Service Levels in (Q,R) Models

ISYE 3104 – Fall 2008

Service objectives

- **Type I** service level (α)
  - The proportion of cycles in which no stockouts occur
  - Example: 90% Type I service level → There are no stockouts in 9 out of 10 cycles (on average)
- **Type II** service level (fill rate, β)
  - Fraction of demand satisfied on time

Service objectives - Example

Order cycle	Demand	Stock-outs
1	180	0
2	75	0
3	235	45
4	140	0
5	180	0
6	200	10
7	150	0
8	90	0
9	160	0
10	40	0
TOTAL:	1450	55

Fraction of periods with no stock-outs = 8/10
→Type I service = 80%
α = 0.8

Fraction of demand satisfied on time = (1450-55)/1450=0.96
→ Type II service = 96%
β = 0.96

In general, is it easier to achieve an x% Type I service or Type II service level?

Type I service level, α

α : Long-run average proportion of cycles with no stock-outs
α : Probability of having no stock-outs in a cycle
α : Probability of having no stock-outs during lead time
α : Probability that demand during lead time is less than R !!!

$$\alpha = P(D \le R)$$

$$\text{Recap:} \quad P(D \le R) = P\left(\frac{D-\mu}{\sigma} \le \frac{R-\mu}{\sigma}\right) = P(Z \le z) = \Phi(z) = \alpha$$

- Set Q=EOQ
- Find z that satisfies $\Phi(z) = \alpha$
- Set $R=\sigma z+\mu$ (safety stock + expected demand during lead time)

Example – Rainbow Colors

- Rainbow Colors paint store uses a (Q,R) inventory system to control its stock levels. For a popular eggshell latex paint, historical data show that the distribution of monthly demand is approximately Normal, with mean 28 and standard deviation 8. Replenishment lead time for this paint is about 14 weeks. Each can of paint costs the store $6. Although excess demands are backordered, each unit of stockout costs about $10 due to bookkeeping and loss-of-goodwill. Fixed cost of replenishment is $15 per order and holding costs are based on a 30% annual interest rate.
  - What is the optimal lot size (order quantity) and reorder level?
  - What is the expected inventory level (safety stock) just before an order arrives?

Example – Rainbow Colors

- Rainbow Colors is not sure whether the $10 estimate for the shortage cost is accurate. Hence, they decided to use a service level approach. What are the optimal (Q,R) values if they want to
  - achieve no stockouts in 90% of the order cycles?
  - satisfy 90% of the demand on time?

Example – Rainbow Colors

- Input
  - Monthly demand Normal mean=28 std.dev.=8
  - τ=14 weeks, c=\$6, K=\$15
  - h=Ic=(0.3)(6)=\$1.8/unit/year
  - α = 0.9 or β = 0.9
- Computed input
  - d=(28)(12)=336 units/year (Expected annual demand)
  - Expected demand during lead time

$$\boxed{\mu} = \frac{(28)(12)\ \text{units / year}}{52\ \text{weeks / year}} \times (14\ \text{weeks}) = \boxed{90}\ \text{units}$$

  - Variance of demand during lead time

$$\text{Annual variance} = (12)(8^2) = 768$$

$$\text{Variance of lead time demand} = 768 \times \frac{14}{52} = 206.77 \Rightarrow \boxed{\sigma = 14.38}$$

Rainbow Colors – Type I service

Find (Q,R) to have 90% Type I service level

- Q=EOQ=75
- $\Phi(z) = \alpha = 0.9 \rightarrow z = 1.28$
- $R = \sigma z + \mu \rightarrow R = (14.38)(1.28) + 90 = 108$
- **For 90% Type I service level (Q,R)=(75,108)**

<u>Remember</u>: With unit penalty cost of \$10, we found (Q,R)=(80,115). What is the Type I service level that corresponds to (Q,R)=(80,115)?

$R = \sigma z + \mu \rightarrow 115 = (14.38)z + 90 \rightarrow z = 1.7385$

$\Phi(1.7385) = 0.96 \rightarrow$

96% Type I service level when (Q,R)=(80,115)

Type II service level

- β : Fraction of demand met on time
- 1- β : Fraction of demand not met on time (stock-outs)

Recap:

$$\text{Expected \# of stockouts per unit time} = \frac{n(R)}{T} = \frac{d\, n(R)}{Q} \qquad \left(\text{since } T = \frac{Q}{d}\right)$$

$$1-\beta = \frac{\text{Expected \# of stockouts per unit time}}{\text{Expected demand per unit time}} = \frac{n(R)}{Q} \Rightarrow \boxed{1-\beta = \frac{n(R)}{Q}} \quad (4)$$

With this information, for a given (Q,R), we can compute β.

