1 Mortgage Basics

We adopt the following continuous-time notation. Let $LB(t)$ denote the outstanding loan balance at time $t$. Let $IP(t)$ and $PP(t)$ denote the interest and principal payments on the loan made at time $t$. The annual loan interest rate is $i$, and the (remaining) life of the loan is $T$. All of the continuous-time formulas have discrete-time counterparts, which we include as we go.

Let $c(t)$ denote the rate of cash flow paid to the mortgage holder (i.e. the bank). With an outstanding loan balance of $LB(t)$ at time $t$, in the next $\Delta t$ units of time the bank increases the loan balance by $iLB(t)\Delta t$, the interest it expects the mortgagee to pay, but will subtract the payment $c(t)\Delta t$ it receives. Therefore, the change in loan balance from $t$ to $t + \Delta t$ is

$$LB(t + \Delta t) - LB(t) = iLB(t)\Delta t - c(t)\Delta t,$$

which, as $\Delta t \to 0$, implies that

$$\frac{d}{dt}LB(t) = iLB(t) - c(t)$$

The solution to the differential equation (2) may be readily verified as

$$LB(t) = e^{it}\{LB(0) - \int_0^t e^{-is}c(s)ds\}. (3)$$

Note that we can “invert” (3) to reveal that

$$LB(0) = e^{-it}LB(t) + \int_0^t e^{-is}c(s)ds,$$

which merely states that present value of all cash the bank receives up to time $t$ including repayment of the outstanding loan balance at time $t$ (which pays off the loan) must equal the original loan balance, i.e., the bank is indifferent to these two cash flow streams.

A conventional fixed-rate mortgage requires the mortgagee to pay a fixed total payment of $M$ per month for a duration of $T$ years. In continuous-time, $c(s) = M$ and $LB(T) = 0$. Substituting these values into (4),

$$LB(0) = \int_0^T e^{-is}Mds = \frac{M}{i}(1 - e^{-iT})$$

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1 Use the product rule of differentiation and the Fundamental Theorem of Calculus.
or that
\[ M = \frac{iLB(0)}{1 - e^{-iT}} = \left( \frac{iLB(0)}{1 - (1 + i/12)^{-12T}} \right). \]  
(6)

The value for \( M \) is, of course, equal to “Annual Equivalent” of \( LB(0) \) over \( 12T \) monthly periods with a rate of interest of \( i/12 \) per month. Both the continuous-time and discrete-time formulas (the one in parentheses) represent the annual payment; to obtain the monthly payment, simply divide by 12.

After replacing \( c(s) = M \) in (3) with its solution (6), and performing a little algebra\(^2\)
\[ LB(t) = LB(0) \left\{ \frac{1 - e^{-iT}}{1 - e^{-i(T-t)}} \right\}. \]  
(7)

Observe that the expression in braces equals the proportion \( LB(t)/LB(0) \) of the original loan balance remaining, and \( T - t \) represents the time (or number of periods) remaining on the loan. This is a very useful fact to remember.

**Remark 1.** Suppose it is time \( t \) and we wish to take out a loan in the amount of \( LB(t) \) for a period of \( T - t \) at the same rate of interest \( i \). What would be the “new” (monthly) mortgage payment \( M_{t \rightarrow T} \)? Intuition suggests the process is regenerative, namely, the “process starts over” and this claim is easily verified. From (6) and (7),
\[ M_{t \rightarrow T} = \frac{iLB(t)}{1 - e^{-i(T-t)}} = \frac{iLB(0)}{1 - e^{-iT}} = M. \]  
(8)

We now proceed to calculate the principal and interest payments at time \( t \). The total payment
\[ M = IP(t) + PP(t) \]  
(9)
is the sum of the principal and interest payments, and since \( IP(t) = iLB(t) \),
\[ PP(t) = M - iLB(t). \]  
(10)

Since we now know how to compute \( M \) and \( LB(t) \) we can compute \( PP(t) \), too. However, we can do better. Since \( iLB(t) = IP(t) \) and \( c(t) = IP(t) + PP(t) \), it follows immediately from (2) that
\[ \frac{d}{dt} LB(t) = -PP(t). \]  
(11)

Now take the time derivative of both sides of (10) and use the identity (11) to obtain that
\[ \frac{d}{dt} PP(t) = iPP(t), \]  
(12)or that
\[ PP(t) = PP(0)e^{it}. \]  
(13)

\(^2\)To obtain the corresponding discrete-time formula simply replace \( e^{-ix} \) with \((1 + i/12)^{-12x}\), exactly as in (6).
This is not too surprising: we know the loan balance is decreasing exponentially, and this is solely due to the principal payments.

