Recap

- Last class (February 3, 2004)
  - One-card poker
  - Cournot competition under incomplete information
- Today (February 5, 2004)
  - Review results from one-card poker game
  - Static Bayesian games (static games of incomplete information)

ISYE 6230 – Spring 2004

- Total number of games played = 230 (224)
- Average = 12.57 (10.27)
- Number of different outcomes:
  - (B,B): 143 (131)
  - (B,P): 37 (40)
  - (P,B): 48 (38)
  - (P,P): 11 (10)
Games of Incomplete Information (Bayesian Games)

- In a game of complete information the players’ payoff functions are common knowledge.
- In a game of incomplete information, at least one player is uncertain about another player’s payoff function.

Normal Form Representation of Static Bayesian Games

- The normal form representation of an n-player static Bayesian game specifies the players’
  - action spaces $A_1, \ldots, A_n$,
  - type spaces $T_1, \ldots, T_n$,
  - beliefs $p_1, \ldots, p_n$, and
  - payoff functions $\pi_1, \ldots, \pi_n$.
- Player i’s type $t_i$ is privately known by player i, determines player i’s payoff function $\pi_i(a_1, \ldots, a_n; t_i)$ and is a member of the set of possible types.
- Player i’s belief $p_i(t_i \mid t_i)$ describes i’s uncertainty about the n-1 other players’ possible types $t_{-i}$, given i’s own type $t_i$. We denote this game by $G=(A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; \pi_1, \ldots, \pi_n)$. 

Example: Cournot Competition with incomplete information

- **Actions**
  - Quantity choices \( q_1 \) and \( q_2 \).

- **Type spaces**
  - Firm 1: \( T_1 = \{ c_1 \} \)
  - Firm 2: \( T_2 = \{ c^H, c^L \} \)

- **Payoffs**
  - Firm 1: \[
  \pi_1(q_1, q_2^L, q_2^H; c_1) = p[a-(q_1 + q_2^L)]q_1 + (1-p)[a-(q_1 + q_2^H)]q_1 - c_1q_1 .
  \]
  - Firm 2: \[
  \pi_1(q_1, q_2^H; c_1) = (a- q_1 - q_2^H - c_1) q_2 .
  \]
  - \[
  \pi_1(q_1, q_2^L; c_1) = (a- q_1 - q_2^L - c_1) q_2 .
  \]

- **Beliefs**
  - Firm 1: \( p_1(c_1|c_1) = p \quad p_1(c_1^L|c_1) = 1-p \)
  - Firm 2: \( p_2(c_1|t_2) = 1 \quad t_2 = c^H, c^L \)

Timing of a Static Bayesian Game

- Nature draws the type vector \( t = (t_1, ..., t_n) \) according to probability distribution \( p(t) \)
- Nature reveals \( t_i \) to player \( i \) but not to any other player
  - Player \( i \) can compute his/her belief \( p_i(t_i|t_i) \) using Bayes’ rule \( P(A|B) = P(A,B)/P(B) \):
    \[
    p_i(t_i | t_i) = p_i(t_i , t_i) / p(t_i) \\
    = p_i(t_i , t_i) / \sum_{t_i \in T_i} p_i(t_i , t_i)
    \]
- The players simultaneously choose actions
- Payoffs are received
Strategies in a Static Bayesian Game

- In the static Bayesian game $G=(A_1,\ldots,A_n; T_1,\ldots,T_n; p_1,\ldots,p_n; \pi_1,\ldots,\pi_n)$, a strategy for player $i$ is a function $s_i(t_i)$, where for each type $t_i \in T_i$, $s_i(t_i)$ specifies the action from the feasible set $A_i$ that type $i$ would choose if drawn by nature.

- Set of possible (pure) strategies for player $i$: $S_i: T_i \rightarrow A_i$.

- In a separating strategy, each type $t_i \in T_i$ chooses a different action from $a_i \in A_i$.

- In a pooling strategy, player $i$ chooses the same action for each type $t_i \in T_i$.

Strategies in the Cournot Game

Firm 1’s strategy: $q_1$

$$q_1 = \frac{(a - 2c_1 + p c^H + (1-p) c^L)}{3}$$

Firm 2’s strategy: $(q_2(c^H), q_2(c^L))$

$$q_2(c^H) = \frac{(a-2c^H+c_1)}{3} + \frac{(1-p)(c^H- c^L)}{6} = q^H_2$$

$$q_2(c^L) = \frac{(a-2c^L+c_1)}{3} - \frac{p(c^H- c^L)}{6} = q^L_2$$
Bayesian Nash Equilibrium

In the static Bayesian game 
\[ G = (A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; \pi_1, \ldots, \pi_n), \]
the strategies \( s^* = (s^*_1, \ldots, s^*_n) \) are a (pure strategy) Bayesian Nash Equilibrium if for each player \( i \) and for each of \( i \)'s types \( t_i \in T_i \), \( s^*_i(t_i) \) solves

\[
\max_{a_i \in A_i} \sum_{t_{i-1} \in T_{i-1}} \pi_i(s^*_1(t_1), \ldots, s^*_i(t_i-1), a_i, s^*_{i+1}(t_{i+1}), \ldots, s^*_n(t_n); t_i) p_i(t_{i-1} | t_i)
\]

That is, no player wants to change his/her strategy, even if the change involves only one action by one type.

