Recap

- Last class (February 18, 2003)
  - Principal-agent problem
    - General model
    - Optimality conditions
    - Two actions-two outcomes graphical solution
- Today (February 20, 2003)
  - Optimality conditions for two actions and two outcomes
  - Sensitivity analysis

Principal's problem: Step 1

For a given action a_i :

$$\max \sum_{j=1}^{m} P_{ij}(x_j - w_j)$$

$$\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) \geq \underline{U} \quad \ldots\ldots \quad \mu$$

$$\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) \geq \sum_{j=1}^{m} P_{kj}U(w_j) - e(a_k) \quad \forall k \neq i \ \ldots. \quad \lambda_k$$

$$L(w,\lambda,\mu) = \sum_{j=1}^{m} P_{ij}(x_j - w_j) + \mu\left(\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) - \underline{U}\right) +$$

$$\sum_{k\neq i} \lambda_k \left(\sum_{j=1}^{m} P_{ij}U(w_j) - e(a_i) - \sum_{j=1}^{m} P_{kj}U(w_j) + e(a_k)\right)$$

Optimality conditions

$$\frac{1}{U'(w_j)} = \mu + \sum_{k\neq i} \lambda_k \left(1 - \frac{P_{kj}}{P_{ij}}\right)$$

$\dfrac{P_{kj}}{P_{ij}}$: likelihood ratio

Two actions and two outcomes

- Suppose the agent has two possible actions, a and b, and there are two possible outcomes, x_1 and x_2.
- Suppose that action b is preferred by the principal

For action b :

$$\max \sum_{j=1}^{2} P_{bj}(x_j - w_j)$$

$$P_{b1}U(w_1) + P_{b2}U(w_2) - e(b) \geq \underline{U} \quad \ldots\ldots \quad \mu$$

$$P_{b1}U(w_1) + P_{b2}U(w_2) - e(b) \geq P_{a1}U(w_1) + P_{a2}U(w_2) - e(a) \quad \ldots. \quad \lambda$$

Two actions and two outcomes

$$\frac{1}{U'(w_j)} = \mu + \lambda\left(1 - \frac{P_{aj}}{P_{bj}}\right) \qquad \frac{P_{aj}}{P_{bj}} : \text{likelihood ratio}$$

<u>If the incentive constraint is not binding :</u>

$\lambda = 0$, $U'(w_j) = 1/\mu$ and $w_j = w$.

Substituting into the incentive constraint (the agent prefers action b over action a) :

$$\sum_{j=1}^{n} P_{bj}U(w) - e(b) \geq \sum_{j=1}^{n} P_{aj}U(w) - e(a) \rightarrow e(a) \geq e(b)$$

Two actions and two outcomes

$$\frac{1}{U'(w_j)} = \mu + \lambda\left(1 - \frac{P_{aj}}{P_{bj}}\right) \qquad \frac{P_{aj}}{P_{bj}} : \text{likelihood ratio}$$

<u>If the incentive constraint is binding :</u>

$\lambda > 0$, and $U'(w_j)$ depends on the likelihood ratio (LR).

The optimal incentive scheme is a linear function of LR.

Sensitivity analysis: Changes in the agent's costs

- What is the impact of the agent's costs on the outcome?

$$L(w,\lambda,\mu)=P_{b1}(x_1-w_1)+P_{b2}(x_2-w_2)+$$
$$\mu(P_{b1}U(w_1)+P_{b2}U(w_2)-e(b)-\underline{U})+$$
$$\lambda(P_{b1}U(w_1)+P_{b2}U(w_2)-e(b)-P_{a1}U(w_1)-P_{a2}U(w_2)+e(a))$$

$$\frac{\partial L(w,\lambda,\mu)}{\partial e(b)}=-(\mu+\lambda) \qquad \frac{\partial L(w,\lambda,\mu)}{\partial e(a)}=\lambda$$

Sensitivity analysis: Changes in the agent's costs

- What is the impact of the agent's costs on the outcome?

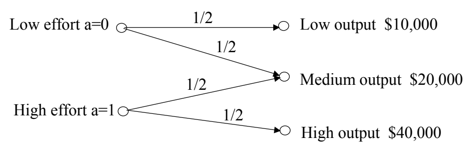
$$\frac{\partial L(w,\lambda,\mu)}{\partial e(b)}=-(\mu+\lambda) \qquad \frac{\partial L(w,\lambda,\mu)}{\partial e(a)}=\lambda$$

- "Carrot": Decrease the cost of the desired action, b
- "Stick": Increase the cost of the undesired action, a
- Carrot vs. stick, which one is better?

Recap

- Last class (February 20, 2003)
  - Optimality conditions for two actions and two outcomes
  - Sensitivity analysis
  - Example
- Today (February 25, 2003)
  - More examples

Example



- Cost of effort to the agent 10,000a
- Agent's reservation utility $\underline{U}$=0
- Agent's expected payoff: E[wage – cost of effort]
- Principal's expected payoff: E[output – wage]

Example (cont)

- Under low effort
  - Agent's cost=0 → Offer w=$\underline{U}$=0 to the agent
  - Principal's expected payoff = 0.5(10,000)+0.5(20,000)-0=15,000
- Under high effort
  - Principal's expected revenue = 0.5(20,000)+0.5(40,000)=30,000
  - Principal's expected payoff = E[revenue – wage]
  - For participation: E[wage] ≥ $\underline{U}$+cost of effort=10,000
- For the principal to prefer high effort over low E[wage]<=15,000

Example: Bonus payment

- Principal's proposed contract
  - If the output is low or medium, w=0
  - If the output is high, w=24,000
- Participation constraint for high effort

0.5(0)+0.5(24,000)-10,000=2,000>0

- Incentive constraint

0.5(0)+0.5(24,000)-10,000=2,000>0.5(0)+0.5(0)=0

- Principal's payoff: 18,000 Agent's payoff: 2,000

- What is the best w that maximizes the principal's payoff?

