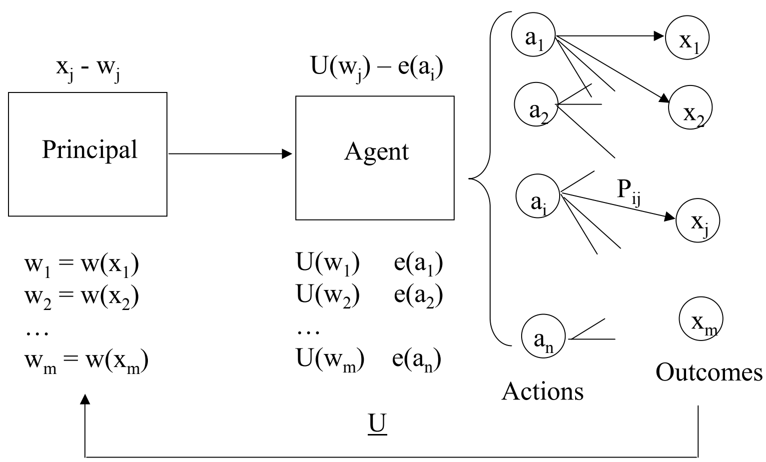
Principal-Agent Problem



Principal-agent problem

- If the agent doesn't accept the contract, his payoff is his reservation utility, U
- If the agent accepts the contract, he chooses between n possible actions: a_1, … , a_n.
- These actions produce m possible outcomes: x_1, … , x_m.
- There is a stochastic relationship between actions and outcomes (called "technology"). When the action is a_i, the principal observes outcome x_j with probability P_{ij}.
- If the principal observes outcome x_j, she pays the agent w_j.
- The agent's payoff is $U(w)-e(a_i)$, where $U(w)$ is the utility of wage w to the agent and $e(a_i)$ is the cost of action a_i to the agent.
  - U is increasing, differentiable and concave.
- Assuming the principal is risk-neutral, her payoff is $x_j – w_j$.

What if the agent's actions can be observed?

- The principal can design a contract where the wages are conditioned on the actions, i.e., $w(a_i)$

$$\max \sum_{j=1}^{m} P_{ij} x_j - w_i \;\equiv\; \min w_i$$

$$\text{Participation constraint}: U(w_i) - e(a_i) \geq \underline{U}$$

Incentive constraint:

$$U(w_i) - e(a_i) \geq U(w_k) - e(a_k) \quad \forall k \neq i$$

To induce the agent to choose action a_i, set w_i such that $U(w_i) = \underline{U} + e(a_i)$ and set all other w_k sufficiently low.

Unobservable actions - Principal's problem: Step 1

- Given an action a_i, how to set the wages such that the agent chooses a_i and the principal's payoff is maximized?

$$\text{Participation constraint}: \sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) \geq \underline{U}$$

Incentive constraint:

$$\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) \geq \sum_{j=1}^{m} P_{kj} U(w_j) - e(a_k) \quad \forall k \neq i$$

Principal's objective (minimize the expected cost of inducing the agent to choose a_i):

$$\max \sum_{j=1}^{m} P_{ij}(x_j - w_j) \;\equiv\; \min \sum_{j=1}^{m} P_{ij} w_j = C(a_i)$$

Principal's problem: Step 1

- $C(a_i)$ is the minimal cost (to the principal) of inducing the agent to take action a_i.
- $C(a_i)$ is convex → the original maximization objective is concave
- Well-behaved mathematical program with a concave objective function (maximization) and linear constraints.

Principal's problem: Step 1

For a given action a_i :

$$\max \sum_{j=1}^{m} P_{ij}(x_j - w_j)$$

$$\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) \geq \underline{U} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \mu$$

$$\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) \geq \sum_{j=1}^{m} P_{kj} U(w_j) - e(a_k) \;\; \forall k \neq i \;\ldots. \quad \lambda_k$$

$$L(w,\lambda,\mu) = \sum_{j=1}^{m} P_{ij}(x_j - w_j) + \mu\left(\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) - \underline{U}\right) +$$

$$\sum_{k \neq i} \lambda_k \left(\sum_{j=1}^{m} P_{ij} U(w_j) - e(a_i) - \sum_{j=1}^{m} P_{kj} U(w_j) + e(a_k)\right)$$