Rainbow Colors

- For 90% Type I service level we found (Q,R)=(75,108)
- What is the Type II service level which corresponds to this policy?

The same policy results in 90% Type I service and 99% Type II service!!

$$\frac{R-\mu}{\sigma} = \frac{108-90}{14.38} = 1.25 = z$$

$$n(R) = \sigma L(z) = \sigma L(1.25) = (14.38)(0.0506) = 0.7276$$

$$1-\beta = \frac{n(R)}{Q} = \frac{0.7276}{75} = 0.0097 \Rightarrow \beta \approx 0.99$$

Finding the optimal (Q,R) for a desired Type II service level

Remember: Optimal solution when we have stock-out cost p:

(1) $$Q = \sqrt{\frac{2d[K + p\,n(R)]}{h}}$$

$$F(R) = 1 - \frac{Qh}{pd}$$ (2)

Finding the optimal (Q,R) for a desired Type II service level

Optimal solution when we have stock-out cost p:

(1) $$Q = \sqrt{\frac{2d[K + p\,n(R)]}{h}}$$

$$F(R) = 1 - \frac{Qh}{pd}$$ (2)

From (2):

$$p = \frac{Qh}{d(1 - F(R))}$$ (5) **Imputed shortage cost**

Substitute p into (1):

$$Q = \frac{n(R)}{1 - F(R)} + \sqrt{\frac{2Kd}{h} + \left(\frac{n(R)}{1 - F(R)}\right)^2}$$ (3)

To be solved simultaneously with $n(R) = (1 - \beta)Q$ (4)

Impact of service level β on R

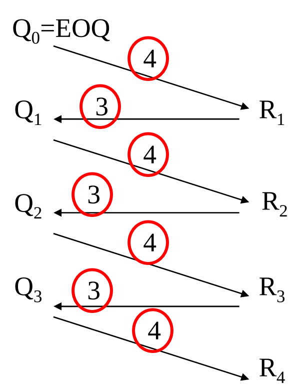
- For a given Q
  - As $\beta \uparrow$ $n(R) = (1-\beta)Q \downarrow$ i.e., $R \uparrow$

As the service level increases, the reorder level increases as well

Finding the optimal (Q,R) for a desired Type II service level

$$Q = \frac{n(R)}{1-F(R)} + \sqrt{\frac{2Kd}{h} + \left(\frac{n(R)}{1-F(R)}\right)^2} \quad (3)$$

$$n(R) = (1-\beta)Q \quad (4)$$



- Start with a Q_0 value and iterate until the Q values (or the R values) converge

Example – Rainbow Colors

- Iteration 0: Compute EOQ

$$\boxed{Q_0} = \sqrt{\frac{2Kd}{h}} = \sqrt{\frac{(2)(15)(336)}{1.8}} = \boxed{75}$$

Example – Rainbow Colors

- Iteration 1: Compute R_1 (given Q_0) and then compute Q_1 (given R_1)

$$n(R_1) = (1-\beta)Q_0 = (1-0.9)(75) = 7.5 = \sigma L(z) \Rightarrow$$

(4)

$$L(z) = 0.5216 \Rightarrow z = -0.22$$

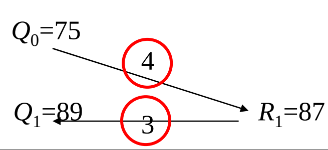
$$R_1 = \sigma z + \mu = (14.38)(-0.22) + 90 = 86.83 \approx 87$$

To find Q_1 we need 1-$F(R_1)$. Look at the Normal table.

$$1 - F(-0.22) = F(0.22) = 0.5871$$

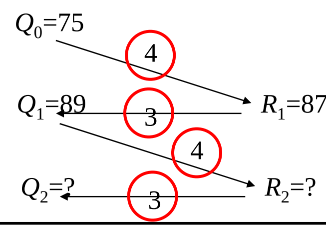
$$Q_1 \approx 89$$

(3)



Example – Rainbow Colors

- Iteration 2: Compute R_2 (given Q_1) and then compute Q_2 (given R_2)



Example – Rainbow Colors

- Iteration 3: Compute R_3 (given Q_2) and then compute Q_3 (given R_3)

STOP! R values converged, optimal (*Q*,*R*)=(?,?)

Exercise Questions

- p=10 → (Q,R)=(80,115) α=0.96 β=?
- α=0.90 → (Q,R)=(75,108) p=? β=0.99
- β=0.90 → (Q,R)=(90,85) p=? α=?