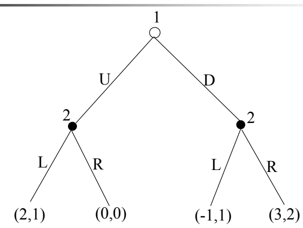
Recap

- Tuesday
  - Stackelberg game
  - Multistage games with observed actions
    - Finite games of perfect information
      - Backward induction
    - Subgame perfect equilibrium
- Today
  - Extensive form of a game
  - Repeated games

Extensive form of a game

- The set of players
- The order of moves
- The players' payoffs as a function of the moves that were made
- The set of actions available to the players when they move
- Each player's information when he makes his move
- The probability distributions over any exogenous events (Nature)

Example 1



Player 1 moves first. After observing player 1's action, player 2 moves
Player 1 action set: {U,D} Player 2 action set:{L,R}
Player 1 strategies: {U,D}
Player 2 strategies: {(L,L), (L,R), (R,L), (R,R)}

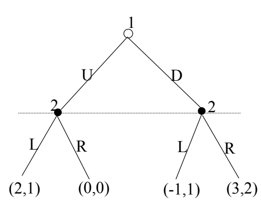
Normal form representation of extensive-form games

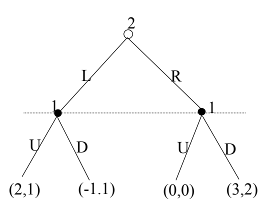
Player 2

Player 1		(L,L)	(L,R)	(R,L)	(R,R)
	U	2,1	2,1	0,0	0,0
	D	-1,1	3,2	-1,1	3,2

- Player 2's strategies correspond to a contingent plan made in advance

Example 2

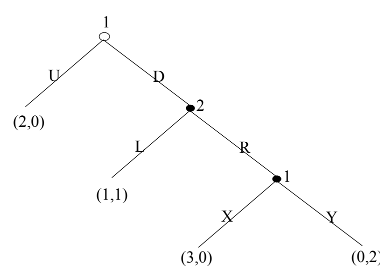




Player 1 moves first, player 2 moves next. Player 2 does <u>not</u> know player 1's action when he chooses his action

Player 2 moves first, player 1 moves next. Player 1 does <u>not</u> know player 2's action when he chooses his action

Classroom exercise



Multistage Prisoner's Dilemma

Prisoner 2

Prisoner 1	C (cooperate)	D (defect)
C	4, 4	0, 5
D	5, 0	1, 1

- The "stage game" will be repeated M times
- The payoffs of the players are the sum of their payoffs from all stages

Two-stage Prisoner's Dilemma

Prisoner 2

Prisoner 1	C (cooperate)	D (defect)
C	4, 4	0, 5
D	5, 0	1, 1

- Stage 2 equilibrium is (D,D) with payoffs (1,1) regardless of stage 1 outcome
- Given the equilibrium outcome of stage 2, update the payoffs of stage 1 and find the equilibrium

Two-stage Prisoner's Dilemma

Prisoner 2

Prisoner 1	C (cooperate)	D (defect)
C	5, 5	1, 6
D	6, 1	2, 2

- The unique equilibrium of the modified stage 1 game is also (D,D)
- The unique subgame-perfect equilibrium of the two-stage Prisoner's Dilemma game is (D,D) in the first stage followed by (D,D) in the second stage

Repeated games

- Let $G = \{A^1,..., A^n; \pi^1,..., \pi^n\}$ denote a static game of complete information in which players $1, ..., n$ simulateneously choose their actions $a^1,...,a^n$ from action spaces $A^1,..., A^n$ and receive payoffs $\pi^1(a^1,...,a^n),..., \pi^n(a^1,...,a^n)$. We call G the *stage game* of the repeated game.
- Given a stage game G, let G(T) denote the *finitely repeated game* in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for G(T) are the (discounted) sum of the payoffs from the T stage games.

Repeated games

- **Result**: If the stage game G has a unique Nash equilibrium then for any finite T, the repeated game G(T) has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage.

