Designing Optimal Pre-Announced Markdowns in the Presence of Rational Customers with Multi-unit Demands

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Abstract

We analyze the optimal design of a markdown pricing mechanism with pre-announced prices. In the presence of limited supply, buyers who choose to purchase at a lower price may face a scarcity in supply. Our focus is on the structure of the optimal markdown mechanisms in the presence of rational or “strategic” buyers who demand multiple units. We first examine a complete information setting where the set of customer valuations is known but the seller does not know the valuation of each individual customer (i.e., cannot exercise perfect price discrimination). We then generalize our analysis to an incomplete valuation information setting where customer valuations are drawn from known distributions. For both settings, we compare the seller’s profit resulting from the optimal markdown mechanism and the optimal single price. We provide a number of managerial insights into designing profitable markdown mechanisms.

Keywords: Pricing, Markdown, Strategic Bidding, Price Discrimination.

1 Introduction

‘Pricing has traditionally been a high-stakes game based on guesses about costs and competitive activities, resulting in either money left on the table or lost sales.’ (Wreden, 2003) In recent years, advances in technology have opened the door for companies to turn their ‘pricing game’ into an intelligent and more profitable strategy with the use of price optimization software (Coy, 2000; Elmaghraby and Keskinocak, 2003). Price optimization software assists sellers to intelligently execute various pricing strategies by studying a wide variety of data, ranging from historical sales to demographics. The increased information available about customers as well as the reduced transaction costs associated with changing prices over time is enabling sellers to explore a variety of pricing schemes aside from the commonly used traditional single (static) pricing policy.

When the seller does not know the identity of a buyer, i.e., cannot exercise first or third degree price discrimination, but knows that she is facing a market where customers differ by their willingness-to-pay, a markdown pricing strategy offers a way to potentially improve profits above single price levels. The idea behind a markdown mechanism is to segment/differentiate customers with diverse valuations by offering different prices over time so as to create scarcity at lower prices, with the goal of inducing high valuation customers to purchase at higher prices. It is clear that the structure of a markdown mechanism influences buyer behavior, and in turn, the seller’s profits.

The rationale behind and the advantages of markdown pricing are well-known in the fashion apparel industry as well as many other business-to-consumer (B2C) markets that sell highly seasonal or short-lived products. As companies begin to explore alternative pricing mechanisms, it is imperative that we understand how they will perform under various market settings. In this paper, we consider a markdown mechanism where the price of a good decreases over time according to a pre-announced schedule. For example, a seller posts a group of items for an opening price, say, $800 per item, and buyers bid the quantities they want at that price. After a short duration, e.g., two days, the price drops according to a pre-announced schedule, e.g., to $600, per item, and then to $300 and...
then $100. That is, markdowns continue until all the items are sold, or until the price drops to the minimum level set by the seller. While there may be only a few buyers interested in purchasing at $800, the threat of many more customers who are willing to pay $300 or $100 may induce the 'high valuation' buyers to purchase at a higher price. Retailers who have been using preannounced markdowns for decades (e.g., Filene’s Basement) are now being joined by others, e.g., Sam’s Club, as a result of the ease-of-use provided by the Internet and increased sophistication of buyers. (An example from Sam’s Club web site and an excerpt from Filene’s Basements’ web site are presented in the Appendix.)

Companies operating in business-to-business (B2B) markets are also seeing the potential advantages of price optimization and markdown pricing, as seen by the pricing tools offered pricing software vendor Zilliant (Williams, 2005) among others (Elmaghraby and Keskinocak, 2003). The majority of the previous research on retail price markdowns and clearance sales assumes that customers are myopic, that is, a customer will make a purchase immediately if the price is below his valuation without considering future prices. While a plausible assumption for some B2C markets, an analysis of B2B markets requires a richer characterization of customer behavior; under B2B settings, customers often act rationally (or strategically), taking into account the entire (expected) price path while deciding when to buy. Furthermore, most papers that previously addressed markdown pricing assume single-unit demand; again, a characterization that is less likely to hold in B2B markets.

In this paper, we analyze the optimal design of a pre-announced markdown mechanism by exploring a setting where strategic/rational customers have multi-unit demands (Engelbrecht-Wiggans and Kahn, 1998; Katzman, 1999; Tenorio, 1997, 1999) and the seller may have a limited supply of goods (Bitran and Mondschein, 1997; Feng and Gallego, 1995; Gallego and van Ryzin, 1994). The goal of this paper is to shed light on three fundamental questions by combining research streams from economics and operations management:

- How will rational buyers behave (bid) when facing a markdown?
- What is the optimal markdown, i.e., how many price steps should it have and what should those prices be? What is the impact of strategic customers with multi-unit (versus single-unit) demands on the design of the optimal markdown?
- Under what conditions would the seller be better off using a single price vs. markdown pricing?

We initially study a markdown mechanism under a setting where the seller has complete information about the buyers’ valuations and demands (Section 3). Under complete information (CI), we find that for \( N \geq 2 \) buyers the optimal markdown has two steps and buyers submit all-or-nothing bids at each price step (Section 3). That is, a buyer never finds it optimal to bid only a portion of his demand at any one price step. We find this to be the case under constant, decreasing, or discounted valuations over time. In Section 3.1, we derive the optimal markdown prices when there are two buyers and identify the circumstances under which a markdown pricing strategy is more profitable for the seller than the optimal single price. In Section 3.2, we extend our analysis to \( N > 2 \) buyers and characterize the buyers’ equilibrium bidding strategies and use numerical examples to illustrate the structure of the optimal markdown.

While a complete information assumption may appear restrictive, we demonstrate that many of the properties of the optimal markdown and equilibrium bidding behavior of the buyers carry over to a more general incomplete information setting (Section 4). Under the incomplete information (IV) setting, each customer’s valuation is private information (drawn randomly from non-overlapping intervals), and the cumulative distribution (CDF) and probability density (pdf) functions from which it is drawn are common knowledge. As before, we examine the design of the optimal markdown mechanism but this time we consider markdowns where there is at most one price step in
any one valuation interval (to be referred to as INT markdowns). We then characterize the buyers’
equilibrium bidding behavior, and compare the performance of the optimal markdown with the
optimal single price policy. As was the case under complete information, we find that the optimal
markdown has very few steps. We find that the seller should never use more than three price
steps in an INT markdown; furthermore, customers continue to submit all-or-nothing bids at each
price step. We conclude with a discussion of managerial insights and future research directions in
Section 5.

1.1 Placement in Literature

Several branches of the economics literature analyze declining price mechanisms, considering (i)
myopic customers (Lazear, 1986; Pashigian, 1988; Pashigian and Bowen, 1991; Wang, 1993; Warner
and Barsky, 1995); (ii) rational/strategic customers (Aviv and Pazgal, 2004; Besanko and Winston,
1990; Harris and Raviv, 1981; Stokey, 1979); and (iii) quantity bid auctions (Carare and Rothkopf,
2001; Kagel and Roth, 1995; Katok and Kwasnica, 2002; Lucking-Reiley, 1999).

The assumption of myopic (non-strategic) customer behavior allows the seller to ignore the effect
of decreasing prices on customer purchases early on, which are detrimental to the seller’s revenue.
However, in many settings, customers act strategically (rationally) taking into account the future
path of prices when making purchasing decisions. In such cases, we need to incorporate customer
rationality or strategic behavior into the seller’s pricing decisions. Recently, Anderson and Wilson
(2003) investigated the impact of using standard revenue management techniques, which ignore
strategic behavior of customers, on airline revenues. They consider the case where the airline
sets protection levels based on expected marginal seat revenue assuming customers will behave
myopically and proceed to show how strategic customers can calculate sell-out probabilities at
different fare classes and make strategic purchasing decisions. They find that the loss of the airline
is higher when the expected demand/capacity ratio is low and when the number of times the
allocation decisions are reset during the sales horizon is low. They conclude that airlines should
explicitly consider strategic customer behavior when making pricing decisions. The bulk of the
operations management literature has focused on the optimal pricing policy of a seller who has a
limited supply (with a possibility of replenishment); Bitran and Mondschein (1997), Federgruen

One of the first papers that considers strategic (rational) customer behavior in a posted price
mechanism is (Stokey, 1979). She studies the case of a seller facing a fixed number of potential
customers and analyzes the optimal markdown structure. Customers have single unit demands
and the seller (monopolist) has unlimited capacity. Besanko and Winston (1990) extend Stokey’s
model by assuming that the seller can only make a finite number of price adjustments, i.e., the
price path is no longer continuous. As in (Stokey, 1979), customers want to purchase at most
one unit and the seller has unlimited capacity. These papers conclude that a markdown pricing
scheme is optimal. Furthermore, they find that the seller strictly prefers that the number of price
adjustments be as few as possible. Harris and Raviv (1981) also analyze a setting where the seller
faces a fixed number of strategic buyers with single unit demands and derives endogenously the
form of an optimal pricing mechanism given that the seller may have capacity constraints. They
assume that customers’ valuations are private information and find that the determining factor in
the structure of the optimal markdown is the seller’s capacity constraint. If capacity is exogenously
determined and exceeds market demand, then a single price is optimal. If capacity is less than the
total demand, then a markdown or Vickrey auction is optimal. If the seller can determine capacity
endogenously then the optimal action is to set capacity equal to the market demand and use a
single price mechanism.
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<td>Single unit</td>
<td>Complete</td>
<td>Incomplete (known common distribution)</td>
<td>Two step markdown mechanism</td>
</tr>
<tr>
<td>Pashigian (1988)</td>
<td>Identical to Lazear (1986), the same setting considered in a perfectly competitive market</td>
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<td>Pashigian, Bowen (1991)</td>
<td>Empirical study, no theoretical model for markdowns</td>
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<td>Gallego, van Ryzin (1994)</td>
<td>Multi-unit</td>
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<tr>
<td>Bitran, Mondschein (1997)</td>
<td>Multi-unit</td>
<td>Single unit</td>
<td>Incomplete (Poisson process with known intensity)</td>
<td>Incomplete (known common distribution)</td>
<td>Continuous time price function, not necessarily decreasing</td>
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<tr>
<td>Federgruen, Heching (1999)</td>
<td>Multi-unit</td>
<td>Aggregate level demand model</td>
<td>Incomplete (general stochastic function of price)</td>
<td>Individual customer valuations are not modeled explicitly</td>
<td>A fixed price for each period which is determined at the beginning of the period</td>
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Table 1: Summary of research on dynamic posted pricing mechanisms with myopic customers

Aviv and Pazgal (2004) study a model where the seller has a fixed initial inventory. Customers have single unit demands and they arrive randomly over time according to a Poisson process. The valuation of a customer who arrives at time $t$ is modeled by a known, deterministic decreasing function. Upon arrival customers decide whether to buy the product at the time of arrival, return later for a lower price, or not to buy at all. They first consider a model where the seller announces the price path and commits to it. They then consider an alternative model where the seller can choose the timing of when to announce the discount (assuming only one price change is allowed) and the discounted price considering the inventory level (optimal dynamic pricing policy with a single price change). They find that the benefits to the seller from employing this sophisticated strategy are minimal, and hence it is unlikely that the expected benefits merit its implementation. More recently, Su (2005) has investigated pricing in a deterministic setting where a monopolist faces a continuous arrival of customers with single unit demands who have low or high valuations and are myopic (who make a purchase or leave immediately) or strategic (stay in the system with the goal of maximizing their surplus through purchasing decisions). When supply is exogenous, whether the optimal policy is a markup or a markdown depends on whether the high or low type customers are relatively more strategic. When supply is endogenously determined, the optimal policy is either to use a single price or to set the price high until the very end of the sales horizon and then drop it to capture the strategic low types.

Wilson (1988) considers a setting where a monopolist faces a known downward sloping demand curve, comprised of a fixed and large number of customers who arrive in random order. Before any of the customers arrive, the seller puts a separate price tag on each unit for sale; these are fixed prices that remain valid until the item is sold. Customers may desire more than one unit, and, upon arrival, will purchase the units at the lowest available price, provided that the marginal benefit of doing so is positive. Although he does not consider a dynamic pricing markdown setting, Wilson (1988) finds that the seller never needs to charge more than two prices to maximize his
revenues. The result we establish for a setting where prices are time-dependent, i.e., the price of each unit changes over time and customers are strategic, resonates with this result. In (Wilson, 1988), rational or myopic customers’ behavior would be identical, due to the static nature of the prices. The same is not true when prices of individual items change over time, as is the case in this paper.

