QUANTITY FLEXIBILITY - (w, δ)

Supplier sell at w per unit. Demand is distributed with cdf $F(x)$ and pdf $f(x)$. Supplier's unit cost is c_S, retailer's unit cost is c_R, and $c = c_S + c_R$. Unit selling price is p.

Centralized supply chain (CSC):

$$\begin{aligned}\Pi(q) &= p \cdot S(q) - cq \\ &= p\left(q - \int_0^q F(x)dx\right) - cq\end{aligned}$$

$$\frac{\partial \Pi(q)}{\partial q} = p - p \cdot F(q) - c = p(1 - F(q)) - c = 0 \Rightarrow \boxed{c = p(1 - F(q))} \tag{1}$$

Quantity Flexibility (QF): Refund the retailer $w + c_R$ for units unsold up to

$$\min\{I, \delta q\}$$

$S(q)$: expected number of units sold by the retailer.

$$\begin{aligned}D \geq q &\Rightarrow \text{ sell } q \\ D < q &\Rightarrow \text{ sell } D\end{aligned}$$

$$S(q) = \underbrace{(1 - F(q))}_{Pr(D \geq q)} \cdot q + \int_0^q xf(x)dx \tag{2}$$

Let us compute the term $\int_0^q xf(x)dx$.

$$\frac{d}{dx} xF(x) = F(x) + xf(x)$$

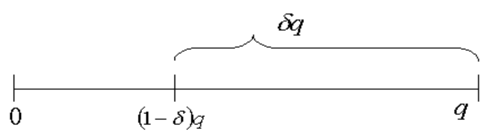
$$\int_0^q \left. \frac{d}{dx} xF(x) \right] dx = q \cdot F(q) = \int_0^q F(x)dx + \int_0^q xf(x)dx$$

$$\Rightarrow \int_0^q xf(x)dx = q \cdot F(q) - \int_0^q F(x)dx \tag{3}$$

From (2) and (3):

$$S(q) = q - \int_0^q F(x)dx \tag{4}$$

What is the retailer's expected compensation?



$$\begin{aligned} D < (1-\delta)q \Rightarrow &\ \text{leftover inventory } > \delta q \Rightarrow \text{ compensated for } \delta q \\ (1-\delta)q < D < q \Rightarrow &\ \text{leftover inventory } < \delta q \Rightarrow \text{ compensated for } q - D \\ D > q \Rightarrow &\ \text{no inventory left.} \end{aligned}$$

Expected # of units for which the retailer gets compensation:

$$I_R = F((1-\delta)q)\delta q + \int_{(1-\delta)q}^q (q-x)f(x)dx$$

Since

$$\int_{(1-\delta)q}^q (q-x)f(x)dx = q\int_{(1-\delta)q}^q f(x)dx - \int_{(1-\delta)q}^q xf(x)dx$$

and

$$q\int_{(1-\delta)q}^q f(x)dx = q\left[F(q) - F((1-\delta)q)\right]$$

we have

$$I_R = F((1-\delta)q)\delta q + qF(q) - qF((1-\delta)q) - \int_{(1-\delta)q}^q xf(x)dx$$

Let's compute $\int_{(1-\delta)q}^{q} xf(x)dx$.

$$\int_{(1-\delta)q}^{q} xf(x)dx = \int_{0}^{q} xf(x)dx - \int_{0}^{(1-\delta)q} xf(x)dx$$

$$= qF(q) - \int_{0}^{q} F(x)dx - \left[(1-\delta)qF((1-\delta)q) - \int_{0}^{(1-\delta)q} F(x)dx\right]$$

$$= qF(q) - (1-\delta)qF((1-\delta)q) - \int_{(1-\delta)q}^{q} F(x)dx \tag{5}$$

Hence, inserting $\int_{(1-\delta)q}^{q} xf(x)dx$ from Equation 5 into I_R, we get

$$I_R = F((1-\delta)q)\delta q + qF(q) - qF((1-\delta)q) - qF(q) + (1-\delta)qF((1-\delta)q) + \int_{(1-\delta)q}^{q} F(x)dx$$

$$\Rightarrow \quad I_R = \int_{(1-\delta)q}^{q} F(x)dx. \tag{6}$$

Retailer's profit function:

$$\Pi_R(q) = p \cdot S(q) - (w + c_R)q + (w + c_R)\int_{(1-\delta)q}^{q} F(x)dx$$

$$\frac{\partial \Pi_R(q)}{\partial q} = p \cdot \frac{\partial S(q)}{\partial q} - (w + c_R) + (w + c_R)(F(q) - F((1-\delta)q)(1-\delta))$$

$$= p(1 - F(q^*)) - (w + c_R)\left[1 - F(q) + F((1-\delta)q)(1-\delta)\right] = 0$$

$$p(1 - F(q)) = (w + c_R)\left[1 - F(q) + F((1-\delta)q)(1-\delta)\right]$$

For q^* to be optimal for the retailer, it needs to satisfy the retailers FOC, i.e.,

$$\Rightarrow w = \underbrace{\frac{p(1 - F(q^*))}{[1 - F(q^*) + F((1-\delta)q^*)(1-\delta)]}}_{\Delta} - c_R$$