0.5(0)+0.5w-10,000≥0 → w=20,000

Principal's payoff: 20,000 Agent's payoff: 0

Example: Revenue sharing

- Principal's proposed contract
  - If the output is less than x=18,000, w=0
  - If the output R ≥ x, w=R-x
- Participation constraint

  0.5(20,000-18,000)+0.5(40,000-18,000)-10,000=2,000>0
- Incentive constraint

  0.5(20,000-18,000)+0.5(40,000-18,000)-10,000=2,000> 0.5(0)+0.5(20,000-18,000)=1,000
- Principal's payoff: 18,000 Agent's payoff: 2,000
- What is the best x that maximizes the principal's payoff?

Example: Revenue sharing (cont.)

- x>20,000: Participation constraint is not satisfied
  - 0.5(0)+0.5(40,000-x)-10,000= 10,000-0.5x<0 if x>20,000
- 10,000<x≤20,000
  - Participation: 0.5(20,000-x)+0.5(40,000-x)-10,000= 20,000-x ≥ 0
  - Incentive: 0.5(20,000-x)+0.5(40,000-x)-10,000 ≥ 0.5(0)+0.5(20,000-x)
  - x≤20,000 → To maximize principal's payoff x=20,000
  - Principal's payoff: 20,000 Agent's payoff: 0
- x ≤ 10,000:
  - Participation: 0.5(20,000-x)+0.5(40,000-x)-10,000= 20,000-x ≥ 0
  - Incentive: 0.5(20,000-x)+0.5(40,000-x)-10,000 ≥ 0.5(10,000-x)+0.5(20,000-x)
  - To maximize principal's payoff x=10,000
  - Principal's payoff: 10,000 Agent's payoff: 10,000
- Optimal x=20,000

Piece-rate system

- Principal's proposed contract: w=αR if the output is R
- Participation constraint for high effort

0.5(α20,000)+0.5(α40,000)-10,000=30,000α-10,000 ≥ 0 → α≥1/3
- Incentive constraint

0.5(α20,000)+0.5(α40,000)-10,000=30,000α-10,000 ≥ 0.5(α10,000)+0.5(α20,000) → α≥2/3
- To maximize the principal's payoff: α=2/3
- Principal's payoff: 10,000 Agent's payoff: 10,000
- Is it optimal for the principal to induce high effort under this contract?

Example

- a is the effort of the agent with cost $c(a)$
- $x=a+\varepsilon$ is the output observed by the principal where ε is Normally distributed with mean zero and variance σ^2.
- Suppose the principal chooses a linear incentive scheme $w(x)=\delta+\gamma x= \delta+\gamma a+\gamma\varepsilon$. δ and γ are parameters to be determined.
- The agent's expected utility is $U(w)=E[w]-r/2(\gamma\sigma)^2$. Agent's reservation utility is $\underline{U}$.
- The principal's expected utility (assuming she is risk-neutral):

  $E[x-w(x)]= E[x]- E[w(x)]=a-(\delta+\gamma a)=(1- \gamma)a- \delta$

Example (cont.)

- The agent's expected utility

$E[U(w)]= E[w]-r/2(\gamma\sigma)^2 = \delta+\gamma a - r/2(\gamma\sigma)^2$

Agent's problem:

Maximize $\delta+\gamma a - r/2(\gamma\sigma)^2-c(a)$

From F.O.C.:

$\gamma- c'(a)=0$

Example (cont.)

- The principal's problem

Maximize $(1- \gamma)a- \delta$

Subject to

$\delta+\gamma a - r/2(\gamma\sigma)^2-c(a)\geq\underline{U}$ (participation constraint)

$\gamma- c'(a)=0$ (incentive constraint)

$\delta=\underline{U}- \gamma a +r/2(\gamma\sigma)^2+c(a)$ $\gamma=c'(a)$

$(1- \gamma)a- \delta=a- \gamma a - \underline{U} +\gamma a - r/2(\gamma\sigma)^2- c(a) = a - \underline{U} - r/2(c'(a)\sigma)^2- c(a)$

Maximize $a - r/2(c'(a)\sigma)^2- c(a)$

Example (cont.)

- Maximize a – r/2(c′(a)σ)^2- c(a)

From F.O.C.:
$1 - r\, c'(a)\, c''(a)\sigma^2 - c'(a) = 0$
$\gamma = c'(a) = 1/(1+r\, c''(a)\, \sigma^2)$

Recall: $w(x)=\delta+\gamma x$
$\sigma^2 = 0 \rightarrow \gamma = 1/(1+r\, c''(a)) \rightarrow \gamma = 1$ and $w(x)=\delta+x$.
$\sigma^2 > 0 \rightarrow \gamma < 1$ and the agent shares some of the risk.