Recap

- Last class (February 3, 2004)
  - One-card poker
  - Cournot competition under incomplete information
- Today (February 5, 2004)
  - Review results from one-card poker game
  - Static Bayesian games (static games of incomplete information)

ISYE 6230 – Spring 2004

- Total number of games played = 230 (224)
- Average = 12.57 (10.27)
- Number of different outcomes:
  - (B,B): 143 (131)
  - (B,P): 37 (40)
  - (P,B): 48 (38)
  - (P,P): 11 (10)

Games of Incomplete Information (Bayesian Games)

- In a game of complete information the players' payoff functions are common knowledge
- In a game of incomplete information, at least one player is uncertain about another player's payoff function

Normal Form Representation of Static Bayesian Games

- The normal form representation of an n-player static Bayesian game specifies the players'
  - action spaces $A_1,\ldots,A_n$,
  - type spaces $T_1,\ldots,T_n$,
  - beliefs $p_1,\ldots,p_n$, and
  - payoff functions $\pi_1,\ldots, \pi_n$.
- Player i's *type* t_i is privately known by player i, determines player i's payoff function $\pi_i(a_1,\ldots,a_n; t_i)$ and is a member of the set of possible types.
- Player i's *belief* $p_i(t_{-i} \mid t_i)$ describes i's uncertainty about the n-1 other players' possible types t_{-i}, given i's own type t_i. We denote this game by $G=(A_1,\ldots,A_n; T_1,\ldots,T_n; p_1,\ldots,p_n; \pi_1,\ldots, \pi_n)$.

Example: Cournot Competition with incomplete information

- Actions
  - Quantity choices q_1 and q_2 .
- Type spaces
  - Firm 1: $T_1=\{c_1\}$
  - Firm 2: $T_2=\{c^H, c^L\}$
- Payoffs
  - Firm 1: $\pi_1(q_1,q^L_2, q^H_2; c_1) = p[a-(q_1+q^H_2)]q_1 + (1-p)[a-(q_1+q^L_2)]q_1 - c_1q_1$.
  - Firm 2: $\pi_1(q_1,q^H_2; c^H) = (a- q_1- q^H_2 - c^H) q_2$.
    $\pi_1(q_1,q^L_2; c^L) = (a- q_1- q^L_2 - c^L) q_2$.
- Beliefs
  - Firm 1: $p_1(c^H|c_1) = p$ $p_1(c^L|c_1) = 1-p$
  - Firm 2: $p_2(c_1|t_2) = 1$ $t_2= c^H, c^L$

Timing of a Static Bayesian Game

- Nature draws the type vector $t=(t_1,\ldots,t_n)$ according to probability distribution $p(t)$
- Nature reveals t_i to player i but not to any other player
  - Player i can compute his/her belief $p_i(t_{-i}|t_i)$ using Bayes' rule $(P(A|B)=P(A,B)/P(B))$:
    $$p_i(t_{-i} \mid t_i) = p_i(t_{-i} , t_i) / p_i(t_i)$$
    $$= p_i(t_{-i} , t_i) / \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} , t_i)$$
- The players simultaneously choose actions
- Payoffs are received

Strategies in a Static Bayesian Game

- In the static Bayesian game $G=(A_1,\ldots,A_n;\ T_1,\ldots,T_n;\ p_1,\ldots,p_n;\ \pi_1,\ldots,\ \pi_n)$ a ***strategy*** for player i is a function $s_i(t_i)$, where for each type $t_i \in T_i$, $s_i(t_i)$ specifies the action from the feasible set A_i that type i would choose if drawn by nature.
- Set of possible (pure) strategies for player i:
  $S_i: T_i \rightarrow A_i$.
- In a ***separating*** strategy, each type $t_i \in T_i$ chooses a different action from $a_i \in A_i$.
- In a ***pooling*** strategy, player i chooses the same action for each type $t_i \in T_i$.

Strategies in the Cournot Game

Firm 1's strategy: q_1

$q_1 = (a - 2c_1 + p\ c^H + (1-p)\ c^L]/3$

Firm 2's strategy: $(q_2(c^H),\ q_2(c^L))$

$q_2(c^H) = (a-2c^H+c_1)/3 + (1-p)(c^H - c^L)/6 = q^H_2$

$q_2(c^L) = (a-2c^L+c_1)/3 - p(c^H - c^L)/6 = q^L_2$

Bayesian Nash Equilibrium

- In the static Bayesian game $G=(A_1,\ldots,A_n;\ T_1,\ldots,T_n;\ p_1,\ldots,p_n;\ \pi_1,\ldots,\ \pi_n)$, the strategies $s^*=(s^*_1,\ldots,\ s^*_n)$ are a (pure strategy) Bayesian Nash Equilibrium if for each player i and for each of i's types $t_i \in T_i$, $s^*_i(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} \pi_i(s_1^*(t_1),\ldots,s_{i-1}^*(t_{i-1}),a_i,s_{i+1}^*(t_{i+1}),\ldots,s_n^*(t_n);t_i)p_i(t_{-i} \mid t_i)$$

That is, no player wants to change his/her strategy, even if the change involves only one action by one type.

