Solutions to Homework 4, prepared by Fred Zahrn (fzahrn@isye)

1. I use Professor Hayter’s notation: $A'$ denotes the complement of $A$ relative to the sample space $S$.

   (a) The test is 40% accurate if the dog has Lyme disease, so $P(B|A) = .4$

   (b) The test is inaccurate with probability $(1 - .4)$ when the dog has Lyme disease: $P(B'|A) = .6$

   (c) $P(B'|A') = .01$

   (d) $P(B'|A') = .99$

   (e) By definition of conditional probability, $P(A ∩ B) = P(A|B)P(B)$, but also $P(A ∩ B) = P(B ∩ A) = P(B|A)P(A)$. Therefore $P(A|B) = P(B|A)P(A)/P(B)$. Also, $P(B) = P(B ∩ A) + P(B ∩ A') = P(B|A)P(A) + P(B|A')P(A')$. We are given that $P(A) = .1$, so $P(A') = .9$. We can now solve: $P(A|B) = .8163$

   (f) Similarly, $P(A|B') = P(B'|A)P(A)/P(B')$. Now $P(B') = 1 - P(B)$, so we can solve: $P(A|B') = .0631$

2. (a) Geometric. We’re counting trials until something special happens. We assume that the trials are independent and identically distributed.

   (b) Poisson. Hayter’s text explicitly refers to the modeling of “number of defects” situations using Poisson distributions. If you read carefully, he hints at why this is true: $B(n, p)$ is approximated by $P(np)$ for $n$ very large and $p$ very small. It is as though we divide the wire into $n$ small segments, each of which (independently, we assume) has a tiny probability $p$ of containing a defect.

   (c) Bernoulli. The tipoff is the phrase “whether or not,” which indicates that the random variable should take only two values. We have a defect with some probability $p$, and no defect with probability $1 - p$.

   (d) Binomial; specifically, $B(25, p)$ for some probability $p$ of producing a defect-free spool. We are assuming 25 independent and identically distributed trials, and we want the probability of observing $k$ successes.

   (e) Binomial, as in (d), but now $p$ is the probability of producing a spool with exactly 3 defects.

3. (a) $1 = P(S) = \sum_{k=0}^{\infty} P(X = k) = e(\frac{1}{n})$ using the formula given. Hence $e = \frac{5}{17}$.

   (b) $P(X \leq 0) = P(X = -1) + P(X = 0) = \frac{15}{22}$

   (c) $P(X = -1) = \frac{5}{22}$

   (d) $P(X = 2 | X \geq 0) = P(X = 2, X \geq 0)/P(X \geq 0) = P(X = 2)/P(X \geq 0)$ since $\{X = 2\} \subset \{X \geq 0\}$. Noting that $P(X \geq 0) = 1 - P(X = -1)$, we find that the answer is $\frac{2}{11}$.

   (e) $P(X^2 = 1) = P(X \in \{-1, 1\}) = P(X = -1) + P(X = 1) = \frac{5}{17}$.

   (f) $P(X \leq k) = \frac{5}{17} \sum_{i=-1}^{k} \frac{1}{1+i^2}$

4. (a) $P(Y = 0) = e^{-1.5}$. Recall that $0! = 1$.

   (b) $P(Y \leq 1) = P(Y = 0) + P(Y = 1) = e^{-1.5} + 1.5e^{-1.5}$.

   (c) $P(Y = 0, Y \leq 1) = P(Y = 0, Y \leq 1)/P(Y \leq 1) = P(Y = 0)/P(Y \leq 1)$, which we can calculate from our answers to (a) and (b). Note that we argue $P(Y = 0, Y \leq 1) = P(Y = 0)$ since $\{Y = 0\} \subset \{Y \leq 1\}$.

   (d) $P((Y - 1)^2 = 1) = P(Y \in \{0, 2\}) = P(Y = 0) + P(Y = 2) = e^{-1.5} + \frac{1.5^2}{2} e^{-1.5}$.

   (e) $P(Y \in \{0, 2, 4, 6, \ldots\}) = e^{-1.5} \sum_{k=0}^{\infty} (1.5)^{2k}/(2k)! = e^{-1.5} \left( \frac{1.5^2 e^{-1.5}}{2} \right) = \frac{1 + e^{-3}}{2}$. To understand the tricky sum, write the first several terms of the Taylor series for $e^{-1.5}$ and $e^{1.5}$, and notice that the even terms are the same but the odd terms are of opposite sign. Therefore $e^{-1.5} + e^{1.5}$ is twice the sum we want.

5. (a) $P(N = k) = (1 - p)^{k-1}p$, for $k \in \{1, 2, 3, \ldots\}$ by definition. Think of $k - 1$ failures, followed by a success.