Liu and van Ryzin (2005) consider a monopolist who uses a two-step markdown with exogenous prices in a market where \( N \) risk-averse customers each have a single unit demand. The seller’s goal is to maximize her profit by deciding on the initial stocking quantity, and implicitly determining the rationing at the second step. The main differences between (Liu and van Ryzin, 2005) and our work is that they ignore the strategic interaction among individual customers, assume that price is given and optimize quantity decisions and do not solve for an equilibrium but assume that the seller can commit to a price and quantity decision.

<table>
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<tr>
<th>Demand Structure</th>
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<tr>
<td>Stokey (1979)</td>
<td>Single unit Complete (known valuations over time)</td>
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<tr>
<td>Besanko, Winston (1990)</td>
<td>Single unit Incomplete (known common distribution)</td>
</tr>
<tr>
<td>Harris, Raviv (1981)</td>
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<td>Aviv, Pazgal (2004)</td>
<td>Single unit Complete (known valuations over time)</td>
</tr>
<tr>
<td>Su (2005)</td>
<td>Single unit Complete (known fixed valuations)</td>
</tr>
<tr>
<td>Liu, van Ryzin (2005)</td>
<td>Single unit Incomplete (known common distribution)</td>
</tr>
<tr>
<td>Our work</td>
<td>Multi-unit Complete (known valuations) and Incomplete (known common distribution) Cannot be associated with individuals</td>
</tr>
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Table 2: Summary of research on dynamic posted pricing mechanisms with strategic customers

Our work also departs from the papers above by considering a multi-unit demand setting. Tables 1 and 2 summarize relevant research with myopic and strategic customers, respectively. With the goal of complementing the papers from economics and operations management, we seek to characterize the optimal markdown mechanism when the seller has a fixed capacity, and faces a (fixed) number of rational customers who demand multiple units.

2 Model

We analyze the pricing decision of a seller (monopolist), who has \( K \) identical units of an item for sale. Seller’s starting inventory is assumed to be exogenously given; this would be the case, for example, if the \( K \) units were comprised of excess inventory for an end-of-season item. We consider mechanisms where the seller announces the prices and the inventory before the sales begin. (The assumption of pre-announced price is an innocuous one under complete information, since buyers will rationally solve for equilibrium prices. The same is not true under incomplete information.) We assume that the valuations of the buyers are constant over time and there is no discounting. (In Section 3, we show that our results easily extend to the case of discounting under complete information.) In addition, we assume that all the buyers are present at the start of the markdown and that each buyer remains until the markdown is over or until his entire demand is satisfied. All potential buyers are present at the start of the selling period and remain until all the units have been sold or their demand has been satiated. This implies that it is never optimal for the seller to increase prices over time, in contrast to pricing policies that could emerge if customers arrived stochastically over time, e.g. (Su, 2005). Hence, in our model the seller’s main decisions are the number of price steps and the price at each step.

The seller wishes to implement an \( m \)-step markdown mechanism with price \( p_k \) at step \( k \), where
The seller faces $N$ rational buyers with valuations (per unit) $v_1 > v_2 > \ldots > v_N$. Buyer $j$ wishes to purchase at most $D_j > 0$ units, $\forall j$, where $D_j, j = 1 \ldots N$, is common knowledge. (The assumption of known (or deterministic) demand is commonly used in the pricing literature in order to highlight the strategic interplay between pricing decisions and sales (Bass et al., 1994; Kalish, 1983; Smith and Achabal, 1998).) We define $D[p] = \sum_{\{j|v_j \geq p\}} D_j$ and refer to it as the ‘market demand’ at price $p$, i.e., $D[p]$ is the maximum possible demand at price $p$. At any (price) step $k$, buyer $j$ submits a quantity bid, $q_{jk}$, indicating the number of units he wishes to purchase at the current price, $p_k$. Let $\bar{q}_{jk}$ denote the quantity awarded to buyer $j$; note that $\bar{q}_{jk}$ may be smaller than $q_{jk}$ if the total bid quantity $(\sum_{j=1}^{N} q_{jk})$ is greater than the remaining inventory at step $k$. In that case, the seller uses the following random rationing rule: Randomly choose a bidder $j$ and assign him the minimum of $q_{jk}$ and the remaining inventory. If there are remaining units, again randomly choose another bidder, and repeat this procedure until the inventory is exhausted.

As noted by Nocke and Peitz (2005) and Elmaghraby et al. (2006), this random allocation rule is consistent with the situation when all markdown items are sold on a first-come-first-serve basis at the end of the season when all customers with unsatisfied demand return to the store at the markdown price.

Both the seller and the buyers are risk-neutral and want to maximize their expected profits. The expected profit (or surplus) of buyer $j$ is the difference between his valuation for the item and the purchase price, that is $\Pi_j = \sum_{k=1}^{m} (v_j - p_k) \bar{q}_{jk}, j = 1, \ldots, N$, where $\sum_{k=1}^{m} \bar{q}_{jk} \leq D_j$. The seller’s expected revenue (or profit) is given by $\Pi_S = \sum_{k=1}^{m} \sum_{j=1}^{N} p_k \bar{q}_{jk}$.

We make the following assumptions, commonly adopted in game-theoretic analysis, concerning buyer behavior:

A1. In the last price step, if a buyer is indifferent between purchasing and not purchasing (i.e., $v_j = p_m$ and the buyer’s expected profit is the same under both alternatives), the buyer prefers to purchase.

A2. If a buyer is indifferent between bidding his entire demand at step $k$ or $k' > k$, he prefers to bid at step $k$.

Observation 1. In an $m$-step markdown mechanism, all customers with $v_j \geq p_m$ bid all their remaining demand at step $m$, i.e., customers do not benefit from withholding any of their demand at the last price step.

3 Markdown Mechanisms under Complete Information (CI)

In designing the optimal markdown mechanism, a seller who has $K$ units for sale must answer the following questions: (1) How many price steps should there be? (2) What should the price be at each step? In this section, we address these questions under the complete information setting.

We call a markdown mechanism effective if it induces positive bids at each price step. Note that if the seller knows ex ante that a price step in a markdown mechanism will be ineffective, then she can remove that price step at which no bids will occur and still obtain the same revenue. We say that there is a scarcity of supply at price $p_k$ if the market demand at that price is higher than the number of units available, i.e., $D[p_k] > K$, or equivalently, if $p_k \leq p_c$, where $p_c$ is the clearing price.

Observation 2. If $p_k > p_c$ and the customers know $p_c$, then customers have no incentive to buy at any step $i < k$, i.e., at any price higher than $p_k$.

This result is quite intuitive, since any customer $j$ can postpone his purchases until price step $p_k$, without the risk of facing a scarcity in supply and receiving less than $D_j$. In particular, if
$D[p_m] \leq K$, then customers can postpone their purchases until the lowest price step $p_m$. In such a case, the markdown mechanism is no different than a single price mechanism with price $p_m$.

Based on Observation 2, a markdown mechanism cannot be effective if $D[p_m] \leq K$. Similarly, the seller would have no need for a markdown if $D_1 > K$ since she could sell all of her units to customer 1 at a price of $p = v_1$. Therefore, as we search for the optimal markdown and markets in which it may be an appropriate pricing strategy, it is sufficient to focus on market settings and markdowns satisfying the following conditions.

**A3-CI.** $D_1 < K$, $D[v_N] > K$, i.e., the total market demand exceeds supply, and $D[p_m] > K$, i.e., there is scarcity at the lowest price step.

Two markdown pricing mechanisms are said to be *equivalent* if the mechanisms yield the same profits (for both the buyers and the seller). A markdown mechanism $M$ is said to *dominate* a markdown mechanism $L$, if $M$ generates more (expected) profits for the seller than $L$.

**Theorem 1.** For any $m$-step markdown mechanism, $m > 2$, there exists a 2-step markdown mechanism which dominates or is equivalent to it, if the customers know the clearing price $p_c$.

**Proof:** Consider an $m$-step mechanism with prices $p_1, \ldots, p_m$.

Case 1: $p_c \geq p_1$. In this case, the maximum profit the seller can obtain is $p_1K$. Consider an alternative markdown with two steps, where $p_1 = p_c + \epsilon$ ($\epsilon \rightarrow 0$ can be thought of as the minimum possible price increment), and $p_2 = p_c$. Under this two-step markdown, the minimum profit of the seller is $p_cK \geq p_1K$. Hence, the seller is no worse off, and may be better off under the alternative two-step markdown.

Case 2: $p_c \leq p_m$. Under the $m$-step markdown, the seller’s profit is given by $p_mK$ (by Observation 2). The same level of profit can be obtained by a two-step markdown mechanism with prices $p_1 = p_{m-1}$ and $p_2 = p_m$.

Case 3: $p_1 > p_c > p_m$. Let $p_k$ be the largest price where $p_k \leq p_c$. If $k > 2$, i.e., $D[p_i] \leq K \forall i < k$, buyers have no incentive to buy until $p_{k-1}$ since they do not expect a scarcity of supply before the price reaches $p_k$. Hence, a markdown mechanism with prices $p_{k-1}, \ldots, p_m$ would result in the same profit for the seller (Observation 2). Now suppose that $k < m$, i.e., $D[p_i] > K, i = k, \ldots, m$. (Note that the cases $k > 2$ and $k < m$ are not necessarily mutually exclusive.) We claim that the seller would be better off by eliminating the last price step $p_m$. Since $D[p_i] > K \forall i \leq p_k$, eliminating $p_m$ does not decrease the number of units sold. We also need to show that the seller is guaranteed to sell all $K$ units at the same or higher prices than with the $m$-step markdown. Customers originally bidding at step $m$ may bid at higher price steps or they may not bid at all in the new markdown with fewer steps. If they do not bid in the new mechanism, then the competition at the higher price levels is unaffected. On the other hand, if they bid at higher price steps, then the competition can only increase since there would be more customers bidding at a price step. As a result, at any price step $j < m$, the competition either increases or remains the same, preventing the seller’s revenue from decreasing. Hence, the revenue of the seller is not made any worse by eliminating the lowest price step from the markdown. Using these arguments repeatedly, any $m$-step markdown mechanism can be reduced to a 2-step markdown mechanism with prices $p_{k-1}, p_k$ yielding the seller equal or higher revenue. □

Although based on a different market model, Theorem 1 parallels some of the earlier results in the literature. Wilson (1988) analyzes a static pricing mechanism that assigns prices for each unit in the inventory. He shows that the seller never needs to charge more than two prices to maximize revenues. Besanko and Winston (1990) find that when customers demand at most one unit, a seller’s expected profit increases as the number of price adjustments decreases. This is because as
the number of possible price adjustments increases, so does the customers’ price elasticity, which dampens the seller’s ability to exercise his market power. We find a similar result for the multi-unit demand case; fewer markdowns are preferred to multiple markdowns. Feng and Gallego (1995) study the problem where a set of allowable prices, as well as the initial price, are given and the goal is to find the optimal timing of the price change. Gallego and van Ryzin (1994) study a similar but more general problem than in (Feng and Gallego, 1995) where the optimal initial price and the price path have to be chosen from a discrete set of allowable prices. They find that under the assumption of a deterministic demand function, the optimal solution has two price points: some $p_k^*$ for a specified period of time and a neighboring price $p_{k+1}$ for the rest of the selling season. We find a similar result in the case of strategic customer behavior and multi-unit demands.

Remark: The results in Observation 2 and Theorem 1 only require that the customers know the clearing price $p_c$, i.e., the informational requirements are less restrictive than CI.

Theorem 1 allows us to narrow our search for the optimal markdown from a general $m$-step mechanism to a specific 2-step markdown mechanism under CI. Before we can characterize the optimal 2-step markdown, we need to analyze the customers’ bidding behavior. For expositional ease, we state our main results in the paper and relegate most of the proofs to the online Appendix.