Principal's problem: Step 1

$$L(w,\lambda,\mu)=\sum_{j=1}^{m}P_{ij}(x_j-w_j)+\mu\left(\sum_{j=1}^{m}P_{ij}U(w_j)-e(a_i)-\underline{U}\right)+$$

$$\sum_{k\neq i}\lambda_k\left(\sum_{j=1}^{m}P_{ij}U(w_j)-e(a_i)-\sum_{j=1}^{m}P_{kj}U(w_j)+e(a_k)\right)$$

$$\frac{\partial L(w,\lambda,\mu)}{\partial w_j}=-P_{ij}+\mu P_{ij}\frac{\partial U(w_j)}{\partial w_j}+\sum_{k\neq i}\lambda_k\left(P_{ij}\frac{\partial U(w_j)}{\partial w_j}-P_{kj}\frac{\partial U(w_j)}{\partial w_j}\right)=0\rightarrow$$

$$-P_{ij}+\frac{\partial U(w_j)}{\partial w_j}\left(\mu P_{ij}+\sum_{k\neq i}\lambda_k(P_{ij}-P_{kj})\right)=0$$

$$-1+\frac{\partial U(w_j)}{\partial w_j}\left(\mu+\sum_{k\neq i}\lambda_k\left(1-\frac{P_{kj}}{P_{ij}}\right)\right)=0\rightarrow\frac{1}{\frac{\partial U(w_j)}{\partial w_j}}=\mu+\sum_{k\neq i}\lambda_k\left(1-\frac{P_{kj}}{P_{ij}}\right)$$

Principal's problem: Step 1

$$\frac{1}{U'(w_j)}=\mu+\sum_{k\neq i}\lambda_k\left(1-\frac{P_{kj}}{P_{ij}}\right)$$

$\frac{P_{kj}}{P_{ij}}$: likelihood ratio

Principal's problem: Step 2

- From step 1:

$$\max \sum_{j=1}^{m} P_{ij}(x_j - w_j) \equiv \sum_{j=1}^{m} P_{ij}x_j - \min \sum_{j=1}^{m} P_{ij}w_j =$$
$$R(a_i) - C(a_i)$$

- What is the action a_i that maximizes $R(a_i)$-$C(a_i)$?

Two actions and two outcomes

- Suppose the agent has two possible actions, a and b, and there are two possible outcomes, x_1 and x_2.
- Suppose that action b is preferred by the principal

For action b:

$$\max \sum_{j=1}^{2} P_{bj}(x_j - w_j)$$

$$P_{b1}U(w_1) + P_{b2}U(w_2) - e(b) \geq \underline{U} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \mu$$

$$P_{b1}U(w_1) + P_{b2}U(w_2) - e(b) \geq$$
$$P_{a1}U(w_1) + P_{a2}U(w_2) - e(a) \quad \ldots. \quad \lambda$$

Two actions and two outcomes

Incentive compatibility constraint:

$$P_{b1}U(w_1)+P_{b2}U(w_2)-e(b)\geq P_{a1}U(w_1)+P_{a2}U(w_2)-e(a)$$

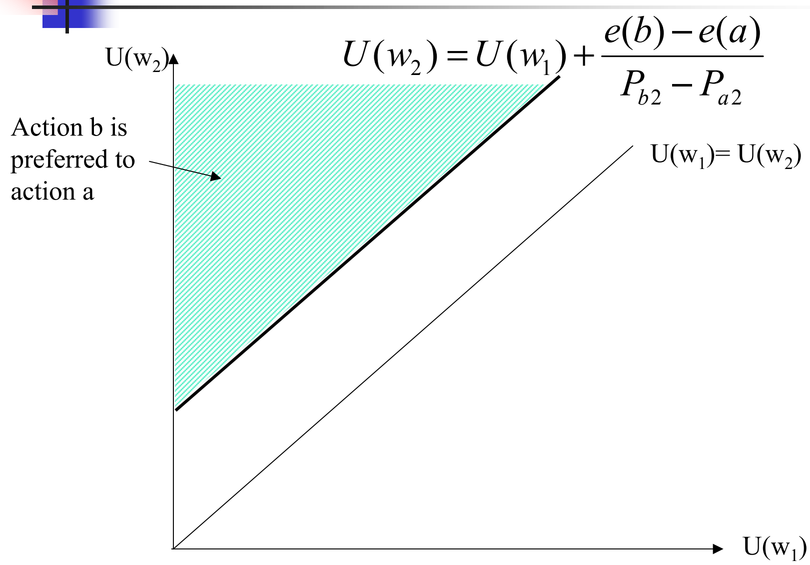
When is the agent indifferent between a and b?