Example

Player 2

Player 1	L	M	R
L	1, 1	5, 0	0, 0
M	0, 5	4, 4	0, 0
R	0, 0	0, 0	3, 3

- The stage game is played twice
- The first-stage outcome is observed before the second stage begins

Example

Player 2

Player 1

	L	M	R
L	1, 1	5, 0	0, 0
M	0, 5	4, 4	0, 0
R	0, 0	0, 0	3, 3

- Partial strategy for stage 2:
  - Play R in stage 2 if stage 1 outcome is (M,M); otherwise, play L in stage 2.

Example

Player 2

Player 1

	L	M	R
L	2, 2	6, 1	1, 1
M	1, 6	7, 7	1, 1
R	1, 1	1, 1	4, 4

- Modified stage 1 game
- Subgame perfect equilibria:

  [(L,L),(L,L)]   [(M,M),(R,R)]   [(R,R),(L,L)]

Observation

- Let G be a static game of complete information with multiple Nash equilibria. There may be subgame-perfect outcomes of the repeated game G(T) in which for any $t < T$, the outcome in stage t is not a Nash equilibrium of G.

Definitions

- In the finitely repeated game G(T), a player's ***strategy*** specifies the player's actions in each stage, for each possible history of play through the previous stages.

- In the finitely repeated game G(T), a ***subgame*** beginning at stage t+1 is the repeated game in which G is played T-t times, denoted by G(T-t).

Example

Player 2

Player 1

	L	M	R
L	1, 1	5, 0	0, 0
M	0, 5	4, 4	0, 0
R	0, 0	0, 0	3, 3

- All possible outcomes (histories) at the end of stage 1:

  (L,L) (L,M) (L,R) (M,L) (M,M) (M,R) (R,L) (R,M) (R,R)
- (M, L, L, L, L, R, L, L, L, L)

  Play M in the first stage; Play L in the second stage unless the first stage outcome is (M,M)

Infinitely Repeated Prisoner's Dilemma

Prisoner 2

Prisoner 1

	C (cooperate)	D (defect)
C	4, 4	0, 5
D	5, 0	1, 1

- The game is repeated infinitely
- For each t, the outcomes of the previous t-1 stage games are observed
- Payoffs?

Discounted payoffs

- Let δ be the value today of a dollar to be received one stage later
  - E.g., $\delta=1/(1+r)$ where r is the interest rate per stage
- Given the discount factor δ the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3, \ldots$ is

$$\pi_1+ \delta\pi_2+ \delta^2\pi_3+ \ldots = \sum_{t=1\to\infty} \delta^{t-1}\pi_t .$$

Discounted payoffs

- Suppose after each stage is played, the game continues to the next stage with probability 1-p and stops with probability p.
- Expected present value of next stage's payoff $(1-p)\pi/(1+r)$.
- Expected present value of the payoff two stages later $(1-p)^2\pi/(1+r)^2$.
- Let $\delta= (1-p)/(1+r)$
- $\pi_1+ \delta\pi_2+ \delta^2\pi_3+ \ldots$ reflects the time value of money and the possibility that the game will end

Average payoffs

- $V = \pi_1+ \delta\pi_2+ \delta^2\pi_3+ \ldots = \sum_{t=1\to\infty} \delta^{t-1}\pi_t$
- If we received an "average" payoff of π in every stage, then
  $V = \pi+ \delta\pi+ \delta^2\pi+ \ldots = \pi(1+ \delta+ \delta^2+ \ldots)= \pi/(1- \delta)$
- $\pi/(1- \delta) = \sum_{t=1\to\infty} \delta^{t-1}\pi_t$.

$$\pi = (1- \delta) \sum_{t=1\to\infty} \delta^{t-1}\pi_t .$$

Example: Payoffs 4 4 4 4 4 ....

Average payoff = 4 Net present value = $4/ (1- \delta)$

Infinitely repeated games

- Given a stage game G, let $G(\infty, \delta)$ denote the *infinitely repeated game* in which G is repeated forever and the players share discount factor δ. For each t, the outcomes of the t-1 preceding plays of the stage game are observed before the t^{th} stage begins. Each player's payoff in $G(\infty, \delta)$ is the present value of the player's payoffs from the infinite sequence of stage games.