Can we claim that if the supplier chooses w in this way, the retailer will choose q^*? This is the case only <u>if</u> the retailer's profit function is concave.

$$\frac{\partial^2 \Pi_R(q)}{\partial q^2} = -p \cdot f(q) + (w + c_R) f(q) - (w + c_R)(1-\delta)^2 f((1-\delta)q)$$

$$\frac{\partial F((1-\delta)q)}{\partial q} = f((1-\delta)q) \cdot (1-\delta)$$

$$\Rightarrow \quad \frac{\partial^2 \Pi_R(q)}{\partial q^2} = -\underbrace{(p - (w + c_R))}_{\geq 0} f(q) - \underbrace{(w + c_R)(1-\delta)^2 f((1-\delta)q)}_{>0} < 0$$

$$\text{as long as} \quad \begin{array}{ll} w + c_R \leq p, \ \text{i.e.,} & w \leq p - c_R \\ w + c_R \geq 0, \ \text{i.e.,} & w \geq -c_R \end{array}$$

What do you think happens to retailer's profit as $\delta \uparrow$? As $\delta \uparrow$, the retailer gets protection for more units but at a higher cost!

Supplier's profit:

$$\Pi_s = (w - c_s)q - (w + c_R) \int_{(1-\delta)q}^{q} F(x)dx$$

$$\delta = 0 \Rightarrow w = \frac{p(1 - F(q))}{1 - F(q) + F(q)} - c_R = p(1 - F(q)) - c_R$$

From equation 1 we have $p(1 - F(q^*)) = c$, i.e., if $\delta = 0$, we have $w = c_S$, $\Pi_S = 0$ and $\Pi_R = \Pi$.

<u>Compliance</u>

If the retailer orders q, he will certainly not accept more than q units from the supplier. However, what if the supplier provides less than q units? This might happen due to shortages or other problems at the supplier, or the supplier might simply find it more profitable to deliver a different quantity given the contractual terms and her profit maximization objective. In *voluntary compliance*, the supplier delivers an amount (not to exceed the retailer's order quantity) to maximize her profits. In *forced compliance*, the supplier delivers exactly the amount ordered by the retailer. Is there a difference in quantities delivered under these two regimes, say, if the retailer does not order the "correct" quantity?

Look at the supplier's FOC. Is q^* the best response for the supplier?

$$\begin{aligned} \tfrac{\partial \Pi_S(q)}{\partial q} &= (w - c_s) - (w + c_R)\left[F(q) - F((1-\delta)q)(1-\delta)\right] \leftarrow \text{ add and subtract } (w + c_R) \\ &= (w - c_s) + (w + c_R)\underbrace{\left[1 - F(q) + F((1-\delta)q)(1-\delta)\right]}_{\Delta} - (w + c_R) \end{aligned}$$

Recall $w = \frac{p(1-F(q))}{\Delta} - c_R \Rightarrow \Delta = \frac{p(1-F(q))}{w+c_R}$

$$\Rightarrow \frac{\partial \Pi_s}{\partial q} = -(c_s + c_R) + p(1 - F(q)) = -c + p(1 - F(q)) = 0$$

$\Rightarrow c = p(1 - F(q))$ same as the FOC of CSC!

Does this mean that q^* is optimal/best response for the supplier? Check 2^{nd} order conditions at q^*.

$$\frac{\partial^2 \Pi_S(q)}{\partial q^2} = -(w + c_R)\left[f(q) - (1-\delta)^2 f((1-\delta)q)\right]$$

If $(1-\delta)^2 f((1-\delta)q) > f(q)$ then $\frac{\partial^2 \Pi_s}{\partial q^2} > 0$, i.e., the function is concave and q^* is a minimizer! This will hold if δ is small or $f(q)$ is small.

Hence, under voluntary compliance, even if the wholesale price is $w(\delta)$, coordination may not be achieved, depending on δ and $f(q)$. Clearly it is achieved under forced compliance.