-large-numbers (FSLLNs) for a driftless Brownian motion, one has, with probability one,

$$\lim_{r \to \infty} \frac{1}{r} \sup_{0 \leq s \leq t} |B(rs)| = 0 \quad \text{for each } t > 0.$$ 

Because the one-dimensional Skorohod map is Lipschitz continuous in the space of continuous paths under the uniform norm on finite intervals, (7.5) implies that, with probability one,

$$\lim_{r \to \infty} \left( r^{-1} \sup_{0 \leq s \leq t} \left| \hat{Z}_1(rs) \right| \right) = 0 \quad \text{and} \quad \lim_{r \to \infty} \sup_{0 \leq s \leq t} \left| r^{-1} \hat{Y}_1(rs) - 1 \right| = 0 \quad \text{for each } t > 0.$$ 

Therefore, with probability one, one has $\lim_{t \to \infty} t^{-1} \xi_2(t) = v_2 > 0$ and $\lim_{t \to \infty} t^{-1} \xi_3(t) = v_3 > 0$, where

$$\xi_2(t) = \theta_2 t + B_2(t) + a\hat{Y}_1(t), \quad t \geq 0,$$

$$\xi_3(t) = \theta_3 t + B_3(t) + b\hat{Y}_1(t), \quad t \geq 0.$$ 

Thus, there exists a constant $T > 0$ such that $\mathbb{P}(A) > 1/2$, where

$$A = \{ \xi_2(t) > 0 \text{ and } \xi_3(t) > 0 \text{ for all } t \geq T \}.$$ 

Choose $N$ large enough so that $\mathbb{P}(B) > 1/2$, where

$$B = \{ N + \xi_2(t) > 0 \text{ and } N + \xi_3(t) > 0 \text{ for all } t \in [0,T] \}.$$ 

Because $A \cap B \subset \{ \hat{\tau} = \infty \}$, one has $\mathbb{P}\{ \hat{\tau} = \infty \} > 0$. 

\[ \square \]

**Lemma 8.** If there is an LCP solution $(u,v)$ in Category II, then $Z$ is transient.

**Proof.** Without loss of generality, we assume that

$$(7.8) \quad u_1 > 0, \quad u_2 \geq 0, \quad u_3 = 0, \quad v_1 = 0, \quad v_2 = 0, \quad v_3 > 0.$$ 

Assume that $R$ is of the form (6.2), and let $\hat{R}$ be the $2 \times 2$ principal sub-matrix of $R$ defined in (6.3). By assumption, $\det(\hat{R}) > 0$. Conditions (2.5) and (7.8) imply that

$$\hat{R}^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \leq 0,$$

$$\mu \equiv \theta_3 - (b,b')\hat{R}^{-1} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} > 0.$$ 

Let $Z(0) = (0,0,N + N_0)'$ for some constants $N_0 \geq 0$ and $N > 0$. Let $\tau = \inf\{ t \geq 0 : Z_3(t) = N_0 \}$ be the first time that $Z_3$ hits $N_0$. We will show that for each $N_0 \geq 0$ and
sufficiently large \(N\), \(\mathbb{P}\{\tau = \infty\} > 0\), and thus \(Z\) is transient. The proof is identical for each \(N_0 \geq 0\). In the rest of this proof, we assume \(N_0 = 0\). On \(\{t < \tau\}\) one has \(Z_3(t) > 0\) and thus \(Y_3(t) = 0\). Because the SRBM \(Z\) satisfies equations (7.2)-(7.4), on \(\{t < \tau\}\) equations (7.2) and (7.3) can be expressed simultaneously as

\[
\begin{pmatrix}
Z_1(t) \\
Z_2(t)
\end{pmatrix} = \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} t + \begin{pmatrix}
B_1(t) \\
B_2(t)
\end{pmatrix} + R Y_1(t) Y_2(t), \quad t < \tau
\]

or equivalently

\[
\begin{pmatrix}
Y_1(t) \\
Y_2(t)
\end{pmatrix} = R^{-1} \left[ \begin{pmatrix}
Z_1(t) \\
Z_2(t)
\end{pmatrix} - \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} t - \begin{pmatrix}
B_1(t) \\
B_2(t)
\end{pmatrix} \right], \quad t < \tau.
\]

Substituting the expression for \((Y_1(t), Y_2(t))'\) into (7.4), one has

\[
Z_3(t) = N + \theta_3 t + B_3(t) + (b, b') R^{-1} \left[ \begin{pmatrix}
Z_1(t) \\
Z_2(t)
\end{pmatrix} - \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} t - \begin{pmatrix}
B_1(t) \\
B_2(t)
\end{pmatrix} \right], \quad t < \tau.
\]

