Recap
- Last Tuesday: Games with continuous action sets
  - The tragedy of the commons
  - Duopoly games
  - Cournot, Bertrand
  - Comparison of duopoly games with monopoly
- Today
  - Continue with duopoly games
    - Stackelberg
  - Multi-stage games with observed actions

Stackelberg Model
- Two competing firms, selling a homogeneous good
  - The marginal cost of producing each unit of the good: $c_1$ and $c_2$
  - Firm 1 moves first and decides on the quantity to sell: $q_1$
  - Firm 2 moves next and decides on the quantity to sell: $q_2$
  - $Q = q_1 + q_2$, total market demand
  - The market price, $P$, is determined by (inverse) market demand:
    - $P = a - bQ$ if $a > bQ$, $P = 0$ otherwise.
  - Both firms seek to maximize profits

Stackelberg Model: Strategy of Firm 2
- Suppose firm 1 produces $q_1$
- Firm 2’s profits, if it produces $q_2$ are:
  $$\pi_2 = (P - c)q_2 = [a - b(q_1 + q_2)]q_2 - c_2q_2 = (\text{Residual}) \text{ revenue} - \text{Cost}$$
- First order conditions:
  $$d\pi_2/dq_2 = a - 2bq_2 - bq_1 - c_2 = 0 \implies q_2 = (a - c_2)/2b - q_1/2 = R^2(q_1)$$

Stackelberg Model: Firm 1’s decision
- Firm 1’s profits, if it produces $q_1$ are:
  $$\pi_1 = (P - c)q_1 = [a - b(q_1 + q_2)]q_1 - c_1q_1$$
- We know that from the best response of Firm 2:
  $$q_2 = (a - c_2)/2b - q_1/2$$
- Substitute $q_2$ into $\pi_1$:
  $$\pi_1 = [a - b(q_1 + (a - c_2)/2b - q_1/2)]q_1 - c_1q_1$$
  $$= [(a + c_2)/2 - (b/2)q_1 - c_1]q_1$$
- From FOC:
  $$d\pi_1/dq_1 = (a + c_2)/2 - bq_1 - c_1 = 0 \implies q_1 = (a - 2c_1 + c_2)/2b$$

Stackelberg Equilibrium
- We have Firm 1’s profits, if it produces $q_1$:
  $$q_1 = (a - 2c_1 + c_2)/2b$$
And firm 2’s best response
  $$q_2 = (a - c_2)/2b - q_1/2$$
- Therefore:
  $$q_2 = (a + 2c_1 - 3c_2)/4b$$
- If $c_1 = c_2 = c$:
  $$q_1 = (a - c)/2b$$
  $$q_2 = (a - c)/4b$$
  $$Q = 3(a - c)/4b$$
Cournot vs. Stackelberg vs. Bertrand

<table>
<thead>
<tr>
<th></th>
<th>Bertrand</th>
<th>Stackelberg</th>
<th>Cournot</th>
<th>Monopoly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>(c)</td>
<td>((a+3c)/4)</td>
<td>((a+2c)/3)</td>
<td>((a+c)/2)</td>
</tr>
<tr>
<td>Quantity</td>
<td>((a-c)/b)</td>
<td>(3(a-c)/4b)</td>
<td>((a-2c)/2b)</td>
<td>((a-c)/2b)</td>
</tr>
<tr>
<td>Total Firm Profits</td>
<td>0</td>
<td>(3(a-c)/16b)</td>
<td>(2(a-c)/9b)</td>
<td>((a-c)/4b)</td>
</tr>
</tbody>
</table>

Example: Stackelberg Competition

- \(P = 130-(q_1+q_2)\), so \(a=130\), \(b=1\)
- \(c_1 = c_2 = c = 10\)
- Firm 2: \(q_2=(a-c_2)/2b-q_1/2 = 60 - q_1/2\)
- Firm 1:
  - Residual demand: \(a-b(q_1+q_2)=70-q_1/2\)
  - \(RMR = (a+c_2)/2-bq_1=70-q_1\)
  - Set \(RMR=MC\), \(70-q_1=10 \rightarrow q_1=60\)
- Market price and demand
  - \(Q=90, P=40\)

Stackelberg competition

Example: Stackelberg Competition

- \(P = 130-(q_1+q_2)\), so \(a=130\), \(b=1\)
- \(c_1 = c_2 = c = 10\)
- Firm 2: \(q_2=(a-c_2)/2b-q_1/2 = 60 - q_1/2\)
- Firm 1:
  - Residual demand: \(a-b(q_1+q_2)=70-q_1/2\)
  - \(RMR = (a+c_2)/2-bq_1=70-q_1\)
  - Set \(RMR=MC\), \(70-q_1=10 \rightarrow q_1=60\)
- Market price and demand
  - \(Q=90, P=40\)

Monopoly vs. Cournot vs. Bertrand vs. Stackelberg

<table>
<thead>
<tr>
<th></th>
<th>Bertrand</th>
<th>Stackelberg</th>
<th>Cournot</th>
<th>Monopoly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>10</td>
<td>40</td>
<td>50</td>
<td>70</td>
</tr>
<tr>
<td>Quantity</td>
<td>120</td>
<td>90</td>
<td>80</td>
<td>60</td>
</tr>
<tr>
<td>Total Firm Profits</td>
<td>0</td>
<td>2700</td>
<td>3200</td>
<td>3600</td>
</tr>
</tbody>
</table>

- Firm profits and prices:
  - Bertrand \(\leq\) Stackelberg \(\leq\) Cournot \(\leq\) Monopoly