Example: Battle of the sexes

\[
\begin{array}{c|cc}
 & \text{Ballet} & \text{Football} \\
\hline
\text{Ballet} & 2, 1 & 0, 0 \\
\text{Football} & 0, 0 & 1, 2 \\
\end{array}
\]

- Pure strategy equilibria: (Ballet,Ballet) (Football,Football)
- Mixed strategy equilibrium
  - Alice: Ballet with \( p=2/3 \), Football with \( 1-p=1/3 \)
  - Bob: Ballet with \( q=1/3 \), Football with \( 1-q=2/3 \)
Example: Battle of the sexes

- $t_A$ is privately known to Alice and $t_B$ is privately known to Bob
- $t_A$ and $t_B$ are independent draws from a uniform distribution on $[0,x]$

Action spaces
- $A_A = A_B = \{\text{Ballet, Football}\}$

Type spaces
- $T_A = T_B = [0,x]$

Beliefs
- $\pi_B(t_A) = \pi_A(t_B) = 1/x$ for all $t_A$ and $t_B$
- Bayesian game: $G=\{A_A, A_B; T_A, T_B; p_A, p_B; \pi_A, \pi_B\}$
### Example: Battle of the sexes

<table>
<thead>
<tr>
<th></th>
<th>Alice</th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ballet</td>
<td>$2 + t_A, 1$</td>
<td>$0, 0$</td>
</tr>
<tr>
<td>Football</td>
<td>$0, 0$</td>
<td>$1, 2 + t_B$</td>
</tr>
</tbody>
</table>

#### Strategies
- **Alice**: Play ballet if $t_A$ exceeds a critical value $c_A$; otherwise, play football
- **Bob**: Play football if $t_B$ exceeds a critical value $c_B$; otherwise, play ballet

#### Alice’s expected payoffs
- Alice plays Ballet
  \[
P(t_b < c_B)(2 + t_A) + P(t_b > c_B)0 = (c_B/x)(2 + t_A) + (1 - c_B/x)0 = (c_B/x)(2 + t_A)
  \]
- Alice plays Football
  \[
P(t_b < c_B)0 + P(t_b > c_B)1 = 1 - c_B/x
  \]
- Play ballet if $(c_B/x)(2 + t_A) > 1 - c_B/x \quad \rightarrow \quad t_A > x/c_B - 3 = c_A$
Example: Battle of the sexes

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</tr>
<tr>
<td>Football</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- Bob’s expected payoffs
  - Bob plays Ballet
    \[ P(t_A > c_A)1 + P(t_A < c_A)0 = (1 - c_A/x)1 + (c_A/x)0 = 1 - c_A/x \]
  - Bob plays Football
    \[ P(t_A > c_A)0 + P(t_A < c_A)(2 + t_B) = (c_A/x)(2 + t_B) \]
  - Play football if \((c_A/x)(2 + t_B) > 1 - c_A/x \rightarrow t_B > x/c_A - 3 = c_B\)

- Alice’s expected payoffs
  - Alice plays Ballet: \((c_B/x)(2 + t_A)\)
  - Alice plays Football: 1 - \(c_B/x\)
  - Play ballet if \((c_B/x)(2 + t_A) > 1 - c_B/x \rightarrow t_A > x/c_B - 3 = c_A\)

- Bob’s expected payoffs
  - Bob plays Ballet: 1 - \(c_A/x\)
  - Bob plays Football: \((c_A/x)(2 + t_B)\)
  - Play football if \((c_A/x)(2 + t_B) > 1 - c_A/x \rightarrow t_B > x/c_A - 3 = c_B\)
Example: Battle of the sexes

- $t_A > x/c_B - 3 = c_A$
- $t_B > x/c_A - 3 = c_B$
- $c_A = c_B = c$
- $c^2 + 3c - x = 0$
  \[ c = \frac{-3 + \sqrt{(9+4x)}}{2} \]
- What is the probability that Alice plays ballet?
  \[ 1 - \frac{c_A}{x} = 1 - \frac{-3 + \sqrt{(9+4x)}}{2x} \]
- What is the probability that Bob plays football?
  \[ 1 - \frac{c_B}{x} = 1 - \frac{-3 + \sqrt{(9+4x)}}{2x} \]

In the limit ($x \to 0$), these probabilities approach 2/3!