Because the 3 \times 3 matrix \(\Sigma\) is positive definite, it follows from Theorem 1.3 of Dai and Williams (1995) that there exists a three-dimensional process \(\tilde{Z}\) that lives in \(\mathbb{R}^2_+ \times \mathbb{R}\) and has the following two properties. First, the process \(\tilde{Z}\), together with a two-dimensional process \(\tilde{Y}\), is defined on some filtered probability space \((\tilde{\Omega}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}})\). Second, \(\tilde{Y}\) and \(\tilde{Z}\) together satisfy the following: \(\tilde{Z}\) and \(\tilde{Y}\) are adapted to \(\{\tilde{\mathcal{F}}_t\}\), and \(\tilde{\mathbb{P}}\)-almost surely

\[
\begin{pmatrix}
\tilde{Z}_1(t) \\
\tilde{Z}_2(t)
\end{pmatrix} = \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} t + \begin{pmatrix}
\tilde{B}_1(t) \\
\tilde{B}_2(t)
\end{pmatrix} + \hat{R} \begin{pmatrix}
\tilde{Y}_1(t) \\
\tilde{Y}_2(t)
\end{pmatrix} \geq 0, \quad t \geq 0,
\]

\[
\tilde{Z}_3(t) = \theta_3 t + \tilde{B}_3(t) \in \mathbb{R}, \quad t \geq 0,
\]

\[
\tilde{Y}_i(0) = 0 \text{ and } \tilde{Y}_i \text{ is nondecreasing, } i = 1, 2,
\]

and

\[
\int_0^\infty \tilde{Z}_i(t) d\tilde{Y}_i(t) = 0, \quad i = 1, 2,
\]

where the 3-dimensional process \(\tilde{B}\) is a driftless Brownian motion with covariance matrix \(\Sigma\), and \(\tilde{B}\) is an \(\{\tilde{\mathcal{F}}_t\}\)-martingale. Let

\[
\xi(t) = \theta_3 t + \hat{B}_3(t) + (b, b') \hat{R}^{-1} \left[ \begin{pmatrix}
\tilde{Z}_1(t) \\
\tilde{Z}_2(t)
\end{pmatrix} - \begin{pmatrix}
\theta_1 \\
\theta_2
\end{pmatrix} t - \begin{pmatrix}
\tilde{B}_1(t) \\
\tilde{B}_2(t)
\end{pmatrix} \right], \quad t \geq 0,
\]

and \(\hat{\tau} = \inf\{t \geq 0 : N + \xi(t) = 0\}\). Because the law of \(\tilde{Z}\) is unique, one has that \(\tau\) and \(\hat{\tau}\) are equal in distribution. We now prove that for sufficiently large \(N\),

(7.9) \(\hat{\mathbb{P}}\{\hat{\tau} = \infty\} > 0\),
which implies that \( P\{ \tau = \infty \} > 0 \).

To prove (7.9), note that \((\tilde{Z}_1, \tilde{Z}_2)\) is a two-dimensional SRBM with data \((\hat{\theta}, \hat{R})\), where \(\hat{\theta} = (\theta_1, \theta_2)'\) and \(\hat{\Sigma}\) is the \(2 \times 2\) principal submatrix of \(\Sigma\) obtained by deleting the 3rd row and the 3rd column of \(\Sigma\). It follows from Lemma 12 in Appendix D that

\[
\lim_{t \to \infty} \frac{1}{t} \tilde{Z}_i(t) = 0 \text{ almost surely, } i = 1, 2.
\]

This result, together the SLLN for Brownian motions, implies that

\[
\lim_{t \to \infty} \frac{1}{t} \xi(t) = \mu \text{ almost surely.}
\]

Because \(\mu > 0\), one has almost surely \(\xi(t) \to \infty\) as \(t \to \infty\). Hence, one can choose \(N\) large enough so that \(P\{ N + \xi(t) > 0 \text{ for all } t \geq 0 \} > 0\), which proves (7.9).

\[\square\]

**Lemma 9.** If there is an LCP solution \((u, v)\) in Category V, then \(Z\) is not positive recurrent.

**Proof.** Without loss of generality, we assume that

\[
(7.10) \quad u_1 > 0, \quad u_2 = 0, \quad u_3 = 0, \quad v_1 = 0, \quad v_2 \geq 0, \quad v_3 > 0.
\]

Then a minor variation of the first paragraph in the proof of Lemma 5 establishes that both \(\theta_1\) and \(\theta_2\) are negative, so \(\theta_1 = \theta_2 = -1\) with our scaling convention. It follows from (6.5) that \(u_1 = 1, a = 1\) and \(v_3 = b + \theta_3 > 0\).

Let \(Z(0) = x = (0, N_2, N_3)'\) for some constants \(N_2, N_3 > 0\). Let

\[
\tau_i = \inf\{ t \geq 0 : Z_i(t) = 0 \}, \quad i = 2, 3,
\]

and \(\tau = \min(\tau_2, \tau_3)\). We would like to prove that \(E_x(\tau) = \infty\) for each \(N_2 > 0\) and sufficiently large \(N_3\), implying that \(Z\) is not positive recurrent.

