Consumption, Investment and the Fisher Separation Principle
Introduction to Financial Engineering—ISyE 6227

1 Consumption with a Perfect Capital Market

We consider a simple 2-period world in which a single consumer must decide between consumption $c_0$ today (in period 0) and consumption $c_1$ tomorrow (in period 1). Our consumer is endowed with money $m_0$ today and $m_1$ tomorrow. Consistent with his endowment, the consumer has the opportunity to borrow or lend $b_0$ today at interest rate $r$.

The equations governing the consumer’s feasible actions today and tomorrow are:

\[ c_0 = m_0 + b_0 \]  
\[ c_1 = m_1 - (1 + r)b_0. \]

It is understood that consumption in both periods must be nonnegative, which puts limits on how much the consumer may borrow or lend today, namely, $-m_0 \leq b_0 \leq m_1/1 + r$. After substituting $c_0 - m_0$ for $b_0$ in (2), the consumer’s budget constraint is

\[ c_0 + \frac{c_1}{1 + r} = m_0 + \frac{m_1}{1 + r} := W_0. \]

We have written the budget constraint as an equality since we shall assume our consumer always prefers more consumption. We interpret the symbol $W_0$ as the consumer’s present value of wealth. Note that the consumer’s future value of wealth $W_1$ would be $(1 + r)W_0$.

To determine the optimal consumption and borrowing plan, we posit a consumer utility function $U(c_0, c_1)$ of the form $\sqrt{c_0} + \beta \sqrt{c_1}$. As we shall see, the parameter $\beta$ serves as a discount factor on future consumption. Smaller values of $\beta$ imply a larger discount factor on future consumption, which implies that our consumer prefers more consumption today.

Formally, the consumer’s optimization problem is given by:

\[ \text{MAX} \{ U(c_0, c_1) : c_0 + \frac{c_1}{1 + r} = W_0 \}, \]

which may be equivalently expressed as

\[ \text{MAX} \{ U(c_0, c_1(c_0)) : 0 \leq c_0 \leq W_0 \}, \]

where

\[ c_1(c_0) := (1 + r)(W_0 - c_0). \]

The form of utility implies that consumption in both periods must be positive so that the optimal choice for $c_0$ necessarily lies strictly between 0 and $W_0$. Accordingly, first-order optimality conditions imply that

\[ 0 = \frac{\partial U}{\partial c_0} + \frac{\partial U}{\partial c_1} \frac{dc_1}{dc_0} = \frac{1}{2\sqrt{c_0}} - \frac{\beta(1 + r)}{2\sqrt{c_1}}. \]
From (7) the optimal consumption plan necessarily satisfies the condition

$$\sqrt{c_1} = \beta(1 + r),$$  \hspace{1cm} (8)

or, equivalently,

$$c_1 = \beta^2(1 + r)^2c_0.$$  \hspace{1cm} (9)

Substituting (9) into the budget constraint (3) and solving for $c_0$ and $c_1$, the optimal consumption plan is given by:

$$c_0 = \left[ \frac{1}{1 + \beta^2(1 + r)} \right] W_0 := \rho_0 W_0$$  \hspace{1cm} (10)

$$c_1 = \left[ \frac{\beta^2(1 + r)^2}{1 + \beta^2(1 + r)} \right] W_0 := \rho_1 W_0.$$  \hspace{1cm} (11)

It is important to point out here that the constants $\rho_0$ and $\rho_1$ are independent of present wealth $W_0$; that is, they are known parameters strictly determined by the two discount factors $\beta$ and $r$. Note further that the optimal utility is of the form

$$U^*(W_0) := \sqrt{\rho_0 + \beta \sqrt{\rho_1}} \sqrt{W_0}.$$  \hspace{1cm} (12)

**Example 1.** Suppose $m_0 = 100$, $m_1 = 990$, $r = 0.10$ and $\beta = 0.60$. Then $c_0 = 0.716W_0$, $c_1 = 0.312W_0$ and $U^*(W_0) = 1.182\sqrt{W_0}$. Here $W_0 = 100 + 990/1.1 = 1000$. Thus $c_0 = 716$, $c_1 = 312$ and $U^*(1000) = 37.36$. Due to the availability of a capital market at which to borrow or loan, the consumer has increased his utility by almost 30% above the level corresponding to the initial endowment $U(100, 990)$. To obtain the optimal consumption plan, the consumer must borrow $b_0 = 716 - 100 = 616$ today, pay back 678 tomorrow, thereby leaving him with $990 - 678 = 312$ to consume in the final period.

