Recap

- Last Tuesday (February 4, 2003)
  - Subgames in extensive form games
  - Games of incomplete information
    - Cournot competition under incomplete information
    - Battle of the sexes under incomplete information
- Today (February 6, 2003)
  - Battle of the sexes under incomplete information
  - First-price sealed-bid auctions
  - Equilibrium recap
  - One-card poker game

Example: Battle of the sexes

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
</tr>
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<tbody>
<tr>
<td>Ballet</td>
<td>2+t_A 1</td>
</tr>
<tr>
<td>Football</td>
<td>0 0</td>
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- t_A is privately known to Alice and t_B is privately known to Bob
- t_A and t_B are independent draws from a uniform distribution on [0,x]

Strategies

- Alice: Play ballet if t_A exceeds a critical value c_A; otherwise, play football
- Bob: Play football if t_B exceeds a critical value c_B; otherwise, play ballet

Alice's expected payoffs

- Alice plays Ballet: \((c_B/x)(2+t_A)\)
- Alice plays Football: \(1- c_B/x\)

Bob's expected payoffs

- Bob plays Ballet: \(1-c_A/x\)
- Bob plays Football: \((c_A/x)(2+ t_B)\)

Example: Battle of the sexes

- What is the probability that Alice plays ballet?
  \(-1- c_A/x = 1 - [-3+\sqrt{(9+4x)}]/2x\)
- What is the probability that Bob plays football?
  \(-1- c_B/x = 1 - [-3+\sqrt{(9+4x)}]/2x\)

In the limit \(x\to0\), these probabilities approach 2/3!

Example: Battle of the sexes

First-Price Sealed-Bid Auction

- Two bidders, one good
- Bidder i's valuation for the good is v_i, is known only by bidder i. Valuations are independently and uniformly distributed on \([0,1]\).
- Each bidder i submits a nonnegative bid b_i. The higher bidder wins and pays his bid. Other bidder pays nothing.
  - In case of a tie, the winner is determined by a coin flip
  - Bidder i's payoff, if wins and pays p, is v_i - p
  - Bidders are risk-neutral
  - All of this information is common knowledge
First-Price Sealed-Bid Auction

- Action spaces
  - $A_1 = A_2 = [0, \infty)$
- Type spaces
  - $T_1 = T_2 = [0, 1]$
- Beliefs
  - $p_1(t_2 | t_1) = p_1(t_2)$
  - $p_2(t_1 | t_2) = p_2(t_2)$
- Player $i$'s (expected) payoff function
  $$\sigma_i(b_1, b_2; v_i, v_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

Strategy for player $i$: $b_i(v_i)$
Strategies $(b_1(v_1), b_2(v_2))$ are a Bayesian Nash equilibrium if for each $v_i$ in $[0, 1]$, $b_i(v_i)$ solves
$$\max (v_i - b_i) \text{Prob}(b_i > b_j(v_j)) + (v_i - b_i) \text{Prob}(b_i = b_j(v_j))/2$$

Consider a linear equilibrium
$$b_i(v_i) = a_i + c_i v_i \quad i = 1, 2$$
Assuming player $j$ adopts the strategy $b_j(v_j) = a_j + c_j v_j$
player $i$'s best response:
$$\max (v_i - b_i) \text{Prob}(b_i > b_j(v_j)) = (v_i - b_i) \text{Prob}(b_i > a_j + c_j v_j)$$

Maximize
$$\max (v_i - b_i)(b_i - a_j)/c_j$$
subject to $b_i \leq \min(a_i + c_i, v_i)$
From F.O.C.: $b_i = v_i$ if $v_i \leq a_j$, $b_i = (v_i + a_j)/2$, otherwise

Player $i$'s best response
$$b_i = v_i \text{ if } v_i \leq a_j \text{, else } (v_i + a_j)/2$$
Can $a_i$ be
- Between 0 and 1?
- Greater than or equal to 1?
- Less than or equal to zero?
- $b_i(v_i) = a_i + c_i v_i$

We have $a_i = a_j = 0$, $c_i = c_j = 1/2$ and $b_i(v_i) = v_i/2$ i=1,2

Equilibrium recap

- Static games of complete information
  - Nash equilibrium
- Dynamic games of complete information
  - Subgame-perfect Nash equilibrium
- Static games of incomplete information (Bayesian games)
  - Bayesian Nash equilibrium
- Dynamic games of incomplete information
  - Perfect Bayesian equilibrium
Example: One-card poker

- Two players
- One deck of cards, half aces, half kings
- Pay $a to play
- Each player is dealt a card face down
- After seeing his/her card, each player (simultaneously)
  - Action B: bets b, or
  - Action P: passes
- Payoffs
  - (B,P) or (P,B) → betting player gets the pot
  - (B,B) or (P,P) → higher card gets the pot; in case of a tie, the pot is split

Record the results

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
<th>Payoffs</th>
</tr>
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<tbody>
<tr>
<td>Game 1</td>
<td>Ace-Bet</td>
<td>King-Bet</td>
<td>(a+b, -(a+b))</td>
</tr>
<tr>
<td>Game 2</td>
<td>King-Pass</td>
<td>King-Bet</td>
<td>(-a, a)</td>
</tr>
<tr>
<td>Game 3</td>
<td>King-Pass</td>
<td>King-Pass</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>etc</td>
<td></td>
<td></td>
<td></td>
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