Assume \(R\) is of the form (6.2). Then the SRBM \(Z\) satisfies equations (7.2)-(7.4). It follows from the definition of \(\tau\) that \(Z_2(t) > 0\) and \(Z_3(t) > 0\) for \(t < \tau\). Thus one has \(Y_2(t) = Y_3(t) = 0\) for \(t < \tau\). Because \(Z(0) = (0, N_2, N_3)'\) and \(a = 1\), (7.2)-(7.4) reduce to

\[
(7.11) \quad Z_1(t) = -t + B_1(t) + Y_1(t), \quad t < \tau,
\]

\[
(7.12) \quad Z_2(t) = N_2 - t + B_1(t) + Y_1(t), \quad t < \tau,
\]

\[
(7.13) \quad Z_3(t) = N_3 + \theta_3 t + B_3(t) + b Y_1(t), \quad t < \tau.
\]

From (7.11) one has \(Y_1(t) = Z_1(t) + t - B_1(t)\) for \(t < \tau\). Substituting \(Y_1(t)\) into (7.12) and (7.13) respectively, one has

\[
Z_2(t) = N_2 + B_2(t) - B_1(t) + Z_1(t), \quad t \leq \tau,
\]

\[
Z_3(t) = N_3 + (\theta_3 + b) t + B_3(t) - b B_1(t) + b Z_1(t), \quad t \leq \tau.
\]
For each \( t \geq 0 \), let \( \hat{Y}_1(t) = \sup_{0 \leq s \leq t} (-s + B_1(s))^\frac{1}{3} \). Define

\[
\hat{Z}_1(t) = -t + B_1(t) + \hat{Y}_1(t), \quad t \geq 0, \\
\hat{Z}_2(t) = N_2 + B_2(t) - B_1(t) + \hat{Z}_1(t), \quad t \geq 0, \\
\hat{Z}_3(t) = N_3 + (\theta_3 + b)t + B_3(t) - bB_1(t) + b\hat{Z}_1(t), \quad t \geq 0,
\]

and let \( \hat{\tau} \) be the first time that either \( \hat{Z}_2 \) or \( \hat{Z}_3 \) hits zero. It is clear that \( \tau \) and \( \hat{\tau} \) are equal on every sample path. By Lemma 10 below, one concludes that \( \mathbb{E}(\hat{\tau}) = \infty \) for each \( N_2 > 0 \) and sufficiently large \( N_3 \), and thus \( Z \) is not positive recurrent.

\[ \square \]

**Lemma 10.** Let \( B = (B_1, B_2, B_3) \) be a three-dimensional Brownian motion with drift zero and covariance matrix \( \Sigma \), which is assumed to be positive definite. Let

\[
Y_1(t) = \sup_{0 \leq s \leq t} (-s + B_1(s))^\frac{1}{3}, \quad t \geq 0.
\]

Define

\[
Z_1(t) = -t + B_1(t) + Y_1(t), \quad t \geq 0, \\
Z_2(t) = N_2 + B_2(t) - B_1(t) + Z_1(t), \quad t \geq 0, \\
Z_3(t) = 3N + 3\mu t + B_3(t) - bB_1(t) + bZ_1(t), \quad t \geq 0,
\]

for some \( N_2, N > 0 \) and \( b \in \mathbb{R} \). Assume \( \mu > 0 \). Then for each \( N_2 > 0 \) and sufficiently large \( N \), one has \( \mathbb{E} (\sigma) = \infty \), where \( \sigma = \inf \{ t \geq 0 : Z_2(t) = 0 \text{ or } Z_3(t) = 0 \} \).

**Remark:** The factor 3 in the definition of \( Z_3 \) is to make the notation easier in the proof.

**Proof.** We prove the case when \( b = -1 \); when \( b < 0 \), the proof is identical with slightly more complicated notation; when \( b \geq 0 \), the proof is actually significantly simpler. Under the assumption that \( b = -1 \), for \( t \geq 0 \),

\[
Z_1(t) = -t + B_1(t) + Y_1(t), \\
Z_2(t) = N_2 + B_2(t) - B_1(t) + Z_1(t), \\
Z_3(t) = (N + \mu t + B_3(t) + B_2(t)) + N_2 + (2N + 2\mu t - (N_2 + B_2(t) - B_1(t)) - Z_1(t)).
\]

Let \( X_2(t) = N_2 + B_2(t) - B_1(t) \) and \( X_3(t) = B_3(t) + B_2(t) \). Then \( X_2 \) is a Brownian motion starting from \( N_2 \) and \( X_3 \) is a Brownian motion starting from 0, and

\[
Z_2(t) = X_2(t) + Z_1(t) \geq X_2(t), \quad t \geq 0, \\
Z_3(t) = (N + \mu t + X_3(t)) + N_2 + (2N + 2\mu t - X_2(t) - Z_1(t)), \quad t \geq 0.
\]

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Define

\[ \tau_0 = \inf\{t \geq 0 : X_2(t) \leq 0\}, \]
\[ \tau_2 = \inf\{t \geq 0 : X_2(t) \geq 2N + 2\mu t - Z_1(t) \text{ or } X_2(t) \geq N + \mu t\}, \]
\[ \tau_3 = \inf\{t \geq 0 : X_3(t) \leq -(N + \mu t)\}, \]
\[ \tau = \tau_0 \land \tau_2 \land \tau_3. \]

Because \( Z_1(t) \geq 0 \), it follows from the definition of \( \tau \) that for each \( t < \tau \), one has \( Z_2(t) > 0 \) and \( Z_3(t) > 0 \). Thus \( \tau \leq \sigma \). To prove the lemma, it suffices to prove that

\[
E(\tau) = \infty.
\]