Example: Battle of the sexes

Bob (columns), Alice (rows)

	Ballet	Football
Ballet	2, 1	0, 0
Football	0, 0	1, 2

- Pure strategy equilibria: (Ballet,Ballet) (Football,Football)
- Mixed strategy equilibrium
  Alice: Ballet with p=2/3, Football with 1-p=1/3
  Bob: Ballet with q=1/3, Football with 1-q=2/3

Example: Battle of the sexes

Bob

Alice	Ballet	Football
Ballet	$2+t_A$, 1	0, 0
Football	0, 0	1, $2+t_B$

- t_A is privately known to Alice and t_B is privately known to Bob
- t_A and t_B are independent draws from a uniform distribution on [0,x]

Example: Battle of the sexes

Bob

Alice	Ballet	Football
Ballet	$2+t_A$, 1	0, 0
Football	0, 0	1, $2+t_B$

- Action spaces
  - $A_A = A_B$ ={Ballet,Football}
- Type spaces
  - $T_A = T_B = [0,x]$
- Beliefs
  - $p_B(t_A) = p_A(t_B) = 1/x$ for all t_A and t_B
- Bayesian game: $G=\{A_A, A_B; T_A,T_B; p_A, p_B; \pi_A, \pi_B\}$

Example: Battle of the sexes

Bob (columns), Alice (rows)

	Ballet	Football
Ballet	$\underline{2+t_A}$, $\underline{1}$	0, 0
Football	0, 0	$\underline{1}$, $\underline{2+t_B}$

- Strategies
  - Alice: Play ballet if t_A exceeds a critical value c_A; otherwise, play football
  - Bob: Play football if t_B exceeds a critical value c_B; otherwise, play ballet

Example: Battle of the sexes

Bob (columns), Alice (rows)

	Ballet	Football
Ballet	$\underline{2+t_A}$, $\underline{1}$	0, 0
Football	0, 0	$\underline{1}$, $\underline{2+t_B}$

- Alice's expected payoffs
  - Alice plays Ballet

    $P(t_B<c_B)(2+t_A) + P(t_B>c_B)0 =$

    $(c_B/x)(2+t_A) + (1- c_B/x)0 = (c_B/x)(2+t_A)$
  - Alice plays Football

    $P(t_B<c_B)0 + P(t_B>c_B)1 = 1- c_B/x$
  - Play ballet if $(c_B/x)(2+t_A)> 1- c_B/x \rightarrow$ $t_A>x/c_B-3 = c_A$

Example: Battle of the sexes

Bob (columns), Alice (rows)

	Ballet	Football
Ballet	$\underline{2+t_A}$, $\underline{1}$	0, 0
Football	0, 0	$\underline{1}$, $\underline{2+t_B}$

- Bob's expected payoffs
  - Bob plays Ballet
    $P(t_A>c_A)1 + P(t_A<c_A)0 =$
    $(1-c_A/x)1 + (c_A/x)0 = 1-c_A/x$
  - Bob plays Football
    $P(t_A>c_A)0 + P(t_A<c_A)(2+ t_B) = (c_A/x)(2+ t_B)$
  - Play football if $(c_A/x)(2+t_B)> 1- c_A/x \rightarrow t_B>x/c_A-3 = c_B$

Example: Battle of the sexes

- Alice's expected payoffs
  - Alice plays Ballet: $(c_B/x)(2+t_A)$
  - Alice plays Football: $1- c_B/x$
  - Play ballet if $(c_B/x)(2+t_A)> 1- c_B/x \rightarrow t_A>x/c_B-3 = c_A$
- Bob's expected payoffs
  - Bob plays Ballet: $1-c_A/x$
  - Bob plays Football: $(c_A/x)(2+ t_B)$
  - Play football if $(c_A/x)(2+t_B)> 1- c_A/x \rightarrow t_B>x/c_A-3 = c_B$

Example: Battle of the sexes

- $t_A > x/c_B - 3 = c_A$
- $t_B > x/c_A - 3 = c_B$
- $c_A = c_B = c$
- $c^2+3c-x=0$
  $c = (-3+\sqrt{(9+4x)})/2$
- What is the probability that Alice plays ballet?
  $1- c_A/x = 1 - [-3+\sqrt{(9+4x)}]/2x$
- What is the probability that Bob plays football?
  $1- c_B/x = 1 - [-3+\sqrt{(9+4x)}]/2x$

In the limit ($x \rightarrow 0$), these probabilities approach 2/3!

