Introduction to Game Theory

A collection of tools for predicting outcomes for a group of interacting agents where an action of a single agent directly effects the payoffs of the other participating agents.

What is a game?

- Many types of games: board games, card games, video games, field games (e.g., football), etc.
  - A zero-sum game is one in which the players' interests are in direct conflict, e.g. in football, one team wins and the other loses.
  - A game is non-zero sum, if players interests are not always in direct conflict, so that there are opportunities for both to gain.
- We focus on games where:
  - There are 2 or more *players*.
  - There is some choice of action where *strategy* matters.
  - The game has one or more *outcomes*, e.g. someone wins, someone loses.
  - The outcome depends on the strategies chosen by all players; there is *strategic interaction*.
- What does this rule out?
  - Games of pure chance, e.g., lotteries, slot machines. (Strategies don't matter).
  - Games without strategic interaction between players, e.g. Solitaire.

Elements of a game

- The *players*
  - how many players are there?
  - does nature/chance play a role?
- A complete description of what the players can do – *the set of all possible actions*.
- A description of *the payoff consequences* for each player for every possible combination of actions chosen by all players playing the game.

Example: NBC vs. ABC

NBC

ABC		News	Sitcom
	News	55% 45%	52% 48%
	Sitcom	50% 50%	45% 55%

- Players: NBC and ABC
- Set of all possible actions: sitcom, news
- Payoffs: market shares for each outcome

Normal form game: Notation

- A set of N players $I = \{1, ..., N\}$
- Each player $i \in N$ has an action set $A^i = \{a_1^i, a_2^i, \ldots, a_{k_i}^i\}$ such that $a^i \in A^i$ is a particular action by i
- Each player i has a payoff $\pi^i(a)$ where $a = \{a^1, a^2, \ldots, a^N\}$ is the game outcome
- $a^{-i} = \{a^1, a^2, \ldots, a^{i-1}, a^{i+1}, \ldots, a^N\}$ denotes the actions of all players except i in the outcome

Assumptions

- Payoffs are known and fixed.
- Players are risk neutral, i.e., maximize expected payoffs.
  - Example: a risk neutral person is indifferent between
    - $25 for certain or
    - a 25% chance of earning $100 and a 75% chance of earning 0.
- All players behave rationally. They understand and seek to maximize their own payoffs.
- The rules of the game are common knowledge
  - Each player knows the set of players, strategies and payoffs from all possible combinations of strategies: call this information "X." *Common knowledge* means that each player knows that all players know X, that all players know that all players know X, that all players know that all players know that all players know X and so on,..., *ad infinitum*.

Classroom game

Player 2 (columns), Player 1 (rows)

	X	Y
X	8, 7	-100, 6
Y	7, 6	6, 5

- Player 1
  - X:9 Y:21
- Player 2
  - X: 18 Y: 12
- Outcomes
  - (X,X):6 (X,Y): 4 (Y,X): 11 (Y,Y): 9

Classroom game

Player 2 (columns), Player 1 (rows)

	X	Y
X	8, 7	5, 6
Y	7, 6	6, 5

- Player 1
  - X:18 Y:12
- Player 2
  - X:21 Y: 9
- Outcomes
  - (X,X):14 (X,Y):3 (Y,X):9 (Y,Y):4

Example: NBC vs. ABC

NBC (columns), ABC (rows)

	News	Sitcom
News	55% 45%	52% 48%
Sitcom	50% 50%	45% 55%

- Regardless of ABC's action, NBC chooses sitcom - Sitcom is a *dominant action* for NBC
- Regardless of NBC's action, ABC chooses news - News is a *dominant action* for ABC

Definition: Dominant and dominated actions

- A particular action $\tilde{a}^i \in A^i$ is a dominant action for player i if, no matter what all other players are playing, playing $\tilde{a}^i$ maximizes i's payoff
  $\pi^i(\tilde{a}^i, a^{-i}) \geq \pi^i(a^i, a^{-i})$ for all $a^i \in A^i$
- A particular action $\tilde{a}^i \in A^i$ is a (weakly) dominated action for player i if, no matter what all other players are playing, there exists another action $\hat{a}^i \in A^i$ such that
  $\pi^i(\tilde{a}^i, a^{-i}) \leq \pi^i(\hat{a}^i, a^{-i})$

Example: NBC vs. ABC

NBC (columns), ABC (rows)

	News	Sitcom
News	55% 45%	52% 48%
Sitcom	50% 50%	45% 55%

- Sitcom is a *dominant action* for NBC
- News is a *dominant action* for ABC
- (News,Sitcom) is an *equilibrium in dominant actions*

Definition: Equilibrium in dominant actions

- An outcome $a = \{\tilde{a}^1, \tilde{a}^2, \ldots, \tilde{a}^N\}$ is an equilibrium in dominant actions
  if $\tilde{a}^i \in A^i$ is a dominant action for player i

Example: Old vs. new technology

Firm 2 (columns), Firm 1 (rows)

	New	Current
New	0, 0	a, -a
Current	-a, a	0, 0

- Regardless of Firm 1's action, Firm 2 chooses new technology – Dominant action for firm 2 is to choose new technology

Example: Old vs. new technology

Firm 2 (columns), Firm 1 (rows)

	New	Current
New	0, 0	a, -a
Current	-a, a	0, 0

- Regardless of Firm 2's action, Firm 1 chooses new technology – Dominant action for firm 2 is to choose new technology

Example: Old vs. new technology

Firm 2 (columns), Firm 1 (rows)

	New	Current
New	0, 0	a, -a
Current	-a, a	0, 0

- New technology is a dominant action for both players
- (New,New) is an *equilibrium in dominant actions*

Classroom game

Player 2 (columns), Player 1 (rows)

	X	Y
X	8, 7	-100, 6
Y	7, 6	6, 5

- Is there a dominant action for player 2? For player 1?
- What is the equilibrium?