It is well known that \( E(\tau_0) = \infty \). When \( N \) is large, it is intuitively clear that \( \tau_0 < \tau_2 \land \tau_3 \) with large probability, which leads to \( E(\tau) = \infty \). To make the argument rigorous, first note that, because \( Z_1 \) is adapted to \( B_1 \), each \( \tau_i \) is a stopping time with respect to the filtration generated by the Brownian motion \( B \). Therefore, \( \tau \) is a stopping time as well. Because \( X_2(0) = N_2 \) and \( X_2 \) is a martingale with respect to the filtration generated by \( B \), by the optional sampling theorem one has for each \( t > 0 \) that

\[
E[X_2(\tau \land t)] = N_2.
\]

We are going to show that for sufficiently large \( N \),

\[
E[X_2(\tau)1_{\{\tau < \infty\}}] \leq \frac{N_2}{2}.
\]

Therefore,

\[
N_2 = E(X_2(\tau \land t)) = E(X_2(\tau)1_{\{\tau < t\}}) + E(X_2(t)1_{\{\tau \geq t\}})
\leq E[X_2(\tau)1_{\{\tau < \infty\}}] + E[(N + \mu t)1_{\{\tau \geq t\}}]
\leq \frac{N_2}{2} + (N + \mu t)P\{\tau \geq t\}, \quad t > 0,
\]

where we have used the fact that \( X_2(t) \leq N + \mu t \) for \( t \leq \tau_2 \) to obtain the inequality. Thus,

\[
P\{\tau \geq t\} \geq \frac{N_2}{2(N + \mu t)}, \quad t > 0,
\]

from which one concludes that \( E(\tau) = \infty \).

Our only remaining task is to prove (7.15). Because \( X_2(\tau_0) = 0 \) when \( \tau_0 \) is finite, one has

\[
E[X_2(\tau)1_{\{\tau < \infty\}}] \leq E[X_2(\tau_2 \land \tau_3)1_{\{\tau_2 \land \tau_3 < \infty\}}]
\leq E[(N + \mu(\tau_2 \land \tau_3))1_{\{\tau_2 \land \tau_3 < \infty\}}]
\leq \sum_{n=0}^{\infty} (N + \mu(n + 1))P\{n < \tau_2 \land \tau_3 \leq n + 1\}.
\]
To bound the probability $P\{n < \tau_2 \land \tau_3 \leq n + 1\}$ for each integer $n \geq 0$, one has

$$\{n < \tau_2 \land \tau_3 \leq n + 1\} \subset \{n < \tau_2 \leq n + 1\} \cup \{n < \tau_3 \leq n + 1\}.$$

On $\{n < \tau_3 \leq n + 1\}$ the Brownian motion $X_3$ must reach level $N + \mu n$ for the first time in the time interval $(n, n + 1]$. By the strong Markov property of $X_3$ and using a reflection-principle-type argument, the probability of the latter event is at most $2P\{X_3(n + 1) > N + \mu n\}$. On $\{n < \tau_2 \leq n + 1\}$ it must be that either $X_2(t)$ reaches level $N + \mu n$ for the first time in the time interval $(n, n + 1]$ or $Z_1(t)$ reaches level $N + \mu n$ at some time in the interval $(n, n + 1]$. Therefore,

$$P\{n < \tau_2 \land \tau_3 \leq n + 1\} \leq P\{n < \tau_2 \leq n + 1\} + P\{n < \tau_3 \leq n + 1\} \leq 2P\{X_3(n + 1) > N + \mu n\} + 2P\{X_2(n + 1) > N + \mu n\} + P\{\sup_{n < s \leq n + 1} Z_1(s) > N + \mu n\}.$$

Note that

$$P\{X_3(n + 1) > N + \mu n\} = P\left\{N(0, 1) > \frac{N + \mu n}{\gamma \sqrt{n + 1}}\right\} \leq \frac{1}{\sqrt{2\pi}} \frac{\gamma \sqrt{n + 1}}{N + \mu n} \exp\left(-\frac{1}{2} \frac{(N + \mu n)^2}{\gamma^2(n + 1)}\right) \leq \frac{\gamma}{\sqrt{2\pi \mu}} \exp\left(-\frac{\mu^2}{2\gamma^2} (N + \mu n)\right),$$

where $N(0, 1)$ denotes the standard normal random variable, $\gamma^2$ is the variance of the Brownian motion $X_3$, and the first inequality is standard for the normal random variable; see, for example, the third display on page 191 of Pollard (1984). Setting $N = N_0 \mu$ for a positive integer $N_0$, one has

$$\sum_{n=0}^{\infty} (N + (n + 1)\mu) 2P\{X_3(n + 1) > N + \mu n\} \leq \sum_{n=N_0}^{\infty} \frac{2\gamma}{\sqrt{2\pi}} (n + 1) \exp\left(-\frac{\mu^2}{2\gamma^2} n\right),$$

which is less than $\frac{N_2}{6}$ for sufficiently large $N_0$. One can prove similarly that, for $N = N_0 \mu$ with sufficiently large $N_0$,

$$\sum_{n=0}^{\infty} (N + (n + 1)\mu) 2P\{X_2(n + 1) > N + \mu n\} \leq \frac{N_2}{6}.$$

To bound the probability $P\left\{\sup_{n \leq s \leq n + 1} Z_1(s) > N + \mu n\right\}$, we use Lemma 11 in Appendix D: there are constants $c_1, c_2 > 0$ such that

$$P\left\{\sup_{n \leq s \leq n + 1} Z_1(t) > x\right\} \leq c_1 \exp(-c_2 x) \quad \text{for } x \geq 0.$$
Setting $N = N_0\mu$, one has
\[
\sum_{n=0}^{\infty} (N + (n + 1)\mu)\mathbb{P}\left\{ \sup_{n<s\leq n+1} Z_1(s) > N + \mu n \right\} \leq \sum_{n=N_0}^{\infty} (n + 1)\mu c_1 \exp(-c_2n\mu),
\]
which is less than $N_2/6$ for sufficiently large $N_0$. Thus we can choose $N$ large enough so that (7.15) is satisfied.