**Theorem 2.** Under a 2-step markdown mechanism, it is a dominant strategy for buyer $j$, $j = 1, \ldots, N$, to submit an all-or-nothing bid, i.e., to submit either all or none of his demand at a price step.

The proof of Theorem 2 is comprised of solving for the buyers’ best response bidding strategies. We show that the expected profit of a buyer as a function of his bid quantity at any step is convex, and is maximized at one of the boundary points regardless of its opponents’ bids. Therefore, submitting all-or-nothing bids is a dominant strategy. The result also holds if the customer valuations decrease over time (i.e., due to a discount factor $0 \leq \delta \leq 1$ or customer $j$’s valuation drops to $\delta v_j$ in step 2, $\forall j$). Furthermore, we will show in Theorem 7 that this result carries over to setting when there is incomplete information about customer valuations (Section 4).

Summarizing our results so far: (1) When $p_c$ is known, as is the case under CI, it is sufficient for the seller to focus on 2-step markdowns, since any additional price steps in the markdown will not improve profits. (2) Under a two-step markdown $(p_1, p_2)$, a buyer will either submit all of his demand at $p_1$ or at $p_2$. Hence we have already answered two out of our initial three questions. The question that remains is; what are the optimal prices in an effective markdown? (Recall that an effective markdown induces positive bids at each price step. If a 2-step markdown is ineffective at one of its prices, then it is equivalent to and can be replaced by a single price policy.)

**Observation 3.** In a 2-step markdown mechanism, the optimal price for the second step, $p_2^*$, is equal to the valuation of some customer, i.e., $p_2^* = v_j$ for some $j$.

Next, we characterize the optimal $p_1$.

**Observation 4.** In an optimal effective 2-step markdown mechanism, the prices satisfy $D[p_1] < K$ and $D[p_2] > K$.

**Proof** Consider a 2-step markdown mechanism $(p_1, p_2)$ where $D[p_1] \geq K$. Consider an alternative mechanism $(\tilde{p}_1, \tilde{p}_2)$ where $p_2 = p_1$ and $\tilde{p}_1 > p_1$ such that $0 < D[\tilde{p}_1] < K$. The maximum profit the seller can obtain with prices $p_1$ and $p_2$ is $Kp_1$. Conversely, with prices $\tilde{p}_1$ and $\tilde{p}_2$, the minimum profit the seller can obtain is $Kp_1$. The fact that $D[p_2] > K$ follows directly from A3-CI. □
Observation 4 implies that in an effective optimal markdown, there is no scarcity at the high price but there is scarcity at the low price, motivating the high types to buy at \( p_1 \).

To find the optimal value of \( p_1 \), we must first identify the buyers’ subgame perfect Nash equilibrium (SPNE) bidding strategies. In Section 3.1, we focus on the case of two customers, characterize the equilibria in a 2-step optimal markdown, and compare the markdown mechanism with the optimal single price in terms of seller’s profits. In Section 3.2, we extend some of these results to multiple customers.

### 3.1 Two Customers

In this subsection, we consider a seller who faces two customers \( (N = 2) \) with valuations \( v_1 > v_2 \) and demands of \( D_1 \) and \( D_2 \) units. Following A3-CI and observations 3 and 4 we assume that \( D_1 < K \) and \( D_1 + D_2 > K \) and that the seller employs a markdown satisfying \( D[p_1] < K \) and \( D[p_2] > K \). Without loss of generality we also assume \( D_2 \leq K \) because additional demand above \( K \) from customer 2 does not further increase the competition at step 2. Hence, we focus our attention on markdown mechanisms that satisfy the following condition: \( v_1 > p_1 > v_2 = p_2 \).

Obviously, \( p_1 \) should be within the range \((v_1, v_2)\) if the markdown mechanism is to be effective. If \( p_1 > v_1 \), neither buyer will submit a positive bid at \( p_1 \); if \( p_1 = v_1 \), the high valuation buyer is made strictly better off waiting and submitting all of his demand at \( p_2 \), rendering the 2-step markdown mechanism ineffective. Borrowing from the language in (Engelbrecht-Wiggans and Kahn, 1998), we label such an equilibrium to be a pooling mechanism ineffective.

We next characterize the set of prices that induce customer 1 to bid at \( p_1 \).

**Proposition 3.** Given a markdown mechanism with prices \( p_1 \) and \( p_2 = v_2 \), customer 1 bids his entire demand at step 1 if and only if \( p_1 \leq \hat{p}_{C1}(v_1, v_2) \) where

\[
\hat{p}_{C1}(v_1, v_2) = p_2 + (v_1 - v_2) \frac{D_1 + D_2 - K}{2D_1}.
\]

**Proof.** Since we have \( p_2 = v_2 \), buyer 2 will bid all of his demand at step 2 (\( q_{21} = 0 \) and \( q_{22} = D_2 \)). From Theorem 2, buyers submit all-or-nothing bids; hence, buyer 1 has two options: (a) Bid \( D_1 \) at step 1 and get a guaranteed surplus of \( \Pi_{11} = D_1(v_1 - p_1) \), or (b) bid zero at step 1 and \( D_1 \) in step 2. If the buyer chooses option (b), with probability 0.5 he has the priority in the random allocation and gets \( D_1 \); similarly, with probability 0.5, buyer 2 has the priority and buyer 1 gets \( K - D_2 \). This leads to an expected surplus of \( \Pi_{12} = \frac{D_1 + K - D_2}{2} (v_1 - v_2) \) for buyer 1 under option (2). Buyer 1 bids in step 1 if and only if \( \Pi_{11} \geq \Pi_{12} \) i.e., \( p_1 \leq \hat{p}_{C1}(v_1, v_2) \). \( \square \)

Customer 1’s expected unmet demand is \( \frac{D_1 + D_2 - K}{2D_1} \) if he bids at step 2. The expression \( \frac{D_1 + D_2 - K}{2D_1} \) in Equation (1) denotes the ratio of customer 1’s expected unmet demand to \( D_1 \), if he chooses to bid at \( p_2 \). Hence, \( (v_1 - v_2) \frac{D_1 + D_2 - K}{2D_1} \) is premium charged by the seller, which is the maximum additional amount (above \( p_2 = v_2 \) per unit customer 1 is willing to pay to secure his demand by bidding at \( p_1 \)). Table 3 summarizes all possible outcomes of a markdown mechanism with prices \( p_1 \) and \( p_2 = v_2 \).

**Proposition 4.** Under an effective markdown, the optimal price at step 1 is \( p_1^* = \hat{p}_{C1}(v_1, v_2) \).

The proof of Proposition 4 follows by showing that the profit function of the seller is strictly monotonically increasing (in \( p_1 \)) up to \( \hat{p}_{C1}(v_1, v_2) \), after which it sharply drops to \( v_2K \) and remains
constant. Hence, this break-point \( \hat{p}_{CI}(v_1, v_2) \) is the optimal price \( p_1^* \) at which customer 1 is indifferent between buying at step 1 or 2. By our earlier assumption (A2) customer 1 bids his entire demand at step 1. The optimal first step price, \( p_1^* \), increases with \( v_1, v_2, D_1, \) and \( D_2 \) and decreases with \( K \). These dynamics indicate that price increases as the customers’ demand or willingness to pay increases, and decreases if the supply increases.

**Note:** Recall that multi-unit demands is an important generalization in our model compared to the previous literature. One may question whether our results trivially follow if we were to replace a single customer who demands \( D_1 \) units with \( D_1 \) customers who each demand one unit. In the case of two customers, we showed that the optimal first step price is \( p_1^* = v_1 - (v_1 - v_2) \frac{D_1 + K - D_2}{2D_1} \) (by rearranging the terms in Equation (1)). Consider instead a setting where there are \( D_1 \) customers with single unit demand and a valuation of \( v_1 \), and \( D_2 \) customers with single unit demand and a valuation \( v_2 \). If the buyer uses a random allocation rule at price \( p_2 \), then the optimal first step price that induces all \( v_1 \) customers to purchase at that price is given by \( p_1^{\text{single}} = v_1 - (v_1 - v_2) \frac{K - D_2 - v_2}{D_2 + 1} \).

As is evident from the optimal prices, setting prices for single unit demand is not identical to pricing for multi-unit demand. Furthermore, we can establish the relationship between \( p_1^{\text{single}} \) and \( p_1^* \). We find that \( p_1^{\text{single}} > p_1^* \) and hence the seller’s revenue is higher from single unit demand customers when \( \frac{K - D_2 + 1}{D_2 + 1} < \frac{D_1 + K - D_2}{2D_1} \); From A3-CI, this condition reduces to \( D_1 > \frac{D_2 + 1}{2} \). Hence, our results for the multi-unit demand case require a separate analysis and do not follow by a transformation of the model to the case of multiple customers with single-unit demands.

**Extensions:** Our results easily extend to the case of decreasing valuations over time (or discounting). Suppose customer \( j \)'s valuation drops to \( \delta v_j \) in step 2, \( 0 < \delta \leq 1 \). In this case the threshold step 1 price becomes:

\[
\hat{p}_{CI}(v_1, v_2, \delta) = (1 - \delta)v_1 + \delta v_2 + \delta(v_1 - v_2) \frac{D_1 + D_2 - K}{2D_1}
\]  

If \( \delta = 1 \), i.e., no discounting, then Equation (2) is equivalent to Equation (1). If \( \delta = 0 \), i.e., the customer receives no value from the good if he waits for the second step, then \( \hat{p}_{CI}(v_1, v_2, \delta) = v_1 \). If \( \delta \in (0, 1) \), we have \( \frac{\partial \hat{p}_{CI}(v_1, v_2, \delta)}{\partial \delta} = -(v_1 - v_2) \frac{D_1 + K - D_2}{2D_1} \), which is always negative (since \( D_2 \leq K \) without loss of generality). Hence, the threshold step 1 price is decreasing in \( \delta \), implying that customer 1 is willing to pay a higher premium in the first step, if his value for the good is less in the second step. If the customers are risk-averse with constant valuations over time, similar observations as in the case of discounting hold, since the portion of the revenue generated in the second step of the markdown will be discounted in a similar way due to risk aversion.

### 3.1.1 Comparing Markdown and Single Price Mechanisms

We compare the performance of our markdown mechanism to that of the commonly used optimal single price mechanism. The structure of the optimal single (monopoly) price is well-studied in the economics literature (Kreps, 1990; Varian, 1992). For the sake of completeness, we restate the following observations about the optimal single price \( p^* \) when the seller faces discrete demand.

<table>
<thead>
<tr>
<th></th>
<th>Non-Pooling</th>
<th>Pooling</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1 \leq \hat{p}_{CI}(v_1, v_2) )</td>
<td>((D_1, 0))</td>
<td>((0, 0))</td>
</tr>
<tr>
<td>( p_1 &gt; \hat{p}_{CI}(v_1, v_2) )</td>
<td>((0, D_2))</td>
<td>((D_1, D_2))</td>
</tr>
</tbody>
</table>

Table 3: Optimal quantity bids of the customers under a 2-step markdown mechanism with \( p_2 = v_2 \).
Observation 5. *In the CI setting, \( p^* \in \{v_1, \ldots, v_N\} \).*

Based on this observation, we can easily compute the optimal single price \( p^* \) as follows: Recall that \( D[v_j] \) denotes the market demand if the price is set equal to \( v_j \), and let \( \Pi^*_S[j] = \min\{D[v_j], K\}v_j \) denote the corresponding total revenue (profit) of the seller. Then, \( p^* = \arg \max_{v_i} \Pi^*_S[j] \). Note that \( p^* \) can be higher than the clearing price \( p_c \), where \( p_c \) is the highest price at which the market demand exceeds supply.

Since the optimal single price is equal to one of the customer valuations, the seller has two choices for the single-price \( p; v_1 \) or \( v_2 \).

If \( p = v_1 \), only the high valuation customer can afford to buy at \( v_1 \), and the seller effectively excludes the low valuation customer from the market. We denote this alternative with (SP1) and it corresponds to a revenue of \( v_1D_1 \). On the other hand, if \( p = v_2 \) then both customers can afford the product. We call this (SP2) and it yields a revenue of \( v_2K \) for the seller.

The seller chooses (SP1), if \( v_1D_1 > v_2K \), and chooses (SP2), otherwise.