### 2 Consumption and Investment with a Perfect Capital Market

We now consider a world in which the consumer now has the opportunity to invest $I_0$ today in production from which he will receive $f(I_0)$ tomorrow. The function $f(I_0)$ encapsulates the investment opportunities in production available to our consumer. For example, the consumer may wish to obtain an education while he is young, expecting a return on this investment in his working years. It is generally assumed that $f(\cdot)$ is strictly increasing (more investment leads to more return) but exhibits *diminishing returns* in that the marginal return on an incremental rise in investment *declines* as the total investment increases. When $f(\cdot)$ is differentiable, these assumptions imply the first derivative is positive and the second derivative is negative. (Such a function is called *concave.*) To ensure at least some investment would be made (to make our subsequent calculations easier), we also assume that the derivative $f'(0)$ is infinite.

The equations governing the consumer’s feasible actions today and tomorrow are now:

$$c_0 + I_0 = m_0 + b_0$$  \hspace{1cm} (13)
Rearranging terms as we did before, the new budget constraint is
\[ c_0 + \frac{c_1}{1 + r} = W_0 + \left[ \frac{f(I_0)}{1 + r} - I_0 \right] := \hat{W}_0. \tag{15} \]

A close examination of (15) reveals a fundamental property: *All consumers, regardless of their utility function, should first determine the optimal investment plan to increase their wealth!* That is, they should select the value of \( I_0 \) such that the marginal return \( f'(I_0) = 1 + r \). When the function \( f(I_0) \) represents the investment opportunities for a firm in which consumers hold stock, then each consumer that holds stock would insist that the firm optimize its investment opportunity, *regardless* of each consumer’s different desires for consumption today versus tomorrow. Because there exists a capital market for each consumer to borrow or lend, each consumer can redistribute the increases in wealth as they desire. This principle (in various forms) is known as the *Fisher Separation Theorem of Finance*.

**Example 2.** Suppose \( f(I_0) = 33\sqrt{I_0} \). Now \( f'(I_0) = 33/(2\sqrt{I_0}) \), and so the optimal choice for \( I_0 = 225 \). The additional wealth created through investment equals \( 495/1.1 - 225 = 225 \) so that \( \hat{W}_0 = 1225 \). From (10) and (11) the optimal consumption plan is \( c_0 = 877 \) and \( c_1 = 382 \) with \( U^*(1225) = 41.34 \). The utility has increased by about 10.7%, which also corresponds to \( 100(\sqrt{1225}/1000 - 1) \). To obtain the optimal consumption plan, the consumer must borrow \( b_0 = 877 + 225 - 100 = 1002 \) today, pay back 1103 tomorrow, thereby leaving him with \( 990 + 495 - 1102 = 382 \) to consume in the final period.

### 3 Consumption and Investment Without a Capital Market

We now consider the situation in which the consumer has investment opportunities as described in the previous section, but no longer has the opportunity to borrow or loan, i.e., \( b_0 = 0 \). We shall see that without access to a capital market our consumer is far worse off.

The equations governing the consumer’s feasible actions today and tomorrow are now:
\[ c_0 + I_0 = m_0 \tag{16} \]
\[ c_1 = m_1 + f(I_0). \tag{17} \]

The new budget constraint may be represented as:
\[ c_1(c_0) = m_1 + f(m_0 - c_0). \tag{18} \]

The first-order optimality conditions (7) imply that
\[ \sqrt{\frac{m_1}{c_0}} = \beta f'(I_0). \tag{19} \]
or, equivalently, that the optimal choice for $I_0$ must satisfy the identity

$$\sqrt{\frac{m_1 + f(I_0)}{m_0 - I_0}} = \beta f'(I_0). \quad (20)$$

After substituting the specific choice for $f$ and performing simple algebra, the optimal choice for $I_0$ must satisfy the identity

$$990 + 33\sqrt{I_0} = \frac{9801}{I_0} - 98.01. \quad (21)$$

Since the left-hand side of (21) is an increasing function of $I_0$ that is finite when $I_0 = 0$ and the right-hand side of (21) is a decreasing function of $I_0$ that is infinite when $I_0 = 0$, a unique solution exists, which may be obtained by bisection search. (Alternatively, identity (21) is essentially a cubic equation, which has a closed-form solution.) The optimal value for $I_0$ is about 8.25 with a corresponding consumption plan of $c_0 = 91.75$ and $c_1 = 1085$ with $U^* = 29.34$. The utility has dropped considerably to almost the level corresponding to the original endowment.

4 Homework Problems

1. Jones is endowed with money $m_0 = 55,000$ today and $m_1 = 88,000$ tomorrow. He desires to consume $c_0 = 80,000$ today and $c_1 = 66,000$ tomorrow.

   (a) If there is no opportunity to borrow or lend can Jones achieve his consumption objective? Explain.

   (b) Suppose there is a perfect capital market in which Jones may borrow or lend as much as he desires at the market interest rate of 10%. Can Jones now achieve his consumption objective? Explain.

   (c) Suppose that in addition to a perfect capital market Jones has an opportunity to invest $I_0 = 100,000$ today and receive $f(I_0)$ tomorrow. Determine the minimum value for $f(I_0)$ for which Jones will be able to exactly achieve his consumption objective.