A Semimartingale reflecting Brownian motions

In this section, we present the standard definition of a semimartingale reflecting Brownian motion (SRBM) in the $d$-dimensional orthant $S = \mathbb{R}_+^d$, where $d$ is a positive integer. We also review the standard definition of positive recurrence for an SRBM, connecting it with the alternative definition used in Section 1.

Recall from section 1 that $\theta$ is a constant vector in $\mathbb{R}_+^d$, $\Gamma$ is a $d \times d$ symmetric and strictly positive definite matrix, and $R$ is a $d \times d$ matrix. We shall define an SRBM associated with the data $(S, \theta, \Gamma, R)$. For this, a triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ will be called a filtered space if $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-field of subsets of $\Omega$, and $\{\mathcal{F}_t, t \geq 0\}$ is an increasing family of sub-$\sigma$-fields of $\mathcal{F}$, i.e., a filtration.

Definition 6 (Semimartingale reflecting Brownian motion). An SRBM associated with $(S, \theta, \Gamma, R)$ is a continuous $\{\mathcal{F}_t\}$-adapted $d$-dimensional process $Z = \{Z(t), t \geq 0\}$, together with a family of probability measures $\{\mathbb{P}_x, x \in S\}$, defined on some filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ such that, for each $x \in S$, under $\mathbb{P}_x$, (1.1) and (1.4) hold, where, writing $W(t) = X(t) - \theta t$ for $t \geq 0$, $W$ is a $d$-dimensional Brownian motion with covariance matrix $\Gamma$, an $\{\mathcal{F}_t\}$-martingale such that $W(0) = x$ $\mathbb{P}_x$-a.s., and $Y$ is an $\{\mathcal{F}_t\}$-adapted $d$-dimensional process such that $\mathbb{P}_x$-a.s. (1.2) and (1.3) hold; here (1.2) is interpreted to hold for each component of $Y$, and (1.3) is defined to be

\[
\int_0^t 1_{\{Z_i(s) \neq 0\}} dY_i(s) = 0 \text{ for all } t \geq 0.
\]

Definition 6 gives the so-called weak formulation of an SRBM. It is a standard definition adopted in the literature; see, for example, Dupuis and Williams (1994) and Williams (1995). Note that condition (A.1) is equivalent to the condition that, for each $t > 0$, $Z_j(t) > 0$ implies $Y_j(t - \delta) = Y_j(t + \delta)$ for some $\delta > 0$. Reiman and Williams (1988) showed that a necessary condition for a $(S, \theta, \Gamma, R)$-SRBM to exist is that the reflection matrix $R$ is completely-$S$ (this term was defined in section 1). Taylor and Williams (1993) showed that when $R$ is completely-$S$, a $(S, \theta, \Gamma, R)$-SRBM $Z$ exists and $Z$ is unique in law under $\mathbb{P}_x$ for each $x \in S$. Furthermore, $Z$, together with the family of probability measures $\{\mathbb{P}_x, x \in \mathbb{R}_+^d\}$, is a Feller continuous strong Markov process.
Let $(\theta, \Gamma, R)$ be fixed with $\Gamma$ being a positive definite matrix and $R$ being a completely-$\mathcal{S}$ matrix. Dupuis and Williams (1994, Definition 2.5) and Williams (1995, Definition 3.1) gave the following definition of positive recurrence.

**Definition 7.** An SRBM $Z$ is said to be positive recurrent if for each closed set $A$ in $S$ having positive Lebesgue measure we have $E_x(\tau_A) < \infty$, for all $x \in S$, where $\tau_A = \inf\{t \geq 0 : Z(t) \in A\}$ and $E_x$ denotes the expectation under $P_x$.

Because each open neighborhood of the origin contains a closed ball that has positive volume, Definition 7 appears to be stronger (that is, more restrictive) than the definition adopted in section 1, but one can show that these two notions of positive recurrence are equivalent for an SRBM. Indeed, the last paragraph on page 698 in Dupuis and Williams (1994) provides a sketch of that proof.

### B Convenient normalizations of problem data

Let $R$ be a $d \times d$ completely-$\mathcal{S}$ matrix and $(X, Y, Z)$ a triple of continuous, $d$-dimensional stochastic processes defined on a common probability space. The diagonal elements of $R$ are necessarily positive. Now let $\tilde{D} = \text{diag}(R)$ and $\tilde{R} = R\tilde{D}^{-1}$ (thus $\tilde{R}$ is a $d \times d$ completely-$\mathcal{S}$ matrix that has ones on the diagonal), and define $\tilde{Y}(t) = \tilde{D}Y(t)$ for $t \geq 0$. If $(X, Y, Z)$ satisfy (1.1)-(1.4) with reflection matrix $R$, then $(X, Y, Z)$ satisfy (1.1)-(1.4) with reflection matrix $\tilde{R}$, and vice-versa. Thus the distribution of $Z$ is not changed if one substitutes $\tilde{R}$ for $R$, and that substitution assures the standardized problem format (1.5).

