Homework 5
October 14, 2005
due on Thursday, October 13th
Graded by Idriss

1. Suppose we have a blood test that is 40% accurate if the dog has Lyme disease, but 98% accurate if the dog does not. (I believe the 40% figure is accurate; I made up the 98% figure since the test is supposed to rarely give false positives.)

Let $A$ be the event that the dog has Lyme disease. Let $B$ be the event that the test is positive. Compute the following: (a) $P(B | A)$, (b) $P(B | A^c)$, (c) $P(B | A)$, and (d) $P(B | A^c)$.

Suppose we have a dog that has a prior probability of 10% of having Lyme disease. (e) What is the posterior probability that this dog has Lyme disease given a positive test? (f) What is the posterior probability that such a dog has Lyme disease if the test is negative?

(a) Directly, $P(B | A) = .4$.
(b) $P(B | A) = 1 - P(B | A) = .6$.
(c) Directly, $P(B | A^c) = .02$
(d) $P(B | A^c) = 1 - P(B | A^c) = .98$.
(e) $P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B | A)} P(B | A) = \frac{.1}{.4(.1) + .02(.9)} = .69$.

(f) $P(A | B^c) = \frac{P(A \cap B^c)}{P(B^c)} = \frac{P(A)}{P(B^c | A)} P(B^c | A) = \frac{.1}{.06(.1) + .98.06} = .063$.

2. Let $X$ be a discrete random variable with p.m.f. $Pr \{X = k\} = c / (1 + k^2)$ for $k = -2, -1, 0, 1$. (a) Determine $c$. (b) Determine $Pr \{X \leq 0\}$. (c) Determine $Pr \{X = -1\}$. (d) Compute $Pr \{X = 1 | X \geq 0\}$. (e) Determine the first moment of $X$. (f) Determine all medians of $X$.

(a) $\sum_{k=-2}^{1} P(X = k) = 1$

(b) $P(X \leq 0) = P(X = -2) + P(X = -1) + P(X = 0) = 5/11(1/5 + 1/2 + 1) = 17/22$.
(c) $P(X = -1) = 5/22$.
(d) $Pr \{X = 1 | X \geq 0\} = \frac{P(X=1 \cap X \geq 0)}{P(X \geq 0)} = \frac{P(X=1)}{P(X \geq 0)} = \frac{5/22}{5/11 + 5/22} = 1/3$
(e) $E(X) = \sum_{k=-2}^{1} k P(X = k) = 5/11(-2(1/5) - 1(1/2) + 0(1) + 1(1/2)) = -2/11$
(f) 

\begin{align*}
P(X \leq -2) &= 1/11 & P(X \geq -2) &= 1 \\
P(X \leq -1) &= 7/22 & P(X \geq -1) &= 10/11 \\
P(X = 0) &= 6/11 & P(X \geq 0) &= 15/22 \\
P(X \leq 1) &= 1 & P(X \geq 1) &= 5/22.
\end{align*}

Whenever, \(P(X \geq k) \geq 1/2\) and \(P(X \leq k) \geq 1/2\), \(k\) is a median. In our case, there is a unique median: 0.

3. Let \(Y\) have pdf \(f(s) = cs^2\) for \(-1 < s < 2\). (a) Determine \(c\). (b) Determine \(\Pr\{Y \leq 0\}\). (c) Determine \(\Pr\{Y = -1/2\}\). (d) Compute \(\Pr\{Y < 1/2 \mid Y \geq 0\}\). (e) Compute \(E[Y]\). (f) Determine all medians of \(Y\).

(a) \(\int_{-1}^{2} f(s)ds = c \int_{-1}^{2} s^2ds = c[s^3/3]_{-1} = c(8/3 + 1/3) = 3c = 1\). Therefore, \(c = 1/3\).

(b) \(\Pr\{Y \leq 0\} = \int_{-1}^{0} f(s)ds = \int_{-1}^{0} s^3ds = 1/9\).

(c) Since \(Y\) is continuous, \(\Pr\{Y = -1/2\} = 0\).

(d) \(\Pr\{Y < 1/2 \mid Y \geq 0\} = \frac{\Pr\{0 \leq Y < 1/2\}}{\Pr\{Y \geq 0\}} = \frac{\int_{0}^{1/2} f(s)ds}{1 - \Pr\{Y \leq 0\}} = \frac{1/72}{8/9} = 1/64\)

(e) \(E[Y] = \int_{-1}^{2} sf(s)ds = \int_{-1}^{2} s^3ds = \int_{-1}^{0} s^3ds = \frac{s^4}{4}\bigg|_{-1}^{0} = \frac{1}{12}(16 - 1) = \frac{5}{4}\)

(f) We need to solve \(P(Y \leq x) = 1/2\). But \(P(Y \leq x) = x^3/9 + 1/9\). So \(x = \sqrt[3]{7/2}\).

4. Let \(Y\) be a Poisson random variable with parameter \(\lambda = 2.5\). (a) Compute \(\Pr\{Y = 0\}\). (b) Compute \(\Pr\{Y \leq 1\}\). (c) Compute \(\Pr\{Y = 0 \mid Y \leq 1\}\). (d) Determine the mean of \(Y\). (e) What is the probability that \(Y\) is an odd number? (This is tricky. After you determine what series to sum, try simultaneously playing with the Taylor series expansions of \(e^{2.5}\) and \(e^{-2.5}\).)