Observation 6. *If (SP2) is the optimal single price mechanism, then the markdown mechanism with prices \( (p^*_1, v_2) \) dominates the optimal single price.*

Observation 7. *If (SP1) is the optimal single price mechanism, then the markdown mechanism with prices \( (p^*_1, v_2) \) dominates the optimal single price if and only if \( (v_1 + v_2)D_1 < (v_1 - v_2)D_2 + K(3v_2 - v_1) \), i.e., \( \frac{v_2}{v_1} > \frac{K + D_1 - D_2}{3K - (D_1 + D_2)} \).*

Observation 6 is quite intuitive: The seller’s profits under the markdown mechanism will be at least \( v_2K \), if both customers were to buy in step 2. But since the optimal markdown is effective, customer 1 will buy in step 1 at price \( p^*_1 > v_2 \), increasing the seller’s profits above \( v_2K \).

When (SP1) is the optimal single price (Observation 7), we find that a two-step markdown mechanism is more likely to dominate the optimal single price when the following are true: (1) \( D_2 \) is very large relative to \( D_1 \), (2) the valuations of the two customers are close, and/or (3) \( K \) is close to the total demand of both customers, \( D_1 + D_2 \). Conversely, SP1 will outperform the optimal markdown if these three conditions are not met. As conditions (1)-(3) become stronger, the optimal single price will switch to (SP2), and a markdown will always be optimal. Figure 6 in the Appendix plots the seller’s revenues under the optimal markdown, and illustrates when a markdown or a single price is optimal.

### 3.2 Multiple Customers

In this section, we generalize some of our earlier results and insights to multiple customers. It is important to recall that Theorems 1 and 2 are general results for any \( N \geq 2 \) customers, i.e., the seller can restrict her search for the optimal markdown to a 2-step markdown and customers submit all-or-nothing bids in equilibrium. A consequence of this is that, as opposed to the case of two customers, we can no longer hope to design a markdown where each customer bids at a different price step. The best that we can achieve under an effective markdown is a partitioning of customers into three groups, with the first group purchasing all of their demand at \( p_1 \), the second group purchasing at \( p_2 \), and the third group (possibly an empty set) not making any purchases.

Following Observation 4, in this section we consider markdown mechanisms \( (p_1, p_2) \) with \( D[p_1] < K \) and \( D[p_2] > K \), i.e., customers are guaranteed to receive their bid quantities in the first step and there is scarcity in the second step. Hence, let \( v_n, n \leq N \), be the smallest valuation greater than or equal to \( p_2 \). Let \( S_3 = \{n + 1, \ldots, N\} \) be the set of customers with valuations less than \( p_2 \). In
equilibrium, the customers in \( S_3 \) bid nothing at either price step. Therefore, in the remainder of this section we focus on customers \( 1, \ldots, n \).

**Theorem 5.** Consider a markdown mechanisms \((p_1, p_2)\) with \(D[p_1] < K\) and \(D[p_2] > K\). Given the conditions \((C1)\) and \((C2)\) below, if there exists a partition \( \{S_1, S_2\} \) of customers \( \{1, \ldots, n\} \) such that \((C1)\) is satisfied for all \( t \in S_1 \) and \((C2)\) is satisfied for all \( s \in S_2 \), then the following bidding strategy is the (subgame perfect) Nash equilibrium for customers \( 1, \ldots, n \).

**Equilibrium Strategy (ES):**

1. If \( i \in S_1 \), bid \( D_i \) at step 1 and nothing at step 2.
2. If \( i \in S_2 \), bid nothing at step 1 and \( D_i \) at step 2.

where

\[
D_i(v_t - p_1) \geq (v_t - p_2) E[A_i] \forall t \in S_1 \tag{C1}
\]

\[
D_s(v_s - p_1) < (v_s - p_2) E[A_s] \forall s \in S_2 \tag{C2}
\]

and \( E[A_j] \) is the expected allocation of customer \( j \) in step 2 if all customers (except \( j \)) bid according to ES and \( j \) bids his entire demand at step 2. We have

\[
E[A_j] = \frac{1}{n!} \sum_{\pi \in U} \min \left\{ (K - \sum_{i \in B_j^k} D_i - \sum_{i \in S_1 \setminus \{j\}} D_i^+ + D_j) \right\},
\]

where \( U \) denotes set of all permutations of \( \{j\} \cup S_2 \) and \( B_j^k \subseteq S_2 \) denotes the set of customers whose bid quantities are satisfied before customer \( j \) in step 2 in some permutation \( \pi \in U \).

Theorem 5 formally states how customers would determine at which step to bid by comparing their expected profits from bidding at step 1 versus step 2. Note that it is possible to have a partition where \( S_1 \) is empty, i.e., all customers bid at step 2.

In general, we would be interested in situations where “high” types bid at step 1 and “low” types bid at step 2. Hence, we would like to determine conditions under which \( S_1 = \{0, 1, \ldots, j\} \) and \( S_2 = \{j + 1, \ldots, n\} \) for some \( j \geq 0 \) (where 0 denotes a dummy customer who always bids at step 1 and \( S_1 = \{0\} \) indicates that all customers bid at step 2). Proposition 6 provides a sufficient condition for such a high-low partitioning of customers to exist.

**Proposition 6.** Consider a markdown mechanisms \((p_1, p_2)\) with \(D[p_1] < K\) and \(D[p_2] > K\). If \( D_1 \geq D_2 \geq \ldots \geq D_n \), then there exists a partitioning of customers into \( \{S_1, S_2\} \) such that \( S_1 = \{0, \ldots, j\} \) and \( S_2 = \{j + 1, \ldots, n\} \) for some \( 0 \leq j < n \) and customers in \( S_i \) bid at step \( i \), \( i = 1, 2 \), in equilibrium.

The search for the optimal prices is equivalent to searching for the optimal sets \( S_1, S_2, \) and \( S_3 \). If the seller wants to induce a particular partition \( \{S_1, S_2, S_3\} \) in equilibrium, using conditions \((C1)\) and \((C2)\), she first needs to characterize the range of prices where the optimal \( p_1 \) falls, for a given \( p_2 \). Given its feasible range, the seller can then determine \( p_1 \) as a function of \( p_2 \). From Observation 3, we know that \( p_2 = v_k \) for some \( k \leq N \). The seller can find the optimal price pair \( \{p_1, p_2\} \) by searching over all valuations for \( p_2 \) (in at most \( N \) steps) and the corresponding optimal \( p_1 \).

The proposed search method for finding the optimal price pair would work efficiently if the seller only needs to consider a reasonable number of partitions and if \( p_1 \) can be found easily for a given \( p_2 \). For example, when \( D_1 = \ldots = D_N = D \) and \( K = rD \) (where \( r \) is a positive integer), the seller only needs to consider partitions \( S_1 = \{0, \ldots, j\} \) and \( S_2 = \{j, \ldots, k\} \) for \( j \leq r < k \leq N \) (from
Proposition 6); hence, the optimal $p_1$ can be found in at most $r$ steps for each possible $p_2 = v_k$, $k > r$. As a result, the seller can identify the optimal markdown efficiently in $O(N^2)$ time, which is polynomial in the number of customer types. The price range for $p_1$ that induces the partition \( \{S_1, S_2\} \) is \( \left( v_{j+1} - (v_{j+1} - p_2) \frac{n-r}{n-j+1}, v_j - (v_j - p_2) \frac{n-r}{n-j+1} \right) \). It is important to note that this range may be empty, implying that the partition \( \{S_1, S_2\} \) cannot be supported in equilibrium under any price $p_1$. For any partition supportable in equilibrium under equal demands, the seller’s profits are maximized by setting $p_1$ as follows:

\[
p_1 = p_2 + (v_j - p_2) \frac{n-r}{n-j+1} \tag{3}
\]

This has an interpretation similar to the optimal $p_1$ for the two customer setting given in Equation (1). The expression $\frac{n-r}{n-j+1}$ in Equation (3) denotes the ratio of customer $j$’s expected unmet demand to his total demand, if he chooses to bid at $p_2$. Hence, $(v_j - p_2) \frac{n-r}{n-j+1}$ is the maximum additional amount (above $p_2$) per unit customer $j$ is willing to pay to secure his entire demand by bidding at $p_1$.

The choice of $p_2$ determines which customers are excluded ($S_3$) and included ($S_1$ and $S_2$) in the market. There are two countervailing forces working in the selection of $p_2$. A lower $p_2$ increases the seller’s market (total demand), and hence increases the scarcity at $p_2$, providing the incentive for a high type customer to bid at $p_1$. But a lower $p_2$ also implies reduced revenues for the seller from the units sold at that step and provides an incentive for a high type to bid at $p_2$.

### 3.3 Numerical Experiments

To gain insights on the structure and the performance of the optimal markdown mechanism, we conducted numerical experiments. We examined how the optimal markdown prices vary according to the distribution of valuations and the supply level (Figure 1(a)-(c)) and the conditions under which markdown revenues exceed single price revenues (Figure 1(d)). In our first set of experiments, we assumed that customers are drawn from two groups, high and low valuation customers. Hence, we randomly selected customer valuations from a bimodal distribution, where half of the customers have valuations drawn from a uniform distribution over $[100, 100 + \delta]$, and the remaining half have valuations drawn from a uniform distribution over $[200 - \delta, 200]$. Such clustering of valuations allows us to model multiple customers while keeping a link to the two-customer case for comparison purposes. Note that as $\delta$ increases, the difference between the valuations of the two customer groups decreases. We ran the experiments with 10 different values of $\delta = 5, 10, 15, ..., 50$. For each $\delta$ value, we tested fifty instances with 20 customers, each with a demand of 10 units; hence, the total demand in the market is $20 \times 10 = 200$. We examined six different supply scenarios where the seller has enough supply to meet $Q\%$ of the total demand in the market (i.e., $K = \frac{SD}{100}$), where $Q \in \{30, 40, 50, 60, 70 and 80\}$. Figure 1 presents the properties of the optimal markdown averaged over 50 randomly drawn instances.

From Figure 1(a), we observe that as $\delta$ increases, i.e., the expected customer valuations get closer to each other, $p_1$ decreases for all supply levels. The effect of $\delta$ on $p_2$ and the depth of the markdown critically depend on the supply level (Figures 1(b) and (c)). When supply levels are low ($Q < 50$), we find that $p_2$ is set very high for $\delta = 5$ and then decreases in $\delta$; in addition, the depth of the markdown $\frac{p_1-p_2}{p_1}$ increases in $\delta$. The opposite is true when supply levels are moderate to high ($Q \geq 50$) where $p_2$ is set low for $\delta = 5$ and both $p_2$ and the depth of the markdown decrease in $\delta$. It is interesting to note that the depth of the markdown is the greatest for $Q = 50$. However, when $\delta = 50$, the depth of the average markdown is between 7-10% for all supply levels. When $\delta = 50$, the customers’ valuations are generally evenly distributed across the entire range.
[100,200] - the distinction between a ‘high’ and ‘low’ types becomes weaker. As a result, inducing a separation between these two groups requires a smaller price difference, which implies that the optimal markdown is almost revenue equivalent to the optimal single price (Figure 1(d)). It is worth pointing out that while the depth of the markdown if fairly consistent across supply levels (when $\delta = 50$), the actual price values do depend on $K$, as is clearly illustrated in Figures 1(a) and (b).

The interplay between $K$ and $p_2$ is a very interesting one. Recall that as the seller decreases $p_2$, she increases the number of customers who are able to purchase at $p_2$ and hence increases the ‘scarcity’ at $p_2$. All else equal, an increase in scarcity at $p_2$ creates greater incentives for a high valuation customer to purchase at $p_1$, while a reduction in $p_2$ increases his incentive to purchase at $p_2$. When supply levels are low ($Q < 50$), the seller’s optimal balancing act between these two forces weighs in favor of a high $p_2$; while the seller includes only a few customers in the market at a high $p_2$, this allows her to keep $p_1$ high as well and maintain an effective markdown. As $K$ increases, the increase in supply induces the seller to include more customers in the market by decreasing $p_2$. If the seller were to decrease $p_2$ by a small amount, the scarcity level at $p_2$ would not be enough to support any purchases at $p_1$, i.e., the markdown would not be effective. Hence, the seller finds it optimal to decrease $p_2$ substantially. While she receives a lower revenue (per unit) from sales at $p_2$, the increased scarcity at $p_2$ allows her to support an effective markdown where $p_1$ is still significantly greater than $p_2$.