Now let $R$ and $(X, Y, Z)$ be as in the previous paragraph, and further suppose that $X$ is a Brownian motion with drift vector $\theta$ and non-singular covariance matrix $\Sigma$. Define a $d \times d$ diagonal matrix $D$ by setting $D_{ii} = 1$ if $\theta_i = 0$ and $D_{ii} = |\theta_i|^{-1}$ otherwise. Setting $\hat{Z} = DZ$, $\hat{X} = DX$ and $\hat{R} = DR$, one sees that if $(X, Y, Z)$ satisfy (1.1)-(1.4) with reflection matrix $R$, then $(\hat{X}, \hat{Y}, \hat{Z})$ satisfy (1.1)-(1.4) with reflection matrix $\hat{R}$, and vice-versa. Of course, $\hat{X}$ is a Brownian motion whose drift vector $\hat{\theta} = D\theta$ satisfies (5.1); the covariance matrix of $\hat{X}$ is $\hat{\Sigma} = D\Sigma D$. Thus our linear change of variable gives a transformed problem in which (5.1) is satisfied. To achieve a problem format where both (1.5) and (5.1) are satisfied, one can first make the linear change of variable described in this paragraph, and then make the substitution described in the previous paragraph.

### C Proof that (1.6) is necessary for stability of $Z$

We consider a $d$-dimensional SRBM $Z$ with associated data $(S, \theta, \Gamma, R)$, defined as in Appendix A, assuming throughout that $R$ is completely-$\mathcal{S}$. Let us also assume until further notice that $R$ is non-singular. Because $R$ is an $S$-matrix, there exist $d$-vectors $v, v > 0$ such that $Rw = v$. That is,

(C.1) \[ R^{-1}v > 0, \text{ where } v > 0. \]
Now suppose that (1.6) does not hold. That is, defining \( \gamma = R^{-1} \theta \), suppose that \( \gamma_i \geq 0 \) for some \( i \in \{1, \ldots, d\} \). For future reference let \( u \) be the \( i^{th} \) row of \( R^{-1} \). Thus (C.1) implies
\[
(C.2) \quad u \cdot v > 0.
\]

Our goal is to show that \( Z \) cannot be positive recurrent. Toward that end, it will be helpful to represent the underlying Brownian motion \( X \) in (1.1) as \( X(t) = W(t) + \theta t \), where \( W \) is a \( d \)-dimensional Brownian motion with zero drift and covariance matrix \( \Gamma \). Pre-multiplying both sides of (1.1) by \( R^{-1} \) then gives
\[
R^{-1}Z(t) = R^{-1}W(t) + \gamma t + Y(t).
\]
The \( i^{th} \) component of that vector equation is
\[
(C.3) \quad u \cdot Z(t) = u \cdot W(t) + \gamma_i t + Y_i(t).
\]

Setting the stage for later discussion of the one-dimensional process
\[
(C.4) \quad \xi(t) = u \cdot Z(t), \quad t \geq 0,
\]
let \( A = \{ z \in S : |z| \leq 1 \} \) and \( B = \{ u \cdot z : z \in A \} \). Then \( B \subset \mathbb{R} \) is a compact interval containing the origin, and from (C.2) we know that \( B \) contains positive values as well, because \( A \) contains \( \alpha v \) for sufficiently small constants \( \alpha > 0 \). Thus \( B \) has the form
\[
(C.5) \quad B = [a, b] \text{ where } a \leq 0 \text{ and } b > 0.
\]

As the initial state \( x = Z(0) = W(0) \), we take
\[
(C.6) \quad x = \beta v, \quad \text{where } v \text{ is chosen as in (C.1) and } \beta > \max(|v|^{-1}, (u \cdot v)^{-1}b).
\]

From (C.1), (C.2), (C.5) and (C.6) we have that
\[
(C.7) \quad x \in S, \quad |x| > 1, \text{ and } u \cdot x > b.
\]

Thus, defining \( \tau_A = \inf\{ t \geq 0 : Z(t) \in A \} \) and \( \sigma = \inf\{ t \geq 0 : \xi(t) \in B \} \), it follows from the definitions of \( A, B \) and \( \xi \), plus (C.5) and (C.7), that
\[
(C.8) \quad \tau_A \geq \sigma = \inf\{ t \geq 0 : \xi(t) = b \} > 0 \quad \mathbb{P}_x\text{-a.s.}
\]

From (C.3) and (C.4) we see that \( \xi \) is bounded below by a one-dimensional Brownian motion with non-negative drift, and \( \xi(0) > b \) \( \mathbb{P}_x\text{-a.s.} \). Thus \( \mathbb{E}_x(\sigma) = \infty \), implying that \( \mathbb{E}_x(\tau_A) = \infty \) as well by (C.8). This establishes that \( Z \) is not positive recurrent.