$$P_{b1}U(w_1)+P_{b2}U(w_2)-e(b)=P_{a1}U(w_1)+P_{a2}U(w_2)-e(a)$$

$$U(w_2)=\frac{P_{a1}-P_{b1}}{P_{b2}-P_{a2}}U(w_1)+\frac{e(b)-e(a)}{P_{b2}-P_{a2}}=U(w_1)+\frac{e(b)-e(a)}{P_{b2}-P_{a2}}$$

since $P_{a1}+P_{a2}=P_{b1}+P_{b2}=1$

Two actions and two outcomes



Two actions and two outcomes

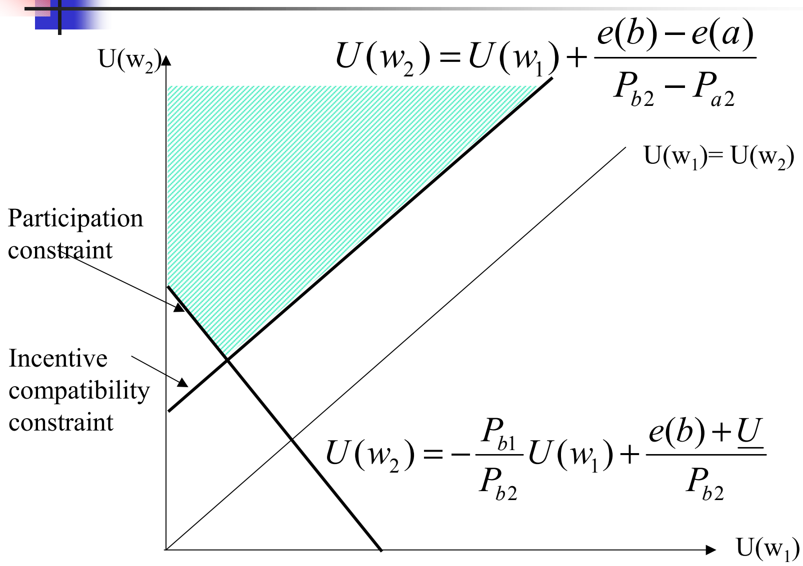
Participation constraints:

Action b: $P_{b1}U(w_1)+P_{b2}U(w_2)-e(b)\geq \underline{U}$

Action a: $P_{a1}U(w_1)+P_{a2}U(w_2)-e(a)\geq \underline{U}$

Action b: $U(w_2)\geq -\dfrac{P_{b1}}{P_{b2}}U(w_1)+\dfrac{e(b)+\underline{U}}{P_{b2}}$

Two actions and two outcomes



Two actions and two outcomes

- Principal's objective function

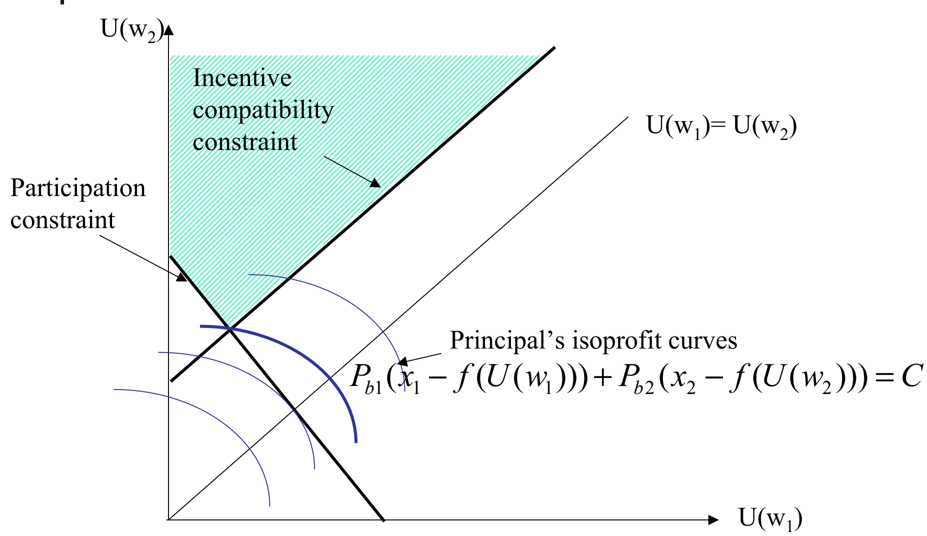
For action b:

$$\max P_{b1}(x_1 - w_1) + P_{b2}(x_2 - w_2)$$

Let f be the inverse of U: $f(U(w_j)) = w_j$

$$\max P_{b1}(x_1 - f(U(w_1))) + P_{b2}(x_2 - f(U(w_2)))$$

Two actions and two outcomes



Two actions and two outcomes

$$\max P_{b1}(x_1 - f(U(w_1))) + P_{b2}(x_2 - f(U(w_2)))$$

Slope of the objective function:

$$-P_{b1}\frac{\partial f(U(w_1))}{\partial U(w_1)} - P_{b2}\frac{\partial f(U(w_2))}{\partial U(w_1)}\frac{\partial U(w_2)}{\partial U(w_1)} = 0$$

$$\frac{\partial U(w_2)}{\partial U(w_1)} = -\frac{P_{b1}\dfrac{\partial f(U(w_1))}{\partial U(w_1)}}{P_{b2}\dfrac{\partial f(U(w_2))}{\partial U(w_1)}} = -\frac{P_{b1}f'(U(w_1))}{P_{b2}f'(U(w_2))}$$