![Figure 1: Average optimal prices and revenues for each valuation distribution and supply scenario. Each data point represents the average of fifty instances.](image_url)
When we look at the difference between the revenues of markdown and single price mechanisms (Figure 1(d)), we see that whether or not a markdown dominates the optimal single price mechanism depends on supply levels as well as dispersion of valuations in the market. When supply levels are low \((Q < 50)\), the markdown and single price are almost equivalent for all \(\delta\). This is fairly intuitive since the depth of the markdown is very small, rendering it equivalent to a single price. When \(Q = 50\) the single price outperforms the markdown for all \(\delta\). When \(Q = 50\), the optimal single price is most likely to be the valuation of the 10\(^{th}\) customer, \(v_{10}\) (lowest valuation ‘high’ type).

However, \(p_2\) must be less than \(v_{10}\) under an effective markdown in order to guarantee scarcity at \(p_2\) and to induce some ‘high’ valuation customers to purchase at \(p_1\). As a result, the performance of the markdown suffers. When supply levels are higher \((Q > 50)\) a markdown yields higher revenues over a wide range of settings. The difference in revenues is a unimodal function of \(\delta\), and the markdown performs best when the depth of the markdown is between 20-25\% for all \(K\). The maximum percentage difference occurs at \(\delta = 25, 30,\) and \(40\) for \(Q = 80, 70,\) and \(60\), respectively.

In summary, we observe that a markdown is almost equivalent to a single price when supply levels are low, and that a markdown will dominate the optimal single price when either (i) the supply is moderate and \(\delta\) is high, or (ii) the supply is high and \(\delta\) is moderate. (Note that condition (i) is similar to the condition in Observation 7 for the case of two customers.)

### 4 Equilibrium Bidding Behavior and Mechanism Design under Incomplete Information (IV)

In the previous section, we answered the following questions for a complete information setting: (1) How will rational buyers bid when facing a markdown? (2) How many price steps are there in an optimal markdown? (3) Under what conditions would the seller be better off implementing a single price vs. markdown pricing? In this section we extend our analysis to an incomplete valuation information (IV) setting. Under IV, we assume that the valuation of customer \(j\) is drawn from \([v_j, \bar{v}_j]\) with probability distribution function \(F_j(\cdot), \forall j\), where \(v_j > \bar{v}_{j+1}\), i.e., valuations are drawn from non-overlapping intervals (Figure 2). Each customer knows his own valuation with certainty and both the seller and the customers know the pdf and CDF of other customers’ valuations.

First, we show that the all-or-nothing bidding result we had for 2-step markdowns under CI (Theorem 2) carries over to the IV setting and \(m\)-step markdowns.

**Theorem 7.** Under IV, in an \(m\)-step markdown with prices \(p_1 > \ldots > p_m\) customers submit all-or-nothing bids in equilibrium.

In Section 3 we showed that under CI, a 2-step markdown was sufficient to maximize the seller’s revenue (Theorem 1). This result relies on the fact that buyers know the clearing price, \(p_c\), under CI. When the customer valuations are unknown, \(p_c\) may not be known with certainty, hence, we cannot claim that a 2-step markdown is always optimal. However, we can easily determine in which customer’s valuation interval \(p_c\) falls and use this information to gain insights on the properties of the optimal markdown. Let \(k\) be such that \(\sum_{j=1}^{k-1} D_j < K\) and \(\sum_{j=1}^{k} D_j \geq K\). That is, \(p_c \in [\bar{v}_k, \bar{v}_k]\).

**Observation 8.** In an effective optimal \(m\)-step markdown

(i) there is at most one price exceeding \(\bar{v}_k\)

(ii) there is at most one price less than \(\bar{v}_k\) and it is in the interval \([v_j, \bar{v}_j]\) for some \(j \in \{k, \ldots, N\}\)

(iii) all other intermediate prices occur in the range \([v_k, \bar{v}_k]\).

Observation 8 states that in an effective optimal markdown, there exists at most one price \((p_1)\) at which the buyers are guaranteed to receive their bid quantities. In addition, the lowest price
$p_m$ is in the valuation interval of some customer $j \geq k$. Note that we cannot simply replace the lowest price $p_m$ with $\bar{v}_k$ (or any other higher price) without potentially having an adverse effect on seller’s revenue (i.e., increasing the lowest price may reduce the competition at the last step, which could prevent customers who were originally bidding at higher prices from doing so). All other (intermediate) prices must occur within the interval $[\bar{v}_k, \bar{v}_k]$, which is the interval containing the clearing price (Figure 2). Note that $p_1$ and $p_m$ may themselves fall within $[\bar{v}_k, \bar{v}_k]$.

![Figure 2: Candidate structure for price steps in an effective optimal $m$-step markdown under IV.](image)

Since all intermediate prices must occur in the interval $[\bar{v}_k, \bar{v}_k]$, we do not know if a two, three, or $m$-step markdown is optimal. In B2C markets, we see that price reductions are typically rather large, e.g., a retailer does not drop the price of a $44$ item to $43$, but rather to $29$. This suggests that the retailer is trying to access a new customer group with each price reduction. Motivated by these practices, we next consider a narrower yet intuitive setting of markdown mechanisms, where there is at most one price step in each customer valuation interval. More formally, we consider a family of Interval markdowns, INT markdowns, defined as follows:

**INT Markdown** There is at most one price drawn from each interval $[\bar{v}_j, \bar{v}_{j+1}]$, $\forall j = 2, \ldots, N+1$, where $\bar{v}_{N+1} = \bar{v}_N$.

**Observation 9.** In an effective INT markdown, there are at most three price steps. Furthermore, if the markdown has three steps, then $p_1 > \bar{v}_k$, $p_2 \in [\bar{v}_k, \bar{v}_k]$, and $p_3 \in [\bar{v}_j, \bar{v}_j]$ for some $j \geq k$, where $[\bar{v}_k, \bar{v}_k]$ is the interval containing the clearing price.

The proof follows directly from Observation 8.

Next, we would like to understand when an optimal INT markdown will have two vs. three steps. While we are not able to answer this question in general, we state a sufficient condition for a 2-step INT markdown to be optimal.

**Observation 10.** If $p_c \in [\bar{v}_N, \bar{v}_N]$, then in an effective INT markdown, there are at most two price steps, where $p_1 > \bar{v}_N$ and $p_2 \in [\bar{v}_N, \bar{v}_N]$.

From Observation 10, when $D_1 + D_2 + \ldots + D_{N-1} \leq K$, in an $m$-step markdown customers are partitioned into two or fewer groups depending on whether or not they may bid at a given price step. For example, if $\bar{v}_j > p_1 > \bar{v}_{j+1}$, customers are partitioned into $\{1, \ldots, j\}$ and $\{j, \ldots, N\}$ where the first group may bid at $p_1$ whereas the second group can only bid at $p_2$.

**Corollary 8.** If $D_1 + D_2 + \ldots + D_{N-1} \leq K$, then a 2-step INT markdown is equivalent to or dominates any 3-step INT markdown.

It remains to characterize the optimal markdown. For the remainder of this section, we characterize the optimal markdown and compare its performance against the optimal single price when $N = 2$. As will quickly become clear, designing the optimal markdown under IV is quite complicated even when $N = 2$. 

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4.1 Two Customers under IV

Given a seller is facing two customers, an INT markdown has 2 steps. While we cannot guarantee that a 2-step markdown will be optimal, we believe this to be a reasonable framework for analysis given their common use in practice (Feng and Gallego, 1995), and the theoretical (Besanko and Winston, 1990; Gallego and van Ryzin, 1994) and empirical results (Feng and Gallego, 1995; Smith and Achabal, 1998) supporting their near-optimal performance under various market settings.

Following the conditions and the intuition laid out in A3-CI, we focus only on market settings where \( D_1 < K \) and \( D_1 + D_2 > K \), and markdown mechanisms where \( \bar{D}[p_1] < K \) and \( \bar{D}[p_2] > K \), i.e., customer 1 is guaranteed to receive his entire quantity bid if he should bid at the first price step, but may face scarcity if he bids at the second price step. These conditions imply that we are searching for the optimal INT markdown that satisfies: \( \bar{v}_1 \geq p_1 > \hat{v}_2 > p_2 \geq v_2 \).

From Observation 1, we know that customer 2 will bid his entire demand at step 2 with probability \( 1 - F_2(p_2) \), i.e., only if \( v_2 \geq p_2 \); otherwise, he does not bid at all. Furthermore, we know from Theorem 7 that customer 1 with valuation \( v_1 \) bids his entire demand at either \( p_1 \) or \( p_2 \). Proposition 9 summarizes customer 1’s bidding behavior under IV.

**Proposition 9.** Given a markdown mechanism with prices \( p_1 \) and \( p_2 \), customer 1 bids \( D_1 \) at step 1 if and only if \( p_1 \leq \hat{P}_{IV}(v_1, p_2) \), or equivalently, \( v_1 \geq \hat{v}(p_1, p_2) \); otherwise, he bids \( D_1 \) at step 2, where

\[
\hat{P}_{IV}(v_1, p_2) = p_2 + (v_1 - p_2)(1 - F_2(p_2)) \frac{D_1 + D_2 - K}{2D_1} \tag{4}
\]

\[
\hat{v}(p_1, p_2) = p_2 + \frac{(p_1 - p_2)}{1 - F_2(p_2)} \frac{2D_1}{D_1 + D_2 - K} \tag{5}
\]

A markdown mechanism \((\hat{P}_{IV}(v_1, p_2), p_2)\) makes customer 1 indifferent between bidding at steps 1 and 2. That is, \( \hat{P}_{IV}(\cdot) \) is the threshold step 1 price above which customer 1 will not bid at step 1. Note the similarity between the threshold step 1 price under incomplete (equation (4)) and complete information (equations (1) and (3)) settings. In equation (4), the term \((1 - F_2(p_2)) \frac{D_1 + D_2 - K}{2D_1}\) indicates the “scarcity” at step 2, i.e., it is the ratio of the expected unmet demand of customer 1 to his entire demand if he bids his entire demand at step 2. Since the scarcity increases in \( D_1 \) and \( D_2 \) and decreases in \( K \), the threshold step 1 price increases in \( D_1 \) and \( D_2 \), and decreases in \( K \). As in the CI setting, the second term of the threshold step 1 price is the premium the high type customer is willing to pay to secure his entire demand at step 1. Similarly, \( \hat{v}(p_1, p_2) \) is the threshold valuation below which customer 1 would not bid at step 1.

For a markdown not to be pooling, the threshold valuation should not exceed \( \hat{v}_1 \), or equivalently, the step 1 price should be below the threshold step 1 price for the highest possible type.

**Corollary 10.** If \( \hat{v}_1(p_1, p_2) > \hat{v}_1 \), equivalently, if \( p_1 > \hat{P}_{IV}(\hat{v}_1, p_2) \), then \( F_1(\hat{v}_1) = 1 \) and customer 1 never bids at \( p_1 \); that is, the markdown is always pooling.

From Corollary 10, a markdown mechanism where \( p_1 > \hat{P}_{IV}(\hat{v}_1, p_2) \) is always pooling (for all possible realizations of customer valuations), and hence, is equivalent to or dominated by the optimal single price. Therefore, in our search for the optimal non-pooling INT markdown, we focus our search to markdowns satisfying the following:

**A4-IV** \( \hat{P}_{IV}(\hat{v}_1, p_2) \geq p_1 > \hat{v}_2 > p_2 \geq v_2 \)

It is also interesting to know when all customer 1 types will bid at \( p_1 \). This happens when the threshold valuation is below the lowest customer 1 type \( v_1 \) (or equivalently, when the step 1 price is below the threshold step 1 price for the lowest customer 1 type).
Corollary 11. If \( \hat{v}_1(p_1, p_2) \leq \bar{v}_1 \), equivalently, if \( p_1 \leq \hat{p}_{IV}(\bar{v}_1, p_2) \), then \( F_1(\hat{v}_1) = 0 \) and every customer 1 type will bid at \( p_1 \).