In conclusion, we need to show that \( Z \) cannot be positive recurrent if \( R \) is singular. In that case there exists a non-trivial row vector \( u \in \mathbb{R}^d \) such that \( uR = 0 \), and we can assume that \( u \cdot \theta \geq 0 \) as well (because \( -u \) can be exchanged for \( u \) if necessary). Pre-multiplying both sides of (1.1) by \( u \) gives the following analog of (C.3):
\[
u \cdot Z(t) = u \cdot W(t) + (u \cdot \theta)t.
\]
Because $R$ is an $S$-matrix, there exist $w, v \in S$ such that

(C.9) \[ u + Rw = v. \]

(There exist $y, v \in S$ satisfying that equation for any $u \in \mathbb{R}^d$; we are focusing on a particular $u$.) After pre-multiplying both sides of (C.10) by $u'$ one obtains

(C.10) \[ u \cdot v = |u|^2 > 0. \]

We choose the initial state $x = Z(0) = W(0)$ exactly as in (C.6), and define the set $A$ as before. The proof that $E_x(\tau_A) = \infty$, and hence that $Z$ is not positive recurrent, then proceeds exactly as in the case treated above, except that now the process $\xi = u \cdot Z$ is itself a Brownian motion with non-negative drift, whereas in the case treated earlier $\xi$ was bounded below by such a Brownian motion.

D Miscellaneous lemmas

In this appendix, we collect a few lemmas that are used in Section 7.

**Lemma 11.** Let $X$ be a one-dimensional Brownian motion with drift $\theta < 0$, variance $\sigma^2$, starting from 0. Let $Z$ be the corresponding one-dimensional RBM; namely,

$$Z(t) = X(t) - \min_{0 \leq s \leq t} X(s) \quad \text{for } t \geq 0.$$ 

There exist constants $c_1 > 0$ and $c_2 > 0$ such that

(D.1) \[ \mathbb{P}\left\{ \sup_{n-1 \leq s \leq n} Z(s) > x \right\} \leq c_1 \exp(-c_2 x) \]

for each integer $n \geq 1$ and $x > 0$.

**Proof.** Since $X$ has negative drift, it is well known (see, for example, Section 1.9 of Harrison (1985)) that for each $t > 0$ and $x > 0$,

$$\mathbb{P}\{Z(t) > x\} = \mathbb{P}\left\{ \max_{0 \leq s \leq t} (X(t) - X(t-s)) > x \right\}$$

$$= \mathbb{P}\left\{ \max_{0 \leq s \leq t} X(s) > x \right\}$$

(D.2) \[ \leq \mathbb{P}\left\{ \sup_{0 \leq s < \infty} X(s) > x \right\} = \exp(-x(2|\theta|/\sigma^2)). \]

Now, for $n \geq 1$,

$$\mathbb{P}\left\{ \max_{n-1 \leq t \leq n} Z(t) > x \right\} \leq \mathbb{P}\{\max(Z(n-1), Z(n)) > x/2\}$$

$$+ \mathbb{P}\{\max(Z(n-1), Z(n)) \leq x/2, \max_{n-1 \leq t \leq n} Z(t) > x\}$$

$$\leq 2 \exp(-x(|\theta|/\sigma^2)) + \mathbb{P}\{Z(n-1) \leq x/2, Z(n) \leq x/2, \max_{n-1 \leq s \leq n} Z(t) > x\}$$

$$\leq 2 \exp(-x(|\theta|/\sigma^2)) + \mathbb{P}\{n-1 < \tau < n, X(n) - X(\tau) < -x/2\},$$

25
where \( \tau = \inf \{ t \geq n - 1 : Z(t) > x \} \). Note that \( \tau \) is a stopping time with respect to the filtration generated by \( Z \), which is equal to the filtration generated by the Brownian motion \( B \), where \( B(t) = X(t) - \theta t \) for \( t \geq 0 \). By the strong Markov property of \( B \),

\[
\mathbb{P}\{ n - 1 < \tau < n, X(n) - X(\tau) < -x/2 \} \leq \mathbb{P}\{ n - 1 < \tau < n, B(n) - B(\tau) < -x/2 + |\theta| \} = \mathbb{E}\left\{ \mathbf{1}_{\{ n - 1 < \tau < n \}} \Phi((-x/2 + |\theta|)/(\sigma \sqrt{n - \tau})) \right\} \leq \Phi((-x/2 + |\theta|)/\sigma) \quad \text{for } x > 2|\theta|,
\]

where \( \Phi \) is the standard normal distribution function. Thus,

\[
\mathbb{P}\{ \max_{n-1 \leq s \leq n} Z(t) > x \} \leq 2 \exp\left(-x(|\theta|/\sigma^2)\right) + \Phi((-x/2 + 1)/\sigma) \quad \text{for } x > 2|\theta|,
\]

from which one readily has the lemma. \( \square \)

The following lemma is used in the proof of Lemma 8. Let

\[
R = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \quad \text{and} \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.
\]

Assume that \( \det(R) > 0 \). For any given \( 2 \times 2 \) positive definite matrix \( \Gamma \), because \( R \) is known as a \( \mathcal{P} \)-matrix and thus a completely-\( \mathcal{S} \) matrix, it follows from Taylor and Williams (1993) that a two-dimensional SRBM \( Z \) with data \( (\theta, \Gamma, R) \) starting from any fixed state \( x \in \mathbb{R}_+^2 \), as defined in Definition 6, is well defined and is unique in law.