First-Price Sealed-Bid Auction

- Two bidders, one good
- Bidder i's valuation for the good is v_i, is known only by bidder i. Valuations are independently and uniformly distributed on [0,1].
- Each bidder i submits a nonnegative bid b_i. The higher bidder wins and pays his bid. Other bidder pays and receives nothing.
- In case of a tie, the winner is determined by a coin flip
- Bidder i's payoff, if wins and pays p, is v_i- p
- Bidders are risk-neutral
- All of this information is common knowledge

First-Price Sealed-Bid Auction

- Action spaces
  - $A_1 = A_2 = [0,\infty)$
- Type spaces
  - $T_1 = T_2 = [0,1]$
- Beliefs
  - $p_1(t_2|t_1) = p_1(t_2)$
  - $p_2(t_1| t_2) = p_2(t_1)$
- Player i's (expected) payoff function

$$\pi_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i \quad , & \text{if } b_i > b_j \\ (v_i - b_i)/2, & \text{if } b_i = b_j \\ 0, & \text{if } b_i < b_j \end{cases}$$

First-Price Sealed-Bid Auction

- Strategy for player i: $b_i(v_i)$
- Strategies $(b_1(v_1), b_2(v_2))$ are a Bayesian Nash equilibrium if for each v_i in [0,1], $b_i(v_i)$ solves

$\max\ (v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} + (v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\}/2$

- Consider a linear equilibrium

  $b_i(v_i) = a_i + c_i v_i \quad i=1,2$
- Assuming player j adopts the strategy $b_j(v_j) = a_j + c_j v_j$, player i's best response:

$\max\ (v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} = (v_i - b_i)\text{Prob}\{b_i > a_j + c_j v_j\}$

First-Price Sealed-Bid Auction

- Assuming player j adopts the strategy $b_j(v_j) = a_j+c_jv_j$, player i's best response:

  max $(v_i- b_i)\text{Prob}\{b_i> a_j+c_jv_j\}$

  s.t. $b_i \le \min\{a_j+c_j , v_i\}$

$\text{Prob}\{b_i> a_j+c_jv_j\} = \text{Prob}\{v_j<(b_i-a_j)/c_j\} = (b_i-a_j)/c_j$

  max $(v_i- b_i)(b_i-a_j)/c_j$

  s.t. $b_i \le \min\{a_j+c_j , v_i\}$

From F.O.C.: $b_i=v_i$, if $v_i\le a_j$, $b_i= (v_i+a_j)/2$, otherwise

First-Price Sealed-Bid Auction

- Player i's best response

  $b_i= v_i$, if $v_i\le a_j$, $b_i= (v_i+a_j)/2$, otherwise

- Can a_j be
  - Between 0 and 1?
  - Greater than or equal to 1?
    - $b_j(v_j) = a_j + c_jv_j \ge 1$
  - Less than or equal to zero?
    - $b_i(v_i)= (v_i+a_j)/2$

We have $a_j\le 0$, $b_i = a_i+c_iv_i = a_j/2+1/2(v_i) \rightarrow$

$a_i= a_j/2$ $c_i=1/2$

First-Price Sealed-Bid Auction

- Player i’s best response
  $a_j \leq 0$, $a_i + c_i v_i = a_j/2 + 1/2(v_i) \rightarrow a_i = a_j/2 \; c_i = 1/2$
- Player j’s best response
  $a_i \leq 0$, $a_j + c_j v_j = a_i/2 + 1/2(v_j) \rightarrow a_j = a_i/2 \; c_j = 1/2$

We have

$a_i = a_j = 0$ and $c_i = c_j = 1/2$ and $b_i(v_i) = v_i/2 \; i=1,2$

Equilibrium recap

- Static games of complete information
  - Nash equilibrium
- Dynamic games of complete information
  - Subgame-perfect Nash equilibrium
- Static games of incomplete information (Bayesian games)
  - Bayesian Nash equilibrium
- Dynamic games of incomplete information
  - Perfect Bayesian equilibrium