When the threshold valuation for a given \((p_1, p_2)\) falls in \((\bar{v}_1, \bar{v}_1)\), then only a strict subset of customer 1 types will submit a quantity bid at \( p_1 \) (Figure 3).

Corollary 12. If \( \underline{v}_1 < \hat{v}_1(p_1, p_2) \leq \bar{v}_1 \), equivalently, if \( \hat{p}_{IV}(\underline{v}_1, p_2) < p_1 \leq \hat{p}_{IV}(\bar{v}_1, p_2) \) then \( 0 < F_1(\hat{v}_1) < 1 \) and customer 1 bids at \( p_1 \) if and only if \( \hat{v}_1(p_1, p_2) \leq v_1 \); that is, the resulting markdown is potentially separating and only some subset of customer 1 types will submit a quantity bid at \( p_1 \).

From Corollaries 11 and 12, the seller’s choice of \( p_1 \), for a given \( p_2 \), determines whether a markdown mechanism will induce all or only some customer 1 types to purchase at \( p_1 \), which we refer to as totally separating (TS) and potentially separating (PS) markdowns, respectively (see Figure 3). In a TS markdown, all customer 1 types bid at \( p_1 \) whereas in a PS markdown only customer 1 types within \([\hat{v}_1, \bar{v}_1]\) bid at \( p_1 \), with the remaining types bidding at \( p_2 \).

Given the customers’ equilibrium behavior, the seller wishes to design a markdown that maximizes his expected profits. The seller’s problem is significantly more complicated under an IV setting than a CI two-customer setting, for the seller must decide on both the optimal \( p_2 \) and the optimal \( p_1 \). Under IV, it is no longer clear that a seller can or should try to design a markdown that is TS (as was the case under CI). Interestingly, we find conditions/examples which illustrate that a seller may be better off with a PS markdown, whereby customer 1 will purchase at \( p_2 \) for some realizations of valuation. An additional challenge facing the seller is the choice of \( p_2 \). Under CI, \( p_2^* = v_2 \) always, and hence customer 2 always bids at the second price step. Under IV, in order to induce all customer 2 types to bid at \( p_2 \) the seller must set \( p_2 = \underline{v}_2 \). However, a higher \( p_2 \) may allow the seller to charge a higher \( p_1 \) and increase his profits. Note that this is somewhat similar to the problem faced by the seller under CI with \( N > 2 \) customers, i.e., the choice of \( p_2 \) determines which customers are excluded from and included in the market. Under IV the effect of increasing \( p_2 \) on the surplus of customer 1 is two-fold: the expected per unit surplus of customer 1 from bidding at step 2 decreases, but his expected allocation increases since the probability that customer 2 can bid at step 2 decreases. Therefore, \( \hat{p}_{IV}(\underline{v}_1, p_2) \) might be decreasing or increasing in \( p_2 \). We provide examples below that illustrate that it may be optimal to set \( p_2 > \underline{v}_2 \).

In solving the seller’s problem, we must

- identify the \( p_2 \) ranges for which TS and PS markdowns are feasible (i.e., exist satisfying A4-IV),
- identify the optimal TS and PS markdowns, namely, \((p_1^{TS}, p_2^{TS})\) and \((p_1^{PS}, p_2^{PS})\), respectively, over their respective feasible regions, and
- determine whether a TS or PS markdown or single price leads to higher (expected) revenues for the seller.

Figure 4 shows (for uniformly distributed valuations) how the existence of PS and TS markdowns depends on the “scarcity” of supply measured by \( \frac{D_1 + D_2 - K}{2D_1} \) (the ratio of customer 1’s expected
unmet demand to his entire demand, if both customers bid at step 2): if scarcity is low, it is more likely to see a pooling outcome. (Figure 4 is based on Results 14 and 15 presented in the online Appendix.)

\[
\begin{array}{cccccc}
(a) & (b) & (c) & (d) & (e) \\
\text{Pooling for all } p_2 & \text{Only PS exists for some } p_2 & \text{PS and TS exist for some } p_2 & \text{PS exists for all } p_2, \text{TS exist for some } p_2 & \text{PS and TS exist for all } p_2 \\
\overline{v}_1 - \overline{v}_2 & \overline{v}_2 - \overline{v}_1 & \overline{v}_2 - \overline{v}_1 & \overline{v}_1 - \overline{v}_2 & D_1 + D_2 - K \\
\end{array}
\]

Figure 4: Existence of INT markdowns satisfying A4-IV as a function of market parameters.

The analysis that goes into answering the three questions above is quite involved and lengthy. So as to not encumber the reader with these derivations, we relegate most of this analysis to our online appendix and provide our main results and insights below via numerical instances that are representative of the general results found in Section 4.3. Given the intractability of finding closed-form solutions for generic distribution functions in this setting, the analysis that follows assumes that the customer valuations are drawn from uniform distributions. Before we present our numerical examples, we characterize the optimal single price under IV.

4.2 Comparison to the Optimal Single Price

As in Section 3, we compare the performance of an INT markdown to that of the optimal single price. We first characterize the optimal single price as a function of the market setting when there are \( N \geq 2 \) customers.

**Observation 11.** In the (IV) setting, \( p^* \in [\overline{v}_j, \bar{v}_j] \) for some customer \( j \), \( j = 1, \ldots, N \).

Note that \( p^* \leq \overline{v}_1 \), otherwise the seller would not be able to sell any units. Assume \( \overline{v}_i > p^* > \bar{v}_{i+1} \) for some \( i < N \). The seller could increase her profits by setting \( p' = \overline{v}_i \), and sell the same number of units as she did at price \( p^* \), thereby contradicting the optimality of \( p^* \).

Using this observation, the seller can easily compute the optimal single price \( p^* \) as follows. Define the maximum potential market demand at price \( p \) as \( \bar{D}[p] = \sum_{i: \overline{v}_i \geq p} D_i \). Let \( p^{*i} \) be the best single price that falls within the valuation interval of customer \( i \), that is \( p^{*i} = \arg \max_{\overline{v}_i \leq p \leq \bar{v}_i} \Pi^0_{\bar{S}[i]} \) where \( \Pi^0_{\bar{S}[i]} = \{ \min\{\bar{D}[p], K\}(1 - F_i(p)) + \min\{\bar{D}[p] - D_i, K\}F_i(p) \} p \). Then the optimal single price is \( p^* = p^{*k} \) where \( \Pi^0_{\bar{S}[k]} \geq \Pi^0_{\bar{S}[i]} \) for all \( i \neq k \).

With \( N = 2 \), the structure of the optimal single price implies that the seller has two choices for the single-price \( p \):

**SP1** : Set \( p \in [\overline{v}_1, \overline{v}_1] \) excluding the low valuation customer from the market to receive a revenue of \( \Pi^0_{\bar{S}[1]} \).

**SP2** : Set \( p \in [\overline{v}_2, \bar{v}_2] \) and possibly sell to both customers and receive revenue \( \Pi^0_{\bar{S}[2]} \). It is important to point out the different interpretations of (SP1) and (SP2) under IV from CI when \( N = 2 \); Under IV, the seller cannot guarantee that customer 1 will purchase under SP1, provided that \( p > \overline{v}_1 \). A similar statement is true for customer 2 and (SP2).

**Proposition 13.** If (SP2) is the optimal single price mechanism with \( p^* \), then a TS or PS markdown mechanism, should it exists for \( p_2 = p^* \), dominates the optimal single price.
**Proof:** The seller’s profits under a markdown mechanism would be the same as $\Pi_S^0[2]$ if all customer 1 types purchased at $p_2$. However, a PS or TS markdown by design leaves some customer 1 type $\hat{v}_1(p_1, p^*)$ indifferent between buying in step 1 or 2. Consequently, a TS or PS markdown - with $p_2 = p^*$ and customer 1 types $v_1 \geq \hat{v}_1(p_1, p^*)$ purchasing in step 1 - yields the seller a higher expected revenue than $\Pi_S^0[2]$; ex post the seller’s expected revenue is bounded below by $\Pi_S^0[2]$.

Proposition 13 is quite intuitive and extends Observation 6 to IV. Note that if a PS or a TS markdown does not exist for $p_2 = p^*$, the optimal markdown revenue may not exceed the (SP2) revenue. When (SP1) is the optimal single price mechanism, we cannot make a theoretical comparison of the markdown and single price revenues since we do not have a closed form solution for the optimal markdown revenue in general. For additional insights on the comparison of markdown vs. single-price revenues, we turn to numerical examples.

### 4.3 Numerical Examples

In this section, we present representative numerical examples (see Table 4.3 for the parameters) to provide insights on the structure and the performance of optimal markdown mechanisms. We first present a base instance, solve for the optimal markdown, identify whether it is totally or partially separating, and compare its performance against the optimal single price.

**(Base Instance) Example 1:** TS optimal. In this setting, $p_2 = 2.41 > 2 = v_2$, i.e., with positive probability, customer 2 will not make a purchase. However, the optimal first price is to induce all customer 1 types to purchase at the first price step, i.e., $p_1 = \hat{p}_IV(12,2.41)$, and hence the optimal markdown in this instance is totally separating. Given the relatively large demand from customer 2 as compared to customer 1, the seller finds it optimal to set the optimal single price to $p^* = 2.76$.

A TS markdown exists for all $p_2 \in [2, 3)$ and we have $p^* = 2.76 \in [2, 3)$; hence, it follows from Proposition 13 that the optimal markdown yields higher profits than SP2.

<table>
<thead>
<tr>
<th>$K, (D_1, D_2), (\underline{v}_2, \overline{v}_2), (\underline{v}_1, \overline{v}_1)$</th>
<th>Markdown type</th>
<th>$\Pi(p_1, p_2)$</th>
<th>Single price</th>
<th>$p^*$</th>
<th>$\Pi_{p^*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1: 20, (3,19), (2,5), (12,18)</td>
<td>TS</td>
<td>(5.17, 2.41)</td>
<td>50.88</td>
<td>SP2</td>
<td>2.76</td>
</tr>
<tr>
<td>Example 2: 20, (3,19), (2,4), (12,18)</td>
<td>TS</td>
<td>(5.33, 2.00)</td>
<td>50.00</td>
<td>SP2</td>
<td>2.18</td>
</tr>
<tr>
<td>Example 3: 20, (3,19), (2,7), (12,18)</td>
<td>PS</td>
<td>(7.01, 3.00)</td>
<td>49.80</td>
<td>SP2</td>
<td>3.94</td>
</tr>
<tr>
<td>Example 4: 20, (3,19), (2,5), (12,23)</td>
<td>PS</td>
<td>(5.37, 2.41)</td>
<td>50.92</td>
<td>SP2</td>
<td>2.76</td>
</tr>
<tr>
<td>Example 5: 20, (8,19), (2,5), (12,18)</td>
<td>TS</td>
<td>(6.38, 2.00)</td>
<td>75.00</td>
<td>SP1</td>
<td>12.00</td>
</tr>
</tbody>
</table>

Table 4: Optimal markdown and single price for specific instances under IV

In each of the subsequent four examples, we alter one parameter from this base instance (highlighted in bold in Table 4.3). These examples are constructed to illustrate general market trends on how the existence and the optimal structure of the PS and TS markdowns may change, as well as the performance of a markdown when compared to the optimal single prices as market parameters vary. These examples were drawn from a larger numerical example set (see the Online Appendix), generated by increasing/decreasing one parameter at a time from the base instance, until one of the following assumptions of our model was violated.

- NP-IV is violated (i.e., INT markdowns do not exist).
- $D_1$ or $D_2$ exceeds $K$ (i.e., there are no leftover units after the first customer’s demand is satisfied).
- $K$ exceeds $D_1 + D_2$ (i.e., there is no scarcity in the market).
- The valuation interval of a customer reduces to a singleton (i.e., there is no uncertainty in valuations).

