

Recap

- Last Thursday
 - Extensive form of a game
 - Classroom exercise – find equilibrium
 - Two-stage Prisoner's Dilemma
 - Repeated games
 - Finitely repeated games
 - Infinitely repeated games
- Today: Continue with Infinitely repeated games
 - Prisoner's dilemma – trigger strategies

Infinitely Repeated Prisoner's Dilemma

		Prisoner 2	
		C (cooperate)	D (defect)
Prisoner 1	C	4, 4	0, 5
	D	5, 0	1, 1

- Strategy:
 - Play C in the first stage. In the t^{th} stage, if the outcome of all $t-1$ preceding stages has been (C,C), then play C; otherwise, play D

Definitions

- In an infinitely repeated game $G(\infty, \delta)$, a player's *strategy* specifies the player's actions in each stage, for each possible history of play through the previous stages.
- In the infinitely repeated game $G(\infty, \delta)$, each *subgame* beginning at stage $t+1$ is identical to the original game $G(\infty, \delta)$.

Trigger strategies for Prisoner's Dilemma

- Assuming player 1 adopts the trigger strategy, what is the best response of player 2?

Player 2 best response in stage $t+1$:

- If the outcome in stage t is (D,D)
 - Play D forever
- If the outcomes of stages $1, \dots, t$ are (C,C)
 - Play D \rightarrow receive 5 in this stage, switch to (D,D) forever after $\rightarrow 5 + \delta \cdot 1 + \delta^2 \cdot 1 + \delta^3 \cdot 1 + \dots = 5 + \delta / (1 - \delta)$
 - Play C \rightarrow receive 4 in this stage, and face the exact same game (same choices) in stage $t+2$!

Trigger strategies for Prisoner's Dilemma

- Let V be the payoff of player 2 from making the optimal choice in the subgame starting in stage $t+1$, given that the outcomes in the previous stages have been (C,C)
- Play C $\rightarrow V = 4 + \delta V \rightarrow V = 4 / (1 - \delta)$
- Play D $\rightarrow V = 5 + \delta / (1 - \delta)$

Play C if $4 / (1 - \delta) \geq 5 + \delta / (1 - \delta) \rightarrow$ if $\delta \geq 1/4$

Trigger strategies for Prisoner's Dilemma

- Two types of subgames:
 - (i) Subgames where the outcomes of all previous stages have been (C,C)
 - The trigger strategies are Nash equilibrium for this class of subgames, as well as for the original game.
 - (ii) Subgames where the outcome of at least one earlier stage differs from (C,C)
 - Player's strategies are to repeat (D,D) forever, which is also a Nash equilibrium for the original game

Observation

- Even if the stage game G has a unique Nash equilibrium, there may be subgame-perfect outcomes of the infinitely repeated game in which no stage's outcome is a Nash equilibrium of G .

Feasible payoffs in the stage game

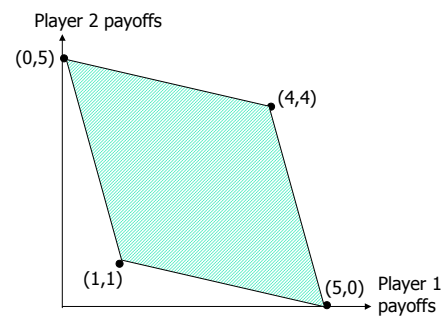
- The payoffs $(\pi^1, \pi^2, \dots, \pi^n)$ are *feasible* in the stage game G if they are a convex combination of the pure-strategy payoffs of G .