   (b) $P(N > k) = (1 - p)^k$ for $k \in \{1, 2, 3, \ldots\}$. Think of $k$ failures. Alternatively, calculate $\sum_{n=k+1}^{\infty} P(N = n) = \sum_{n=k+1}^{\infty} (1 - p)^{n-1}p = p(1 - p)^k \sum_{i=0}^{\infty} (1 - p)^i = p(1 - p)^k \frac{1}{1 - (1 - p)} = (1 - p)^k$. If you want to interpret this as a function over all reals, you would say that $P(N > k) = (1 - p)^{\lfloor k \rfloor}$ for $k \geq 0$, and $P(N > k) = 1$ for $k < 0$; this interpretation can be extended to parts (c) and (d) below.
(c) \( P(N > k+1 \mid N > 1) = P(N > k+1, N > 1)/P(N > 1) = P(N > k+1)/P(N > 1) = \frac{(1-p)^{k+1}}{(1-p)} = (1-p)^k \)

(d) \( P(N > k + l \mid N > l) = P(N > k + l, N > l)/P(N > l) = P(N > k + l)/P(N > l) = \frac{(1-p)^{k+l+1}}{(1-p)} = (1-p)^k \)

(e) The answers to (b), (c), and (d) are the same; (d) is the most general statement. Based on the fact that \( P(N > k + l \mid N > l) = P(N > k) \) we say that the geometric random variable is memoryless.

6. (a) \( \int_{-1}^{1} cs^2 \, ds = 1 \), so \( c = \frac{1}{2} \)

(b) \( P(X \leq 0) = P(-1 \leq X \leq 0) = \frac{3}{2} \int_{-1/2}^{0} s^2 \, ds = \frac{1}{2} \). No need to integrate if you notice the symmetry.

(c) \( P(X = -\frac{1}{2}) = \frac{3}{2} \int_{1/2}^{1} s^2 \, ds = 0. \)

(d) \( P(X < \frac{1}{2} \mid X \geq 0) = \frac{P(0 \leq X < 1/2)}{P(X \geq 0)} = \frac{\frac{1}{2} \int_{0}^{1/2} s^2 \, ds}{1 - P(X < 0)} = \frac{1/16}{1/2} = \frac{1}{8} \)

(e) \( P(X^2 > \frac{1}{4}) = P(X < -\frac{1}{2}) + P(X > \frac{1}{2}) = 2 \times P(X > \frac{1}{2}) = \frac{2}{5} \), using the symmetry.

(f) For \( x \in (-1,1) \) we have \( P(X \leq x) = \frac{3}{2} \int_{-1}^{x} s^2 \, ds = \frac{1}{2}(x^3 + 1) \). For \( x \leq -1 \), \( P(X \leq x) = 0 \), and for \( x \geq 1 \), \( P(X \leq x) = 1 \).

7. (a) The cdf \( F \) of \( Y \) is a continuous function, so \( Y \) is a continuous random variable. The pdf is \( f(t) = \frac{d}{dt} F(t) = \lambda e^{-\lambda t} \) for \( t > 0 \). Note that \( f(t) = 0 \) for \( t \leq 0 \).

(b) We recognize from the form of \( f \) (or \( F \)) that this is an exponential distribution with parameter \( \lambda \).

8. (a) The cdf is a step function, so \( Z \) is a discrete random variable. The pmf is given by \( P(Z = k) = \frac{1}{5} \) for \( k \in \{1,2,3,4,5,6\} \). Each step is of height \( \frac{1}{5} \).

(b) \( Z \) takes any value from \( S = \{1,2,3,4,5,6\} \) with equal probability, like rolling a die.

9. \( P(X = k) = \left(\frac{4}{5}\right)^{k-1} \binom{5}{2}/\binom{5}{2} \) for \( k \in \{0,1,2,3,4\} \). There are \( \binom{5}{2} \) distinct five-card hands; we must count the number of such hands with \( k \) aces. We choose \( k \) of 4 aces, and for each of these sets of aces we select \( 5 - k \) of the 48 cards that are not aces.

10. \( P(Y = k) = \binom{10}{k} \left(\frac{15}{15-2k}\right)^{15-2k} / \binom{10}{15} \) for \( k \in \{0,1,2,3,4,5,6,7\} \). There are \( \binom{10}{15} \) ways to choose 15 socks from 20 different pairs, if we consider the left and right socks in a pair to be distinct. We must count the collections with exactly \( k \) colors such that the left and right socks of those colors are both present.

We choose \( k \) of 20 colors to have both the left and right sock represented. For each of these selections, we choose \( 15 - 2k \) of the \( 20 - k \) remaining colors to have only one sock represented. For each of these \( 15 - 2k \) present but unmatched colors, we select either the left or the right sock to be in our collection. (Note that \( k \) may not be larger than 7, since this would require more than 15 socks in our collection.)

For additional assistance, visit my office hours: Mon 9–10, Wed 10–11, Fri 11–12 in ISyE Main Bldg, Room 103.