ISyE 3104 Fall 2005
Homework 5 Answers

1. 
\[ C(Q) = \begin{cases} 
4000Q, & 0 < Q \leq 200 \\
800,000 + 3800(Q - 200) = 40,000 + 3800Q, & 200 < Q \leq 400 \\
1,560,000 + 3500(Q - 400) = 160,000 + 3500Q, & 400 < Q 
\end{cases} \]

\[ \frac{C(Q)}{Q} = \begin{cases} 
4000, & 0 < Q \leq 200 \\
3800 + 40,000 / Q, & 200 < Q \leq 400 \\
3500 + 160,000 / Q, & 400 < Q 
\end{cases} \]

\[ G(Q) = D \cdot C(Q) / Q + KD / Q + I[C(Q) / Q] Q / 2 \]

\[ G_0(Q) = 5600 \cdot 4000 + 10,000 \cdot 5600 / Q + (0.2) \cdot (4000) \cdot Q / 2 \]

which is minimized at \( Q^{(0)} = \frac{2DK}{Ic_0} = \sqrt{\frac{2 \cdot 5,600 \cdot 10,000}{400 \cdot (0.2)}} \approx 375 \)

\[ G_1(Q) = 5600 \cdot (3800 + 40,000 / Q) + 10,000 \cdot 5600 / Q + (0.2) \cdot (3800 + 40,000 / Q) \cdot Q / 2 \]

which is minimized at \( Q^{(1)} = \frac{2DK}{Ic_1} = \sqrt{\frac{2 \cdot 5,600 \cdot 50,000}{3800 \cdot (0.2)}} \approx 859 \)

\[ G_2(Q) = 5600 \cdot (3500 + 160,000 / Q) + 10,000 \cdot 5600 / Q + (0.2) \cdot (3500 + 160,000 / Q) \cdot Q / 2 \]

Which is minimized at \( Q^{(2)} = \frac{2DK}{Ic_2} = \sqrt{\frac{2 \cdot 5,600 \cdot 170,000}{3500 \cdot (0.2)}} \approx 1650 \)

i. We see that only \( Q^{(2)} \) is realizable. Therefore the optimal solution to the incremental discount schedule is to order 1650 units at a time.

ii. \( G_2(1650) = 5600 \cdot (3500 + 160,000 / 1650) + 10,000 \cdot 5600 / 1650 + (0.2) \cdot (3500 + 160,000 / 1650) \cdot 1650 / 2 = \$20,770,470 \)
2. a. \( Q^* = \sqrt{\frac{2DK}{h}} \sqrt{\frac{b + h}{b}} = \sqrt{\frac{2 \times 500 \times 250}{1200}} \sqrt{\frac{1200 + 1500}{1500}} = 19.36 \approx 20 \)

c. \( S^* = Q^* \left( \frac{h}{h + b} \right) \approx 9 \)

b. \( Q^* - S^* = 20 - 9 = 11 \)

d. \( T = \left( \frac{Q^*}{D} \right) = 0.04 \text{ years} \)

e. \( TC = \frac{DK}{Q^*} + \frac{(Q^* - S^*)^2 h}{2Q^*} + \frac{(S^*)^2 b}{2Q^*} = 6250 + 3630 + 3037.5 = $12917.5 \)

f. \( Q^* = \sqrt{\frac{2DK}{h}} = \sqrt{\frac{2 \times 500 \times 250}{1200}} \approx 15 \)

\( TC = \frac{DK}{Q^*} + \frac{Q^* h}{2} = 8333.3 + 9000 = $17333.3 \)

g. In MMC case, backordering reduces the inventory holding cost and ordering cost significantly. With backorders, \( Q^* \) is always greater, which decreases the number of orders. Some of this \( Q^* \) is not held in the inventory and there is a certain time in each cycle, where the inventory level is zero. This reduces the inventory holding cost. Together these two savings are larger than the extra cost introduced due to backordering. Therefore, backordering case is cheaper.