Two actions and two outcomes

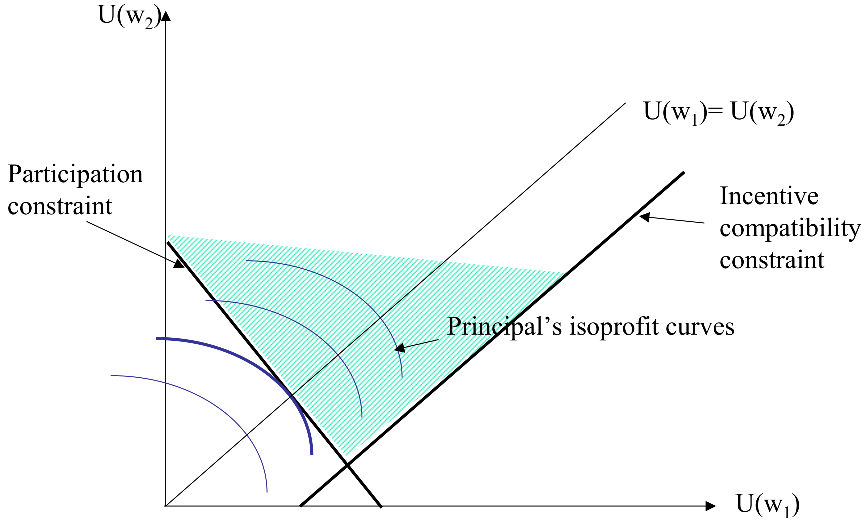
If $U(w_1) = U(w_2)$ (45^0 line)

$$\frac{\partial U(w_2)}{\partial U(w_1)} = -\frac{P_{b1}f'(U(w_1))}{P_{b2}f'(U(w_2))} = -\frac{P_{b1}}{P_{b2}}$$

Same as the slope of the participation constraint!

The principal's isoprofit curve is tangent to the agent's participation constraint along the 45^0 line.

Two actions and two outcomes



Two actions and two outcomes

$$\frac{1}{U'(w_j)} = \mu + \lambda\left(1 - \frac{P_{aj}}{P_{bj}}\right) \qquad \frac{P_{aj}}{P_{bj}}: \text{likelihood ratio}$$

<u>If the incentive constraint is not binding:</u>

$\lambda = 0,\ U'(w_j) = 1/\mu$ and $w_j = w$.

Substituting into the incentive constraint

(the agent prefers action b over action a):

$$\sum_{j=1}^{n} P_{bj}U(w) - e(b) \geq \sum_{j=1}^{n} P_{aj}U(w) - e(a) \rightarrow e(a) \geq e(b)$$

Two actions and two outcomes

$$\frac{1}{U'(w_j)} = \mu + \lambda\left(1 - \frac{P_{aj}}{P_{bj}}\right) \qquad \frac{P_{aj}}{P_{bj}} : \text{likelihood ratio}$$

If the incentive constraint is binding:

$\lambda > 0$, and $U'(w_j)$ depends on the likelihood ratio (LR).

The optimal incentive scheme is a linear function of LR.

Sensitivity analysis: Changes in the agent's costs

- What is the impact of the agent's costs on the outcome?

$$L(w,\lambda,\mu) = P_{b1}(x_1 - w_1) + P_{b2}(x_2 - w_2) +$$
$$\mu(P_{b1}U(w_1) + P_{b2}U(w_2) - e(b) - \underline{U}) +$$
$$\lambda(P_{b1}U(w_1) + P_{b2}U(w_2) - e(b) - P_{a1}U(w_1) - P_{a2}U(w_2) + e(a))$$

$$\frac{\partial L(w,\lambda,\mu)}{\partial e(b)} = -(\mu + \lambda) \qquad \frac{\partial L(w,\lambda,\mu)}{\partial e(a)} = \lambda$$

Sensitivity analysis: Changes in the agent's costs

- What is the impact of the agent's costs on the outcome?

$$\frac{\partial L(w,\lambda,\mu)}{\partial e(b)} = -(\mu+\lambda) \qquad \frac{\partial L(w,\lambda,\mu)}{\partial e(a)} = \lambda$$

- "Carrot": Decrease the cost of the desired action, b
- "Stick": Increase the cost of the undesired action, a
- Carrot vs. stick, which one is better?