Example

		Prisoner 2	
		C (cooperate)	D (defect)
Prisoner 1	C	4, 4	0, 5
	D	5, 0	1, 1

- What are the pure-strategy payoffs?
 - $(4,4)$ $(0,5)$ $(5,0)$ $(1,1)$

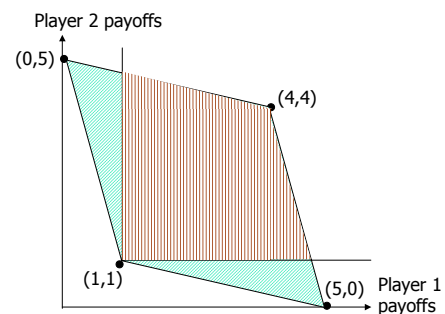
Feasible payoffs in the Prisoner's Dilemma



Friedman's Theorem

- Let G be a finite static game of complete information. Let (e^1, e^2, \dots, e^n) denote the payoffs from a Nash equilibrium of G and let (x^1, x^2, \dots, x^n) denote any other feasible payoffs from G . If $x^j > e^j$ for every player j and if δ is sufficiently close to 1, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves (x^1, x^2, \dots, x^n) as the average payoff.

Feasible payoffs in the Prisoner's Dilemma



Proof of Friedman's Theorem

- Let $(a_{e1}, a_{e2}, \dots, a_{en})$ be the Nash equilibrium of G that yields the equilibrium payoffs (e^1, e^2, \dots, e^n) .
- Let $(a_{x1}, a_{x2}, \dots, a_{xn})$ be the collection of actions that yields the equilibrium payoffs (x^1, x^2, \dots, x^n) .
- Trigger strategy for player i :
 - Play a_{xi} in the first stage. In the t^{th} stage, if the outcome of all $t-1$ preceding stages has been $(a_{x1}, a_{x2}, \dots, a_{xn})$ then play a_{xi} ; otherwise, play a_{ei} .
- Show that the trigger strategies induce a NE
- Show that the equilibrium is subgame perfect

Proof of Friedman's Theorem (cont.)

- Suppose all other players other than player i use the trigger strategy.
- Best response of player i in stage t :
 - If the outcome of the previous stage differs from $(a_{x1}, a_{x2}, \dots, a_{xn})$
 - Play a_{ei} forever
 - If the outcomes of all previous stages are $(a_{x1}, a_{x2}, \dots, a_{xn})$

$$\max_{a_i \in A_i} \pi^i(a_{x1}, \dots, a_{x,i-1}, a_i, a_{x,i+1}, \dots, a_{xn}) = d^i$$

$$d^i \geq \pi^i(a_{x1}, \dots, a_{x,i-1}, a_{xi}, a_{x,i+1}, \dots, a_{xn}) > \pi^i(a_{e1}, \dots, a_{en}) = e^i$$

Proof of Friedman's Theorem (cont.)

- If player i deviates in stage t by choosing a_{di} :
 - Payoff in stage t : d^i
 - Payoff in future stages:
 - $\delta e^i + \delta^2 e^i + \dots = \delta e^i / (1 - \delta)$
 - Total (discounted) payoff: $V^i = d^i + \delta e^i / (1 - \delta)$
- If player i plays a_{xi} in stage t :
 - Receive a payoff x^i in this stage, face the same game in the next stage.

$$V^i = x^i + \delta V^i \rightarrow V^i = x^i / (1 - \delta).$$
- Playing x^i is optimal iff

$$x^i / (1 - \delta) \geq d^i + \delta e^i / (1 - \delta) \rightarrow \delta \geq (d^i - x^i) / (d^i - e^i)$$

Proof of Friedman's Theorem (cont.)

- It is Nash equilibrium for all players to play the trigger strategy if and only if

$$\delta \geq \max_i (d^i - x^i) / (d^i - e^i)$$
- Subgame perfectness:
 - If the outcome of the previous stage differs from $(a_{x1}, a_{x2}, \dots, a_{xn})$
 - Play a_{ei} forever
 - If the outcomes of all previous stages are $(a_{x1}, a_{x2}, \dots, a_{xn})$
 - Play the trigger strategy

Repeated Cournot Game

- Cournot stage game
 - Two competing firms, selling a homogeneous good
 - The *marginal cost* of producing each unit of the good: c
 - The market price, P is determined by (inverse) market demand:
 - $P = a - Q$ if $a > Q$, $P = 0$ otherwise.
 - Each firm decides on the quantity to sell (market share): q_1 and q_2
 - $Q = q_1 + q_2$ total market demand
 - Both firms seek to maximize profits
- Unique NE of the stage game: $q^c = (a - c) / 3$ $Q = 2(a - c) / 3$
- Monopoly quantity: $q^M = (a - c) / 2$

Repeated Cournot Game (cont.)