3.

a. Assuming 52 weeks in a year:

\[ Q^{(0)} = \sqrt{\frac{2DK}{Ic_0}} = \sqrt{\frac{2 \times 60 \times 52 \times 100}{0.2 \times 14}} \approx 473 \]

\[ Q^{(1)} = \sqrt{\frac{2DK}{Ic_1}} = \sqrt{\frac{2 \times 20 \times 52 \times 100}{0.2 \times 18}} \approx 241 \]

\[ Q^{(2)} = \sqrt{\frac{2DK}{Ic_2}} = \sqrt{\frac{2 \times 100 \times 52 \times 100}{0.2 \times 16}} \approx 571 \]

\[ Q^{(3)} = \sqrt{\frac{2DK}{Ic_3}} = \sqrt{\frac{2 \times 30 \times 52 \times 100}{0.2 \times 20}} \approx 280 \]

But 473 + 241 + 571 + 280 = 1565 > 1000. Therefore the optimum \( Q^* \)'s are of the form:

\[ Q^{(i)} = \sqrt{\frac{2DK}{Ic_i + 2Iw_i}}, \text{ where } I = 0.2 \text{ and } w_i = 1 \text{ for all } i. \]
The ratio w/h is constant for all 4 products. Multiplier \( m = \frac{1000}{1565} = 0.639 \)

\[
\begin{align*}
Q_{(0)}^* &= m \times Q_{(0)} = 0.639 \times 473 = 302 \\
Q_{(1)}^* &= m \times Q_{(1)} = 0.639 \times 241 = 153 \\
Q_{(2)}^* &= m \times Q_{(2)} = 0.639 \times 571 = 364 \\
Q_{(3)}^* &= m \times Q_{(3)} = 0.639 \times 280 = 178 \\
\end{align*}
\]

\( \theta^* \)'s corresponding to these \( Q^* \)'s would be: 2.02, 2.64, 2.32, 2.92

min \( \theta = 2.02 \) \hspace{1em} max \( \theta = 2.92 \)

Using bisection search, if we start \( \theta = 2.47 \), we find the following \( Q^* \)'s:

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( \theta )</th>
<th>( Q^0 )</th>
<th>( Q^1 )</th>
<th>( Q^2 )</th>
<th>( Q^3 )</th>
<th>TOTAL</th>
</tr>
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<tr>
<td>1</td>
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<td>284</td>
<td>157</td>
<td>358</td>
<td>187</td>
<td>986</td>
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<tr>
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<td>293</td>
<td>161</td>
<td>368</td>
<td>192</td>
<td>1014</td>
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<td>1001</td>
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<td>363</td>
<td>190</td>
<td>1000</td>
</tr>
</tbody>
</table>

Since 984.26 < 1000, this implies \( \theta < 2.47 \).

Using spreadsheet to find the \( \theta \), after couple of iteratations below we find the optimal \( Q^* \)'s for constrained space:

![Bisection search iterations](image-url)
b.

Maximum investment in the inventory would be:
473*14 + 241*18 + 571*16 + 280*20 = $25,696
m=20000/25696=0.7783

All new optimal order quantities below are rounded “down”:
\( Q^{(0)*} = m \times Q^{(0)} = 0.7783 \times 473 = 368 \)
\( Q^{(1)*} = m \times Q^{(1)} = 0.7783 \times 241 = 187 \)
\( Q^{(2)*} = m \times Q^{(2)} = 0.7783 \times 571 = 444 \)
\( Q^{(3)*} = m \times Q^{(3)} = 0.7783 \times 280 = 217 \)

Now total maximum investment is:
368*14 + 187*18 + 444*16 + 217*20 = $19,962

Remaining $38 can be used to increase the lot sizes of formal products (1 and 3), to
\( Q^{(1)*} = 188 \) and \( Q^{(3)*} = 218 \) that results in a maximum investment of $20,000.