First-Price Sealed-Bid Auction

- Two bidders, one good
- Bidder i’s valuation for the good is $v_i$, is known only by bidder i. Valuations are independently and uniformly distributed on $[0,1]$.
- Each bidder i submits a nonnegative bid $b_i$. The higher bidder wins and pays his bid. Other bidder pays and receives nothing.
- In case of a tie, the winner is determined by a coin flip
- Bidder i’s payoff, if wins and pays $p$, is $v_i - p$
- Bidders are risk-neutral
- All of this information is common knowledge
First-Price Sealed-Bid Auction

- Action spaces
  - \( A_1 = A_2 = [0, \infty) \)
- Type spaces
  - \( T_1 = T_2 = [0,1] \)
- Beliefs
  - \( p_1(t_2|t_1) = p_1(t_2) \)
  - \( p_2(t_1|t_2) = p_2(t_1) \)
- Player i’s (expected) payoff function
  \[
  \pi_i(b_i, b_j; v_i, v_j) = \begin{cases} 
  v_i - b_i & \text{if } b_i > b_j \\
  (v_i - b_j)/2 & \text{if } b_i = b_j \\
  0 & \text{if } b_i < b_j 
  \end{cases}
  \]

Strategy for player i: \( b_i(v_i) \)

Strategies \((b_1(v_1), b_2(v_2))\) are a Bayesian Nash equilibrium if for each \( v_i \) in \([0,1]\), \( b_i(v_i) \) solves
\[
\max (v_i - b_i) \operatorname{Prob}\{b_i > b_j(v_j)\} + (v_i - b_i) \operatorname{Prob}\{b_i = b_j(v_j)\}/2 
\]

Consider a linear equilibrium
\( b_i(v_i) = a_i + c_i v_i \quad i=1,2 \)

Assuming player j adopts the strategy \( b_j(v_j) = a_j + c_j v_j \), player i’s best response:
\[
\max (v_i - b_i) \operatorname{Prob}\{b_i > b_j(v_j)\} = (v_i - b_i) \operatorname{Prob}\{b_i > a_j + c_j v_j\} 
\]
First-Price Sealed-Bid Auction

- Assuming player \( j \) adopts the strategy \( b_j(v_j) = a_j + c_j v_j \), player \( i \)'s best response:
  \[
  \max (v_i - b_i) \text{Prob}\{b_i > a_j + c_j v_j\}
  \text{ s.t. } b_i \leq \min\{a_j + c_j, v_i\}
  \]
  \[
  \text{Prob}\{b_i > a_j + c_j v_j\} = \text{Prob}\{v_j < (b_i - a_j)/c_j\} = (b_i - a_j)/c_j
  \]
  \[
  \max (v_i - b_i)(b_i - a_j)/c_j
  \text{ s.t. } b_i \leq \min\{a_j + c_j, v_i\}
  \]

From F.O.C.: \( b_i = v_i \), if \( v_i \leq a_j \), \( b_i = (v_i + a_j)/2 \), otherwise.

First-Price Sealed-Bid Auction

- Player \( i \)'s best response
  \( b_i = v_i \), if \( v_i \leq a_j \), \( b_i = (v_i + a_j)/2 \), otherwise

- Can \( a_j \) be
  - Between 0 and 1?
  - Greater than or equal to 1?
    - \( b_j(v_j) = a_j + c_j v_j \geq 1 \)
  - Less than or equal to zero?
    - \( b_i(v_i) = (v_i + a_j)/2 \)

We have \( a_j \leq 0, b_i = a_i + c_i v_i = a_j/2 + 1/2(v_i) \rightarrow a_i = a_j/2 c_i = 1/2 \)
First-Price Sealed-Bid Auction

- Player i’s best response
  \[ a_i \leq 0, a_i + c_i v_i = a_j / 2 + 1/2(v_i) \rightarrow a_i = a_j / 2, c_i = 1/2 \]

- Player j’s best response
  \[ a_j \leq 0, a_j + c_j v_j = a_i / 2 + 1/2(v_j) \rightarrow a_j = a_i / 2, c_j = 1/2 \]

We have
\[ a_i = a_j = 0 \text{ and } c_i = c_j = 1/2 \text{ and } b_i(v_i) = v_i / 2 i=1,2 \]

Equilibrium recap

- Static games of complete information
  - Nash equilibrium
- Dynamic games of complete information
  - Subgame-perfect Nash equilibrium
- Static games of incomplete information (Bayesian games)
  - Bayesian Nash equilibrium
- Dynamic games of incomplete information
  - Perfect Bayesian equilibrium