**Example 2: TS optimal.** Example 2 complements the base case presented above. Here the optimal markdown is also TS and it performs better than the optimal single price. However, unlike the base case, the shrinking support for customer 2’s valuations (by a decrease in $\bar{v}_2$) allows the optimal second price step under TS to decrease to $p_2 = \bar{v}_2 = 2$, i.e., all customer 2 types are now guaranteed to bid at $p_2$. Since the optimal markdown remains a totally separating one, both customers’ bids are independent of their valuation realization and the seller receives a guaranteed revenue. A decrease in $\bar{v}_2$ causes a decrease in the expected valuation of customer 2. As a result:
  (i) The seller finds it optimal to decrease $p_2$ to $\bar{v}_2$ to increase the competition at the second step by including all customer 2 types. In the mean time, given the increased competition at a lower $p_2$, the seller finds it optimal to increase $p_1$ from 5.17 to 5.33. Hence, as the gap between the expected valuations increases, so does the gap between the price steps. (ii) The optimal single price is of type SP2, but the price decreases to 2.18. (iii) Both the expected optimal markdown and the single price revenues decrease. (iv) TS markdown now exists for all $p_2 \in [2, 4]$, and hence Proposition 13 implies that the optimal markdown should yield a higher profit than SP2.

As $\bar{v}_2 \to 2$, a TS markdown with $p_2 = 2$ continues to be optimal and dominate the optimal single price (which is of type SP2). Hence, as the uncertainty about the lower valuation customer’s valuation decreases, the seller is better able to design a both effective as well as separating markdown, similar to the CI setting.

**Example 3: SP2 optimal.** In contrast to Example 2, here we expand the support of customer 2’s valuations by increasing $\bar{v}_2$. Since customer 2’s valuation range is now wider, the seller is no longer able to design a TS markdown; this is because the first price step necessary to induce all customer 1 types to bid at $p_1$ falls into $(\bar{v}_2, \bar{v}_2)$, thereby violating A4-IV. The seller’s ability to design a PS markdown is also constricted by this increased uncertainty in customer 2’s valuation since the set of feasible PS markdown decreases. We find that a PS markdown exists for only very low values of $p_2$, specifically, $p_2 \in [2, 3]$. However, the increase in customer 2’s expected valuation positively affects the seller’s optimal single price. A SP2 single price continues to be optimal, but now at the increased price of $p^* = 3.94$, and a single price mechanism outperforms the optimal PS markdown.

In summary, an increased range and expectation of customer 2 valuations implies that it is difficult for the seller to design an effective INT markdown. The only effective INT markdowns require such substantially low $p_2$ prices (relative to SP2), that the seller is better off using SP2.

As $\bar{v}_2$ continues to grow, then the seller is incapable of designing an INT markdown, i.e., the optimal markdown will be characterized by a $p_1 < \bar{v}_2$. This implies that there is a positive probability of competition at each price step and hence the buyer is no longer able to guarantee any high valuation customer that his entire demand will be satisfied at $p_1$.

**Example 4: PS optimal.** In this case $\bar{v}_1$ increases by 5 units compared to Example 1. Let us first remind the reader that the optimal prices in a TS markdown do not depend on $\bar{v}_1$, but rather are driven by $\bar{v}_1$. Hence, as the range of customer 1 valuation increases, as well as its expected value, a PS markdown becomes more attractive since it offers the seller the freedom to set a higher $p_1$. Intuitively, given the increase in the expected valuation of customer 1, the seller finds it more profitable to increase $p_1$ (i.e., increase the gap between the price steps) and induce only a subset of customer 1 types (12.97, 23) to bid at $p_1$, and increase her expected markdown revenue. Despite the increase in $\bar{v}_1$, the large value of $D_2$ implies that the optimal single price is still SP2 with $p^* = 2.76$ and $\Pi^*_{\text{SP2}}[2] = 43.31$, and hence the PS markdown outperforms the single price.

As $\bar{v}_1$ continues to increase, a PS markdown continues to be optimal (for the reasons articulated
above) and dominate the optimal single price for a wide range of $\bar{v}_1$. We find that $\bar{v}_1$ needs to be greater than 40 for the optimal single price to switch to SP1 and and it must be greater than 63 for SP1 to be optimal. In such a setting, the expected value of customer 1’s valuation is so high as to make targeting that customer alone optimal.

**Example 5: SP1 optimal.** While the valuation ranges in this example are the same as in the base case, the increase in $D_1$ from 3 to 8 results in an increase in the optimal first price step under TS; recall that we observed a similar effect of $D_1$ under CI (see Equation (1)). Furthermore, we find that the demand from the high valuation customers is now large enough to have SP1 become the optimal single price. Under this particular market setting, we find that SP1 dominates the optimal (TS) markdown. From our expanded example set, we find that SP1 outperforms the optimal TS markdown when $D_1 \geq 5$ (given the other parameters in the base instance).

Although we do not have any theoretical results that rank the performance of SP1 with the optimal markdown in general, this example clearly illustrates that an optimal markdown may fail to perform better than SP1 when the demand from the high valuation customer is relatively large, the valuation ranges of the two customers are far apart and $K$ is small relative to $D_1 + D_2$. Conversely, a markdown tends to dominate the optimal single price if $D_1$ is small relative to $D_2$ and the space between the customers’ valuations, $(\bar{v}_1 - \bar{v}_2)$ is ‘moderately’ far apart. Note that these conditions are the same as those found in Observation 7 for the optimal markdown to dominate the optimal single price under CI.

## 5 Conclusions and Future Research Directions

As businesses operating in both the B2C and B2B markets face increasingly sophisticated buyers, there is a need for sellers to consider buyers’ strategic behavior in their pricing strategies. In this paper, we study the optimal design of markdowns with pre-announced prices and their suitability in the presence of strategic buyers with multi-unit demands. We find that,

- The optimal markdown has 2 steps if buyers know the clearing price (the price at which the demand exceeds the available supply, $p_c$).
- If the buyers do not know $p_c$, but know that at most one price will occur within any one customer’s valuation range (INT markdown), then the optimal INT markdown has no more than 3 steps.
- Under either of these settings, it is optimal for a buyer to submit all-or-nothing bids at each price step.

There is a common intuition behind the first step prices under complete information (CI) (equations (1) and (3)) and incomplete information (IV) (equation (4)). “The ratio of a customers expected unmet demand to his total demand if he chooses to bid at $p_2$” can be perceived as a measure of scarcity, which the seller can use to induce purchases at the first step. Hence, the seller can charge a premium proportional to the scarcity that is equal to the maximum additional amount (above $p_2$) per unit the customer is willing to pay to secure his demand by bidding at $p_1$. We would also like to point out the similarity of trade-offs involved in the case of CI with multiple customers and IV. In both cases, design of the optimal markdown involves deciding which customers/types to exclude from the market and which to encourage to bid at higher price steps.

**Future Research Directions** Future work in this area could consider the optimal markdown design under IV when $N > 2$. From the results established in this paper, we know that the
optimal INT markdown will have at most 3 steps (Observation 9) and that customers will submit all-or-nothing bids (Theorem 7). We conjecture that the performance of a markdown will improve (vis-a-vis a single price policy) as the number of customer types increases; that is, the added pricing flexibility offered by markdowns will outweigh the strategic opportunities it creates for the buyers.

We were able to prove that a seller would never need to use more than 3-steps in an optimal INT markdown (Observation 9) under IV, which brings up the question: How well does a 2-step markdown perform compared to a 3-step markdown? Under which market settings is the difference in seller profits negligible? Conversely, when can they be substantial?

One of the limitations of the model considered in this paper is the assumption that the seller and buyers have complete information on customer demand. When customer valuations are also known by the seller, this assumption enables the seller to identify the clearing price with certainty. However, if the customer demand information is incomplete, both the seller and the buyers are unable to identify at which price the market would clear. We believe an important extension of this work is to the setting where customer demands are private information (Elmaghraby et al., 2004). Furthermore, we assume that all customers are present at the start of the markdown and remain until either they meet all of their demand, or the markdown is over. Our 2- or 3-step markdown result is partly due to this longevity of customers. Future work could consider the random arrival of customers to the system and its effect on the optimal number of markdown steps. (Note that in the Sam’s Club example in the appendix there are 5 price steps. This is possibly a result of the stochastic arrival of customers with unknown valuations to their website.)

We focused on the design of INT markdowns under the IV setting; this implied that the high valuation customer was guaranteed to have his entire demand satisfied if he bid at $p_1$. For some market settings, in particular when the uncertainty surrounding low valuation customer valuations is high, INT markdowns fail to exist. Under these settings, we would need to understand how customer will behave when they may face competition at each price step; and compare the performance of a single price mechanism to the optimal markdown.

Finally, we study the optimal design and use of markdown with pre-announced prices. While this format has been adopted by some companies, there are many other applications where price drops are not announced. Therefore, a natural extension of our work is to study the design and performance of unannounced markdowns, and to contrast them with preannounced price drops. This form of analysis would allow us to understand if and when a seller is better off sharing the price path information with his strategic buyers.

Given the recent increase in the popularity and use of dynamic pricing, we believe that exploration of these research directions will be crucial in designing and participating in such mechanisms.

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Appendix

Examples of Markdowns with Pre-Announced Prices from Sam’s Club and Filene’s Basement

“Filene developed a revolutionary way to price merchandise called the ‘Automatic Mark Down System.’ The price tag on each item was marked with the date it hit the selling floor. The longer an item remained unsold, the more the price would automatically be reduced, first 25%, then 50% and finally 75%.” http://www.filenesbasement.com/master.html

Figure 5: An example of a markdown mechanism with pre-announced prices from the website of Sam’s Club, under the name of Plunging Prices.

Example for the Two Customer Case (Section 3.1)

To illustrate the results of Section 3.1, we present a numerical example with two customers. We consider an instance where the customers are willing to buy up to 10 and 18 units \((D_1=10, D_2=18)\) and are willing to pay \(v_1=20\) and \(v_2=10\) per unit, respectively. We assume that the seller has an exogenously determined initial inventory of 20 units for sale \((K=20)\).

For this instance, the optimal single price is not unique. The seller can either set the price equal to 20 and only sell to customer 1, or set the price equal to 10 and sell the entire supply. With either one of these prices, the optimal single price revenue of the seller is 200.
In the optimal 2-step markdown, $p_2 = v_2 = 10$ (Observation 3). Optimal step 1 price, $p_1^* = 20 - \frac{10 + 20 - 18}{20 - 10} = 14$ (Proposition 4). Customer 1 buys 10 units at step 1, and the remaining 10 are sold to customer 2 at step 2. The revenue of the seller is 240. The markdown mechanism with prices $(p_1^*, p_2)$ dominates the single price as stated in Observation 6.

Figures 6(a) and 6(b) illustrate how the seller’s revenue under the markdown and the optimal single price mechanisms changes as a function of $(D_1)$ and $(v_1)$, respectively. In Figure 6(a), for $D_1 \leq v_2K/v_1 = 10$, markdown pricing dominates the optimal single price, as stated in Observation 6. For $D_1 > v_2K/v_1 = 10$, markdown pricing dominates the optimal single price only if $(v_1 + v_2)D_1 < (v_1 - v_2)D_2 + K(3v_2 - v_1)$ (as stated in Observation 7), that is if $D_1 < 12.7$. In Figure 6(b), we observe a similar dominance between the two pricing mechanisms as we keep $v_2$ fixed at 10 but vary $v_1$. From Figures 6(a) and 6(b) we also make the following observation: The revenues under the optimal single price remain constant and then start increasing after either $D_1$ or $v_1$ reach a threshold where it becomes more profitable to sell only to high valuation customers. However, revenues under markdown pricing monotonically increase with $D_1$ and $v_1$.

Figures 6(c) and 6(d) reaffirm the conditions stated in Observation 7, under which the optimal markdown mechanism dominates the optimal single price: Markdown pricing dominates the optimal single price when (i) $D_2$ is large relative to $D_1$ (Figure 6(c)), (ii) $K$ is close to the total quantity demanded by both customers $(D_1 + D_2)$ (Figure 6(c)) and (iii) the valuations of the two customers are close to each other (Figure 6(d)).
References


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L. Williams. Vice President Pricing Science, Zilliant, presentation at University of Maryland, 2005.