(a) \(\Pr\{Y = 0\} = e^{-2.5} = 0.08\)

(b) \(\Pr\{Y \leq 1\} = \Pr\{Y = 0\} + \Pr\{Y = 1\} = e^{-2.5}(1 + 2.5) = 7/2e^{-2.5} = .29\)

(c) \(\Pr\{Y = 0 \mid Y \leq 1\} = \frac{\Pr\{Y = 0, Y \leq 1\}}{\Pr\{Y \leq 1\}} = \frac{\Pr\{Y = 0\}}{\Pr\{Y \leq 1\}} = 2/7\)

(d)
\[
E(Y) = \sum_{k=0}^{\infty} e^{-\lambda k} \frac{\lambda^k}{k!} \\
= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\
= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^\lambda = \lambda = 2.5
\]

(e) \(P(Y\text{ is odd}) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{2k+1}}{(2k+1)!}\). Note that
\[
e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!}\] (1)
\[
e^{-\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!}\] (2)
(1) - (2) yields \( e^\lambda - e^{-\lambda} = 2 \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} \). Therefore, \( \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}}{(2k+1)!} = 1/2(e^\lambda - e^{-\lambda}) \). So,

\[
P(Y \text{is odd}) = e^{-\lambda}(1/2)(e^\lambda - e^{-\lambda}) = (1/2)(1 - e^{-2\lambda}).
\]

5. Let \( X \) be the length of time measured in minutes until the next request for an ambulance. Suppose the mean of \( X \) is 1/2 hour. (a) What is the p.d.f. of \( X \)? (b) What is the Pr \{ \( X > 2 \) \}? (c) What is the Pr \{ \( X > 5 \mid X > 3 \) \}? (d) What is the Pr \{ \( X > t + 2 \mid X > t \) \} where \( t \geq 0 \)? (e) On the average, how much longer will it be until the next request given that we’ve already been waiting for 20 minutes?

(a) The p.d.f. of \( X \) is \( f(t) = \frac{1}{30}e^{-\frac{t}{30}} \).

(b) \( \text{Pr} \{ X > 2 \} = \int_{2}^{\infty} \frac{1}{30}e^{-\frac{t}{30}} = e^{-\frac{1}{15}} \)

(c) We can verify the memoryless property:

\[
\text{Pr} \{ X > 5 \mid X > 3 \} = \frac{\text{Pr} \{ X > 5 \}}{\text{Pr} \{ X > 3 \}} = \frac{\int_{5}^{\infty} \frac{1}{30}e^{-\frac{t}{30}} dt}{\int_{3}^{\infty} \frac{1}{30}e^{-\frac{t}{30}} dt} = \frac{e^{-\frac{5}{30}}}{e^{-\frac{330}{30}}} = e^{-\frac{1}{15}}
\]

(d)

\[
\text{Pr} \{ X > t + 2 \mid X > t \} = \frac{\text{Pr} \{ X > t + 2 \}}{\text{Pr} \{ X > t \}} = \frac{e^{-\frac{t+2}{30}}}{e^{-\frac{t}{30}}} = e^{-\frac{2}{15}}
\]

(e) We use the memoryless property of the exponential random variable. The next request will be 1/2 hour from now.

6. Let \( X \) be the number of clicks in an \( n \times n \) version of the game at [this url](#) for a player with perfect memory. (a) If \( n = 2 \), what is the probability mass function of \( X \)? (b) What is the minimum number of clicks needed to win an \( n \times n \) version of the game? (c) Determine \( \text{Pr} \{ X = n^2 \} \)?

(a) We click once on any square. Then we either get a matching square on the second click with probability 1/3 or not with probability 2/3. If we got the matching square, then we clicked 4 times to win the game so \( P(X = 4) = 1/3 \). Assume the second click is wrong, then we have full information on the grid and need to reclick on the all the squares to win the game (totalling 4+2=6 clicks). Hence, \( P(X = 6) = 2/3 \).

(b) The minimum number of clicks is \( n^2 \) as we need to click on at least all the squares to win.

(c) \( X = n^2 \) only if all the pairings are realized on the first trial. For the first pair, it happens with probability \( 1/(n^2 - 1) \). For the second pair, it happens with probability \( 1/(n^2 - 3) \) and so on. Therefore, since we have \( n^2/2 \) pairs,

\[
\text{Pr} \{ X = n^2 \} = \frac{1}{(n^2 - 1)(n^2 - 3) \ldots (3)(1)}.
\]

7. Suppose we have an aisle with storage racks on both sides of the aisle. The aisle is 50 feet long. A worker is stationed at one end of the aisle. The worker needs to retrieve an item from storage. Assume that all storage locations are equally likely. Let \( L \) be the distance that the worker needs to walk along the aisle to reach the retrieval location. (a) What would be a reasonable distribution for \( L \)? (b) What is the probability density function of \( L \)?

(a) The distribution of \( L \) is probably uniform between 0 and 50 as all items are equally likely to be picked.

(b) The p.d.f. or \( L \) is \( f(s) = (1/50)1_{10 \leq s \leq 50} \), where \( 1_{A} = 1 \) is \( A \) is true, 0 otherwise.

8. Consider the situation in the previous problem. Suppose an industrial engineer determines that 20% of the items cause 85% of the activity. Call this most active 20%, Type \( A \), and the others Type \( B \). Assume that Type \( A \) items and Type \( B \) items are the same size. (a) Where should the Type \( A \) items be placed along the aisle? (b) What would the density function of \( L \) become when the Type \( A \) items are stored in those best locations?
(a) As Type A items are more likely to be picked, they need to be placed closer to the worker. So Type A items occupy 20\%(50) = 10 feet being closest to the operator and Type B items the rest.

(b) We need to condition of the type of items being picked. If Type A items are to be picked, $L$ is uniform between 0 and 10. Otherwise, $L$ is uniform between 10 and 50. Therefore,

$$
f(s) = 0.85(1/10)_{10 \leq s \leq 10} + 0.15(1/40)_{10 \leq s \leq 50}.
$$