Monopoly vs. Cournot vs. Bertrand vs. Stackelberg

<table>
<thead>
<tr>
<th>Consumer surplus</th>
<th>7200</th>
<th>4050</th>
<th>3200</th>
<th>1800</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deadweight loss</td>
<td>0</td>
<td>450</td>
<td>800</td>
<td>1800</td>
</tr>
<tr>
<td>Total Firm Profits</td>
<td>0</td>
<td>2700</td>
<td>3200</td>
<td>3600</td>
</tr>
</tbody>
</table>

- Consumer surplus
- Deadweight loss
- Total Firm Profits
Multi-Stage Games with Observed Actions

These games have "stages" such that:
- In each stage $k$, every player knows all the actions (including those by Nature) that were taken at any previous stage.
- Players move simultaneously in each stage $k$.
- Some players may be limited to action set "do nothing" in some stages.
- Each player moves at most once within a given stage.
- No information set contained in stage $k$ provides any knowledge of play in that stage.

Stackelberg game

- Stage 1:
  - Firm 1 chooses its quantity $q_1$; Firm 2 does nothing.
- Stage 2:
  - Firm 2, knowing $q_1$, chooses its own quantity $q_2$; Firm 1 does nothing.

Example: Stackelberg competition

- $P = 130 - (q_1 + q_2)$, $c_1 = c_2 = c = 10$

Backward induction

- Determine the optimal action(s) in the final stage $K$ for each history $h^K$.
  - For each stage $j = K-1, ..., 1$:
    - Determine the optimal action(s) in stage $j$ for each possible $h_j$ given the optimal actions determined for stages $j+1, ..., K$.
  - The strategy profile constructed by backward induction is a Nash Equilibrium.
  - Each player’s actions are optimal at every possible history.

Finite games of perfect information

- A multistage game has perfect information if:
  - For every stage $k$ and history $h^k$, exactly one player has a nontrivial action set, and all other players have one-element action set "do nothing".
  - Each player knows all previous moves when making a decision.
- In a finite game of perfect information, the number of stages and the number of actions at any stage are finite.
- Theorem (Zemelo 1913; Kuhn 1953): A finite game of perfect information has a pure-strategy Nash equilibrium.

Multi-Stage Games with Observed Actions

- $h^k$: History at the start of stage $k$:
  - $h^k = (a^0, a^1, ..., a^{k-1})$, $k = 1, ..., K$.
- $A^i(h^k)$: Set of actions available to player $i$ in stage $k$ given history $h^k$.
- $s^i$: Pure strategy for player $i$ that specifies an action $a \in A^i(h^k)$ for each $k$ and each history $h^k$. 

Example: Stackelberg competition

- $P = 130 - (q_1 + q_2)$, $c_1 = c_2 = c = 10$

Backward induction

- Firm 2 strategy: $s^2(q_1) = q_2 = 60 - q_1/2$
- Firm 1 strategy: $q_1 = 60$
- The outcome (60,30) is a Nash equilibrium (Stackelberg outcome).

Is (60,30) the unique equilibrium in this game?

- Cournot equilibrium (40,40) is also an equilibrium for the Stackelberg game!
  - $s^2(q_1) = 40$, $q_1 = 40$. 

Classroom exercise: Strategic investment

- Duopoly: Firm 1 and Firm 2
- Each firm has unit cost 2
- By paying $f$, Firm 1 can install new technology and reduce its unit cost to zero
- Once Firm 1’s investment decision is observed, both firms simultaneously choose output levels $q_1$ and $q_2$ as in Cournot competition
- $P=14-Q$
- Recall: Cournot best response
  \[
  q_1=\frac{a-c_1}{2b} - \frac{q_2}{2} \\
  q_2=\frac{a-c_2}{2b} - \frac{q_1}{2}
  \]
- Subgame-perfect equilibrium
  - A strategy profile $s$ of a multistage game with observed actions is a subgame-perfect equilibrium if, for every $h^k$, the restriction $s|^{h^k}$ is a Nash equilibrium of subgame $G(h^k)$.
  - $G(h^k)$: game from stage $k$ on with history $h^k$
  - For each player $j$, $s_j|^{h^k}$ is the restriction of $s_j$ to the histories consistent with $h^k$

Classroom exercise: Strategic investment

- Firm 1 does not invest
  \[
  q_1=\frac{a-c_1}{2b} - \frac{q_2}{2} = 6 - \frac{q_2}{2} \\
  q_2=\frac{a-c_2}{2b} - \frac{q_1}{2} = 6 - \frac{q_1}{2} \rightarrow (q_1,q_2)=(4,4)
  \]
  Payoffs: (16,16)

- Firm 1 does invest
  \[
  q_1=\frac{a-c_1}{2b} - \frac{q_2}{2} = 7 - \frac{q_2}{2} \\
  q_2=\frac{a-c_2}{2b} - \frac{q_1}{2} = 6 - \frac{q_1}{2} \rightarrow (q_1,q_2)=(16/3,10/3)
  \]
  Payoffs: (256/9-f,100/9)

- Firm 1 choice:
  - Invest if $256/9-f>16$, i.e., if $f<112/9$

Example: Stackelberg competition

- $P = 130-(q_1+q_2), \quad c_1 = c_2 = c = 10$
- By backward induction the outcome $(60,30)$ is a subgame-perfect equilibrium
- The outcome $(40,40)$ is NOT subgame perfect, because the strategy $s^2(q_1)=40$ does not induce a Nash equilibrium in stage 2 for player 2, for histories other than $q_1=40$