**Lemma 12.** If \( R^{-1}\theta \leq 0 \), then each fluid path \( (y, z) \) starting from \( z(0) = 0 \) remains at zero; namely, \( z(t) = 0 \) for \( t \geq 0 \); hence, the two-dimensional SRBM \( Z \) is rate stable in the sense that almost surely,

\[
\lim_{t \to \infty} \frac{Z(t)}{t} = 0.
\]

**Proof.** Note that

\[
R^{-1} = \frac{1}{1 - ab} \begin{pmatrix} 1 & -b \\ -a & 1 \end{pmatrix}.
\]

Since \( \det(R) = 1 - ab > 0 \) and \( R^{-1}\theta \leq 0 \),

\[
ab < 1, \quad \theta_2 - a\theta_1 \leq 0, \quad \theta_1 - b\theta_2 \leq 0.
\]

Let \( (y, z) \) be a fluid path with \( z(0) = 0 \). By the oscillation inequality (see, for example, Lemma 4.3 of Dai and Williams (1995)), \( (y, z) \) is Lipschitz continuous. Choose positive constants \( c_1 \) and \( c_2 \) such that \( c_1\theta_1 + c_2\theta_2 \leq 0 \). (Such \( c_1 \) and \( c_2 \) exist because of condition (D.4).) Let \( g(t) = c_1 z_1(t) + c_2 z_2(t) \). We would like to show that \( g(t) = 0 \) for \( t \geq 0 \). Since \( g \) is absolutely continuous and \( g(0) = 0 \), it suffices to prove that

\[
\dot{g}(t) \leq 0 \quad \text{for each } t \quad \text{with } g(t) > 0 \quad \text{and } (y, z) \text{ being differentiable at } t,
\]

\[ (D.5) \]
where \( \dot{g}(t) \) is the derivative of \( g \) at \( t \). To prove (D.5), we consider several cases, depending on the signs of \( z_1(t) \) and \( z_2(t) \). When \( z_1(t) > 0 \) and \( z_2(t) > 0 \), (2.1)-(2.4) imply that \( \dot{z}_1(t) = \theta_1 \) and \( \dot{z}_2(t) = \theta_2 \); thus, \( \dot{g}(t) = c_1 \dot{z}_1(t) + c_2 \dot{z}_2(t) \leq 0 \). When \( z_1(t) = 0 \) and \( z_2(t) > 0 \), \( \dot{y}_2(t) = 0 \) and \( \dot{z}_1(t) = 0 \), from which one has \( \dot{y}_1(t) = -\theta_1 \) and \( \dot{z}_2(t) = \theta_2 - a\theta_1 \leq 0 \). Because \( \dot{z}_1(t) = 0 \) and \( \dot{z}_2(t) \leq 0 \), one readily has \( \dot{g}(t) \leq 0 \). When \( z_2(t) = 0 \) and \( z_1(t) > 0 \), one can similarly argue that \( \dot{g}(t) \leq 0 \).

Now let \( Z \) be a two-dimensional SRBM with data \((\theta, \Gamma, R)\) having a fixed initial point \( Z(0) = x \in \mathbb{R}_+^2 \). By Definition 6, \( Z \), together with associated \((X, Y)\), satisfies (1.1)-(1.4), where \( X \) is a Brownian motion. For each \( r > 0 \) and each \( t \geq 0 \), let

\[
\bar{X}^r(t) = \frac{1}{r}X(rt), \quad \bar{Y}^r(t) = \frac{1}{r}Y(rt), \quad \bar{Z}^r(t) = \frac{1}{r}Z(rt).
\]

By the functional strong-law-of-large-numbers (FSLLN) for a Brownian motion, almost surely,

\[
(D.6) \lim_{r \to \infty} \sup_{0 \leq s \leq t} |\bar{X}^r(s) - x(s)| = 0 \quad \text{for each } t > 0,
\]

where \( x(t) = (\theta_1 t, \theta_2 t)' \) for \( t \geq 0 \). Fix a sample path that satisfies (D.6). Let \( \{r_n\} \subset \mathbb{R}_+ \) be a sequence with \( r_n \to \infty \). The FSLLN (D.6) implies that \( \{\bar{X}^{r_n}\} \) is relatively compact in \( C(\mathbb{R}_+, \mathbb{R}) \), the space of continuous functions on \( \mathbb{R}_+ \) that is endowed with the topology of uniform convergence on compact sets. By the oscillation inequality, \( \{(\bar{Y}^{r_n}, \bar{Z}^{r_n})\} \) is relatively compact. Let \((y,z)\) be a limit point of \( \{(\bar{Y}^{r_n}, \bar{Z}^{r_n})\} \). It is clear that \((y,z)\) is a fluid path associated with the data \((\theta, R)\) that satisfies \( z(0) = 0 \). By the first paragraph of this proof, \( z(t) = 0 \) for \( t \geq 0 \). Therefore, the limit point \( z \) is unique, satisfying \( z(t) = 0 \) for \( t \geq 0 \). Thus, almost surely, for each \( t > 0 \), \( \sup_{0 \leq s \leq t} |\bar{Z}^r(s)| \to 0 \) as \( r \to \infty \), which implies that

\[
\lim_{r \to \infty} \frac{1}{r}Z(r) = \lim_{r \to \infty} \bar{Z}^r(1) = 0,
\]

proving (D.3). \( \square \)

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**References**