- The stage game is repeated infinitely many times
- The firms have discount factor δ
- Trigger strategy
 - Produce half the monopoly quantity, $q^M / 2$, in the first stage. In the t^{th} stage, produce $q^M / 2$ if both firms have produced $q^M / 2$ in all previous stages; otherwise, produce q^c .
- Show that the trigger strategy induces a subgame perfect NE.

Repeated Cournot Game (cont.)

- Profit of one firm
 - If both produce $q^M/2$: $(a-c)^2/8 = \pi^M/2$
 - If both produce q^C : $(a-c)^2/9 = \pi^C$
- Best response of firm i:
 - If the last stage outcome is other than $(q^M/2, q^M/2)$
 - Play q^C forever
 - If all previous stages' outcomes are $(q^M/2, q^M/2)$
 - Deviate
 - $\max (a-q_i-q^M/2-c) q_i \rightarrow$
 - $q_i = 3(a-c)/8 \quad \pi^D = 9(a-c)^2/64$
 - $V^i = \pi^D + \pi^C \delta / (1-\delta)$
 - Play $q^M/2$
 - $V^i = \pi^M/2 + \delta V^i \rightarrow V^i = \pi^M / 2(1-\delta)$

Repeated Cournot Game (cont.)

- Profit of one firm
 - If both produce $q^M/2$: $(a-c)^2/8 = \pi^M/2$
 - If both produce q^C : $(a-c)^2/9 = \pi^C$
- Best response of firm i:
 - If the last stage outcome is other than $(q^M/2, q^M/2)$
 - Play q^C forever
 - If all previous stages' outcomes are $(q^M/2, q^M/2)$
 - Deviate: $V^i = \pi^D + \pi^C \delta / (1-\delta)$
 - Play $q^M/2$: $V^i = \pi^M / 2(1-\delta)$
 - Playing the trigger strategy is NE iff
 - $\pi^M / 2(1-\delta) \geq \pi^D + \pi^C \delta / (1-\delta) \rightarrow \delta \geq 9/17$

Repeated Cournot Game (cont.)

- What if $\delta < 9/17$?
- Trigger strategy
 - Produce q^* , in the first stage. In the t^{th} stage, produce q^* if both firms have produced q^* in all previous stages; otherwise, produce q^C .
- Profit of one firm
 - If both produce q^* : $(a-2q^*-c) q^* = \pi^*$
 - If both produce q^C : $(a-c)^2/9 = \pi^C$
 - If firm j produces q^* and firm i deviates:
 - $\max (a-q_i-q^*-c) q_i \rightarrow q_i = (a-q^*-c)/2$
 - $\pi^D = (a-q^*-c)^2/4$

Repeated Cournot Game (cont.)

- Best response of firm i:
 - If the last stage outcome is other than (q^*, q^*)
 - Play q^C forever
 - If all previous stages' outcomes are (q^*, q^*)
 - Deviate: $V^i = \pi^D + \pi^C \delta / (1-\delta)$
 - Play q^* : $V^i = \pi^* + \delta V^i \rightarrow V^i = \pi^* / (1-\delta)$
 - Playing the trigger strategy is NE iff
 - $\pi^* / (1-\delta) \geq \pi^D + \pi^C \delta / (1-\delta)$
 - Substitute and solve for q^* :
 - $q^* = (9-5\delta)(a-c)/3(9-\delta)$