Definition: Best response

• In an N player game, the best response function of player i is such that

$$R^i(a^{-i}) = \arg\max \pi^i(a^i, a^{-i})$$

In other words, given the actions a^{-i} of players $1, 2, \ldots, i\text{-}1, i+1, \ldots, n$, $R^i(a^{-i})$ is the best re-action from player i

Example

Player 2 (columns), Player 1 (rows)

	X	Y
X	2, 1	0, 0
Y	0, 0	1, 2

- Player 1 plays X → Player 2's best response: $R^2(X)=X$
- Player 1 plays Y → Player 2's best response: $R^2(Y)=Y$
- Player 2 plays X → Player 1's best response: $R^1(X)=X$
- Player 2 plays Y → Player 1's best response: $R^1(Y)=Y$
- What is the dominant action for player 1? Player 2?

Example (cont.)

Player 2

Player 1

	X	Y
X	2, 1	0, 0
Y	0, 0	1, 2

- There is no equilibrium in dominant actions
- What is the likely outcome of this game?

Nash Equilibrium

- An outcome $a = \{\hat{a}^1, \hat{a}^2, ..., \hat{a}^N\}$ is a Nash Equilibrium (NE)

if no player would find it beneficial to deviate from a strategy - provided that all other players do not deviate from their strategies played at the Nash extreme:

$\pi^i(\hat{a}^i, \hat{a}^{-i}) \geq \pi^i(a^i, \hat{a}^{-i})$ for all $a^i \in A^i$

Finding Nash Equilibrium

- Calculate the best response functions of every player
- Check which outcomes lie (if any) on the best response function of every player

Example

Player 2

Player 1

	X	Y
X	2, 1	0, 0
Y	0, 0	1, 2

- The outcomes (X,X) and (Y,Y) are Nash equilibrium

Observation

- What is the relationship between an equilibrium in dominant actions and a NE?
- An equilibrium in dominant actions is a NE, but the converse is not necessarily true
- There will be only one equilibrium in dominant actions while there can be multiple NEs

Example

Player 2

Player 1

	X	Y
X	2, 0	0, 2
Y	0, 2	2, 0

- There are neither equilibrium in dominant actions nor (pure strategy) Nash equilibrium

Example: Mixed strategy equilibrium

Player 2 (columns), Player 1 (rows)

	X	Y
X	2, 0	0, 2
Y	0, 2	2, 0

- Player 2: Play X with probability 0.5 and Y with probability 0.5
  - Player 1 (expected) payoff if plays X: (0.5)(2)+(0.5)(0)=1
  - Player 1 (expected) payoff if plays Y: (0.5)(0)+(0.5)(2)=1
  - Player 1 is indifferent between playing X and Y!
- Player 1: Play X with probability 0.5 and Y with probability 0.5
  - Player 2 is indifferent between playing X and Y
- Neither player has an incentive to deviate from the strategy of randomly selecting between X and Y

Example: Mixed strategies

Player 2 (columns), Player 1 (rows)

	X	Y	Z
L	1, 4	1, 1	1, 0
R	1, 0	1, 1	1, 3

- Are there any dominant or dominated pure strategies for player 2?
- Player 2 mixed strategy:
  - Play X with probability 3/7, Z with probability 4/7
- Player 1's expected payoff is 1 whether he plays L or R
- Expected payoff of player 2
  - Player 1 plays L: (3/7)(4)+(4/7)(0)=12/7
  - Player 1 plays R: (3/7)(0)+(4/7)(3)=12/7
- The mixed strategy dominates the pure strategy Y!

Existence of mixed strategy equilibrium

- Theorem: Every finite normal-form game has a mixed strategy equilibrium

Finding the mixed strategy equilibrium

Bob (columns), Alice (rows)

	Ballet	Football
Ballet	2, 1	0, 0
Football	0, 0	1, 2

- Alice: (q,1-q) Bob: (p,1-p)
- Alice's expected payoff
  - If Alice chooses ballet: 2p   If Alice chooses football:1-p
  - 2p=1-p → p=1/3 → Bob chooses football with probability 2/3
- Bob's expected payoff
  - If Bob chooses ballet: q   If Bob chooses football:2(1-q)
  - q=2(1-q) → q=2/3 → Alice chooses ballet with probability 2/3
- Two pure strategy and one mixed strategy equilibria

Comparison of outcomes

- If there are multiple equilibria, are some of them preferable for all players?

Definition: Pareto dominance

- The outcome $\hat{a}$ Pareto dominates the outcome a if, for every player i

$$\pi^i(\hat{a}) \geq \pi^i(a)$$

and there exists at least one player j where

$$\pi^j(\hat{a}) > \pi^j(a)$$

- An outcome a^* is called Pareto efficient if there does not exist any outcome which Pareto dominates outcome a^*.

Example: Prisoner's dilemma

Prisoner 2 (columns); Prisoner 1 (rows)

	C (cooperate)	D (defect)
C	-1, -1	-3, 0
D	0, -3	-2, -2

- (-1,-1) Pareto-dominates (-2,-2)
- The rest are Pareto noncomparable
- Even though (-1,-1) Pareto-dominates (-2,-2), (-2,-2) is the unique Nash equilibrium

Administrivia

- Teaching assistant:
  Martin Smith
  gtg593d@prism.gatech.edu
- Office hours
  Tuesday 3:30-4:30
  Wednesday 2:00-3:00
  Location: ISYE 403