   (d) Explain exactly what Jones must do using the capital market and investment opportunity available to him so that he may exactly achieve his consumption objective.

2. Jones is endowed with money $m_0 = 40,000$ today and $m_1 = 99,000$ tomorrow. He desires to consume $c_0 = 100,000$ today and $c_1 = 55,000$ tomorrow.

   (a) If there is no opportunity to borrow or lend can Jones achieve his consumption objective? Explain.

   (b) Suppose there is a perfect capital market in which Jones may borrow or lend as much as he desires at the market interest rate of 10%. Can Jones now achieve his consumption objective? Explain.
(c) Suppose that in addition to a perfect capital market Jones has an opportunity to
invest \( I_0 = 50,000 \) today and receive \( f(I_0) \) tomorrow. Determine the minimum value
for \( f(I_0) \) for which Jones will be able to exactly achieve his consumption objective.

(d) Explain exactly what Jones must do using the capital market and investment oppor-
tunity available to him so that he may exactly achieve his consumption objective.

3. Smith’s utility function is \( U(c_0, c_1) = \ln c_0 + 0.4 \ln c_1 \). Smith is endowed with money
\( m_0 = 90,000 \) today and \( m_1 = 500,000 \) tomorrow. There is a perfect capital market for
borrowing and lending at the market rate of interest of 25% per period.

(a) Determine the optimal consumption plan for Smith. Explain exactly what Smith
must do each period to achieve his optimal consumption plan. (Recall that \( \frac{d}{dx} \ln x = \frac{1}{x} \).)

(b) Smith has an opportunity to invest \( I_0 = 80,000 \) today and receive \( f(I_0) = 135,000 \)
tomorrow. Should he take advantage of this opportunity? If not, explain why not. If
so, explain exactly what he should do each period to achieve his new optimal
consumption plan.

(c) If Smith no longer has access to the capital market (he cannot borrow or lend), then
should he take advantage of the investment opportunity presented in (b)? Explain
your reasoning.

5 Homework Solutions

Problem 1

a. No. Desired consumption \( c_0 \) exceeds his endowment \( m_0 \).

b. No. The \( PV \) of his wealth \( 55+88/1.1 = 135 \) is less than the \( PV \) of his desired consumption
\( 80 + 66/1.1 = 140 \).

c. Jones needs to add 5 to the \( PV \) of his wealth. Consequently, \( \frac{f(I_0)}{1.1} - 100 = 5 \), which means
that the smallest value for \( f(I_0) \) is 115.5.

d. In the first period, Jones consumes 80, invests 100 and borrows 125. In the second period,
his endowment of 88 plus his investment payout of 115.5 generates a supply of 203.5 of
which 137.5 is needed to pay back the loan, thereby leaving 66 for consumption, as desired.

Problem 2

a. No. Desired consumption \( c_0 \) exceeds his endowment \( m_0 \).

b. No. The \( PV \) of his wealth \( 40+99/1.1 = 130 \) is less than the \( PV \) of his desired consumption
\( 100 + 55/1.1 = 150 \).
c. Jones needs to add 20 to the \( PV \) of his wealth. Consequently, \( \frac{f(I_0)}{1 + i} - 50 = 20 \), which means that the smallest value for \( f(I_0) \) is 77.

d. In the first period, Jones consumes 100, invests 50 and borrows 110. In the second period, his endowment of 99 plus his investment payout of 77 generates a supply of 176 of which 121 is needed to pay back the loan, thereby leaving 55 for consumption, as desired.

**Problem 3**

a. The first-order optimality conditions imply that

\[
\frac{1}{c_0} + \frac{0.4}{c_1}(-1.25) = 0, \tag{22}
\]

which implies that \( c_1 = 0.5c_0 \). Since \( c_0 + 0.8c_1 = W_0 = 490 \), we conclude that \( c_0 = W_0/1.4 \); thus, Smith should consume 350 now and 175 next period. To achieve this plan, Smith needs to borrow 260 today, and pay back 325 next period from the 500 supply, which leaves 175 to consume, as desired.

b. Since the \( NPV \) created by the investment opportunity is \( 135/1.25 - 80 = 28 > 0 \), Smith should invest in the project. His new \( PV \) of wealth will be 518. He should consume \( 518/1.4 = 370 \) now and 185 next period. To achieve this plan, Smith needs to borrow 360, and pay back 450 next period from the 500 + 135 total supply, which leaves 175 to consume, as desired.

c. Since Smith cannot use a capital market he is faced with 2 choices: either invest in the project, which will give him a utility of \( \ln(10) + 0.4 \ln(635) = 4.884 \), or simply consume his initial endowment, which will give him a utility of \( \ln(90) + 0.4 \ln(500) = 6.986 \). Smith should not invest.