Remark 2. To obtain the discrete-time value for \( PP(t) \) a one-period adjustment must take place, as follows. Now \( PP(0) = M - iLB(0) \), which exactly equals the first principal payment in discrete-time (replace \( i \) with \( i/12 \) since the time period is a month). So you can define \( PP(0) \) to be \( e^{-i} \) times the first discrete-time principal payment, or simply define it as \( PP(1)e^{-i(T-1)} \), for \( T \geq 1 \), where \( PP(1) \) equals the first principal payment in discrete-time.

We now address two practical questions often asked by a mortgagee:

1. “If I want to pay off my loan in \( S \) years, then how much more do I have to pay per month?” To answer this question, use (6); that is,

\[
M + A = \frac{e^{iLB(0)}}{1 - e^{-iS}},
\]

from which one may solve for \( A \).

2. “If I pay an additional \( A \) dollars per month, then how many years \( S \) will it take to pay off my loan?” To answer this question, use (5); that is,

\[
LB(0) = \frac{M + A}{i}(1 - e^{-iS}),
\]

from which one may solve for \( S \).

In both cases the \( M \) is calculated from (6) using \( T \) instead of \( S \) (and, of course, setting \( A = 0 \).)

2 Examples

Consider a 30-year, fixed-rate mortgage for 125,000 at 6.75%.

1. What is the monthly payment?

\[
M = \frac{(0.0675/12)(125,000)}{1 - (1 + 0.0675/12)^{-360}} = 810.75.
\]

The annual payment in continuous-time is 9720.55, which is 810.05 per month.

2. What is the loan balance after 10 years, 8 months?

The value for \( t \) is 128 with 232 payment periods remaining.

\[
\frac{LB(128)}{LB(0)} = \frac{1 - (1 + 0.0675/12)^{-232}}{1 - (1 + 0.0675/12)^{-360}} = 0.8392387,
\]

which implies the loan balance is 104,904.84. In continuous-time, the value for \( t \) is 10\( \frac{2}{3} \) with 19\( \frac{1}{3} \) years remaining. Thus, \( LB(10\frac{2}{3})/LB(0) = 0.839657 \), which implies the loan balance is 104,957.13.
3. Suppose the remaining duration of the loan is 19 years and 4 months. If we add 100 to our payment each month, then how quickly will the loan be paid off?

Let $S$ denote the remaining life of the loan. We have

$$104,904.84 = \frac{910.75}{0.0675/12}[1 - (1 + 0.675/12)^{-S}],$$

which gives $S = 186.1012$ or $15.5084$ years. In continuous-time, we have

$$104,957.13 = \frac{(910.05)(12)}{0.0675}(1 - e^{-0.0675S}),$$

which gives $S = 15.4996$ years.

4. Suppose the remaining duration of the loan is 19 years and 4 months. How much do we have to add to our monthly payment to pay off the loan in 10 years (or 120 payments periods)?

Let $A$ denote the additional amount. We have

$$(M + A) = \frac{104,904.84(0.0675/12)}{1 - (1 + 0.675/12)^{-120}} = 1205.56,$$

which means that $A = 393.81$. In continuous-time, we have

$$(M + A) = \frac{104,957.13(0.0675/12)}{1 - e^{-0.0675(10)}} = 1202.79,$$

which means that $A = 392.74$.

### 3 Homework Problems

Consider a 15-year fixed-rate mortgage for 200,000 at 6.25%. In what follows, the first solution is the discrete-time answer while the solution in brackets is the continuous-time answer.

1. What is the monthly payment? $M = 1714.85$ [M = 1712.16]

2. What is the loan balance after 4 years, 3 months? $LB(51) = 160,792.27$ [LB(4.25) = 160,833.17]

3. Suppose the remaining duration of the loan is 10 years and 9 months. If we pay 2000 each month how quickly will the loan be paid off? Let $S$ denote the remaining life of the loan. $S = 104.44$ months or 8.70 years [$S = 8.68$ years]

4. Suppose the remaining duration of the loan is 10 years, 9 months. How much do we have to add to our monthly payment to pay off the loan in 5 years? Let $A$ denote the additional amount. $A = 1412.44$ [$A = 1409.01$]