Online Appendix

Proof of Theorem 2 Given the bid quantities \((q_1, q_2)\) of all customers \(i \neq j\), we want to find the best response (optimal bid quantities) of customer \(j\). If \(v_j < p_2\), customer \(j\) bids zero at both price steps. If \(p_1 > v_j \geq p_2\), then from Observation 1 customer \(j\) bids his entire demand at step 2. If \(v_j \geq p_1\), customer \(j\)'s expected profit from bidding \((q_{j1}, q_{j2} = D_j - q_{j1})\) can be expressed as follows

\[
\Pi_j = (v_j - p_1)q_{j1} + (v_j - p_2)E[A_j]
\]

where \(E[A_j]\) denotes the expected quantity allocated to customer \(j\) in step 2. Note that since we consider effective markdowns, customer \(j\) is guaranteed to receive all his bid quantity at step 1, i.e., his profit at step 1 is \((v_j - p_1)q_{j1}\). Customer \(j\)'s expected profit from bidding at step 2 depends on the (expected) quantity that will be allocated to that customer at step 2. Recall that when the total bid quantity exceeds supply at a given step, the seller uses the random allocation rule. This is equivalent to choosing a permutation of \(N\) customers randomly and satisfying the customers’ demand in sequence based on their position in this permutation. There are \(N!\) distinct permutations of \(N\) customers and the seller will choose any of these permutations with equal probability \(1/N!\). Let us denote the set of all permutations by \(U\) and let \(B_k^i\) denote the set of customers whose bid quantities are satisfied before customer \(j\) in some permutation \(\pi \in U\). \(B_k^i = \emptyset\) indicates that customer \(j\)'s demand is satisfied first.

\[
E[A_j] = \frac{1}{N!} \sum_{\forall \pi \in U} \min\{(K - \sum_{i \in B_k^1} D_i - \sum_{i \notin B_k^1} q_{i1} - q_{j1})^+, (D_j - q_{j1})\},
\]

We will show that the profit function \(\Pi_j\) is convex in \(q_{j1}\) by analyzing its slope \(\Pi_j' = \frac{\partial \Pi_j}{\partial q_{j1}} = (v_j - p_1) + (v_j - p_2)(E[A_j])'\), where \((E[A_j])' = \frac{\partial E[A_j]}{\partial q_{j1}}\) evaluates to \(-\frac{x}{n}\) for some positive real \(x\). The higher the value of \(q_{j1}\), the higher is the number of permutations for which \((K - \sum_{i \in B_k^1} D_i - \sum_{i \notin B_k^1} q_{i1} - q_{j1})^+\) is equal to zero, i.e., the lower is the value \(x\). Hence we observe that as \(q_{j1}\) increases, the slope of the profit function either increases (possibly changing from negative to positive) or remains the same.

Based on these observations, we conclude that the profit function is convex, and will be maximized at one of the extreme points \(\{0, D_j\}\), implying an all-or-nothing bidding strategy at each step. \(\square\)

Proof of Proposition 4 Since \(p_2 = v_2\), using the equilibrium bidding strategies from Table 3 we can write the total profit of the seller as follows:

\[
\Pi_S(p_1) = \begin{cases} 
 p_1D_1 + v_2(K - D_1) & \text{if } p_1 \leq \hat{p}_{CI}(v_1, v_2) \\
 v_2K & \text{if } p_1 > \hat{p}_{CI}(v_1, v_2)
\end{cases}
\]

\(\Pi_S\) is strictly monotonically increasing in \(p_1\) up to \(\hat{p}_{CI}(v_1, p_2)\), after which it sharply drops to \(v_2K\). Hence, this break-point \(\hat{p}_{CI}(v_1, p_2) = p_1^*\) is the optimal price. At \(p_1^*\), customer 1 is indifferent between buying at step 1 or 2. By assumption (A2) customer 1 bids his entire demand at step 1. \(\square\)

Proof of Theorem 5 Suppose there exists some partition \(\{S_1, S_2\}\), which satisfies conditions (C1) and (C2) simultaneously. We will demonstrate that the proposed bidding strategies constitute an equilibrium by considering any profitable deviations. Suppose that all buyers, except buyer \(k\), bid according to ES. From Theorem 2, we know that it is always optimal for customer \(k\) to bid all of his potential demand in one step.
For \( k \in S_1 \), if buyer \( k \) bids according to ES, his profit is given by \( \Pi_{k1} = D_k(v_k - p_1) \). On the other hand, if customer \( k \) deviates from ES and bids \( D_k \) at step 2, he will compete with the other customers in \( S_2 \) for the allocation of the remaining units; in this case customer \( k \)'s expected profit is \( \Pi_{k2} = (v_k - p_2) \cdot E[A_k] \) where \( E[A_k] \) is customer \( k \)'s expected allocation at step 2.

We know that \( \Pi_{k1} = D_k(v_k - p_1) > (v_k - p_2) \cdot E[A_k] = \Pi_{k2} \) (by the definition of the partition \( \{S_1, S_2\} \) and condition (C1)). Therefore, given all other buyers bid according to ES, it is an optimal response for buyer \( k \in S_1 \) to do the same.

Similarly, for \( k \in S_2 \), if buyer \( k \) bids according to ES, his profit is given by \( \Pi_{k2} = (v_k - p_2) \cdot E[A_k] \). If buyer \( k \) deviates and bids \( D_k \) in step 1, his profit is given by \( \Pi_{k1} = D_k(v_k - p_1) \). From the definition of \( \{S_1, S_2\} \) and condition (C2), we know that \( \Pi_{k1} = D_k(v_k - p_1) < (v_k - p_2) \cdot E[A_k] = \Pi_{k2} \). Therefore, given all other buyers bid according to ES, it is an optimal response for buyer \( k \in S_2 \) to do the same. \( \square \)

**Proof of Proposition 6** We can rewrite condition (C1) as follows:

\[
\frac{v_k - p_1}{v_k - p_2} \geq \frac{E[A_k]}{D_k} \tag{C1}
\]

As we increase \( D_k \) by one unit, the denominator of \( \frac{E[A_k]}{D_k} \) increases by one unit, but the numerator increases by at most one unit, i.e., as \( D_k \) increases, \( \frac{E[A_k]}{D_k} \) decreases (or remains the same). On the other hand, \( \frac{v_k - p_1}{v_k - p_2} \) is increasing in \( v_k \). Then we have

\[
\frac{v_j - p_1}{v_j - p_2} > \frac{v_k - p_1}{v_k - p_2} > \frac{E[A_k]}{D_k} \geq \frac{E[A_j]}{D_j}
\]

for any \( j < k \) since \( v_j > v_k \) and \( D_j \geq D_k \). This implies that if condition (C1) holds for customer \( k \) for a given partition \( \{S_1, S_2\} \), then it should hold for all customers \( j < k \). \( \square \)

**Proof of Theorem 7** Let \( q_{jt} \) denote customer \( j \)'s bid quantity at step \( t, t \in \{1, \ldots, m\} \). Note that for any customer \( j \) who can afford to bid at \( p_m \), we have \( q_{jm} = D_j = \sum_{t < m} q_{jt} \).

The expected profit of customer \( j \) can be expressed as:

\[
\Pi_j(q_{j1}, \ldots, q_{jm}) = \sum_{t=1}^{m} (v_j - p_t) \cdot E[A_{jt}]
\]

where \( E[A_{jt}] \) is the expected allocation to customer \( j \) at step \( t \). Recall that when the total bid quantity exceeds supply at a given step, the seller uses a random allocation rule and this is equivalent to choosing a permutation of \( N \) customers randomly and satisfying the demands based on their sequence in this permutation. The seller will choose each of \( N! \) permutations with equal probability \( \frac{1}{N!} \). We denote the set of all permutations with \( U \) and define \( B^j_k \) as the set of customers whose bids are satisfied before customer \( j \) in some permutation \( \pi \in U \). Using this notation, we can define the expected allocation of customer \( j \) at step \( t \) as follows:

\[
E[A_{jt}] = \frac{1}{N!} \sum_{\forall \pi \in U} \min\{K^+_t, q_{jt}\}
\]

where

\[
K_1 = K - \sum_{\forall i \in B^j_1} q_{i1}, \quad K_m = K - \sum_{\forall i \in B^j_m} D_i - \sum_{\forall i \in B^j_k} \sum_{k < m} q_{ik} - \sum_{k < m} q_{jk}
\]
\[ K_t = K - \sum_{i \in B_t^k} \sum_{k \leq t} q_{ik} - \sum_{i \notin B_t^k} \sum_{k < t} q_{ik} - \sum_{k < t} q_{jk}, \quad m > t > 1 \]

We look at the derivative of \( \Pi_j \) with respect to \( q_{jk} \) and show that the profit is maximized at the end-points of the range for \( q_{jk} \).

\[
\frac{\partial \Pi_j(q_{j1}, \ldots, q_{jm})}{\partial q_{jk}} = \sum_{t=1}^{m} (x_j - p_t) \frac{\partial E[A_{jt}]}{\partial q_{jk}}
\]

\( \frac{\partial E[A_{jt}]}{\partial q_{jk}} \) evaluates to zero for \( k > t \). For \( k < t \), \( \frac{\partial E[A_{jt}]}{\partial q_{jk}} \) evaluates to \( -\frac{x_k}{N_t!} \) for some \( x_k \). The higher the value of \( q_{jk} \), the higher is the number of permutations for which \( K_t^+ \) is equal to zero, i.e., the lower is the value of \( x_t \). For \( k = t \), \( \frac{\partial E[A_{jt}]}{\partial q_{jk}} \) evaluates to \( \frac{w_t}{N_t!} \) for some \( w_t \). Note that \( K_t^+ \) is constant with respect to \( q_{jt} \), hence, as \( q_{jt} \) increases, the number of permutations for which \( \min\{K_t^+, q_{jt}\} = q_{jt} \) decreases (or remains the same), implying that \( \frac{\partial E[A_{jt}]}{\partial q_{jk}} \) is non-decreasing in \( q_{jt} \). In summary, as \( q_{jk} \) increases, \( \frac{\partial E[A_{jt}]}{\partial q_{jk}} \) increases or remains the same. From this observation, it follows that the slope of the profit function in the direction of \( q_{jk} \) in non-decreasing in \( q_{jk} \) implying that customer \( j \)'s profit is maximized at one of the end-points 0 or \( D_j - \sum_{t \leq t} q_{ji} \).

Since the optimal bid quantity at step \( t \) is 0 or \( D_j - \sum_{t \leq t} q_{ji} \), customer \( j \) submits 0 or \( D_j \) at step 1. By induction, if the customer submits 0 or \( D_j \) at steps \( 1, \ldots, t, t < m \), then he will submit 0 or \( D_j \) at step \( t + 1 \). Hence, it is optimal for customer \( j \) to submit all-or-nothing bids at any step. □

**Proof of Observation 8:** Consider a markdown where \( p_1 > p_2 > \ldots > p_j > \bar{v}_k > p_{j+1} > \ldots \) where \( p_c \in [\bar{v}_k, \bar{v}_k] \).

(i) Since the total demand is less than \( K \) for any price higher than \( \bar{v}_k \), customers will not purchase but simply wait until the markdown reaches \( p_j \), which is the lowest price above \( \bar{v}_k \). Hence, one can eliminate \( p_1, p_2, \ldots, p_{j-1} \) from the markdown without affecting the expected profit of the seller.

(ii) Suppose \( p_1 > \ldots > \bar{v}_k > p_l > \ldots > p_m \), i.e, \( p_l \) is the highest price which is smaller than \( \bar{v}_k \). We claim that the seller would be better off by eliminating the last price step \( p_m \). Since \( D[p_l] > K \) \( \forall p_t \leq p_l \), eliminating \( p_m \) does not decrease the number of units sold. We also need to show that the seller is guaranteed to sell all \( K \) units at the same or higher prices than with the \( m \)-step markdown. Customers originally bidding at step \( m \) may bid at higher price steps or they may not bid at all in the new markdown with fewer steps. If they do not bid in the new mechanism, then the competition at the higher price levels is unaffected. On the other hand, if they bid at higher price steps, then the competition can only increase since there would be more customers bidding at a price step. As a result, at any price step \( j < m \), the competition either increases or remains the same, preventing the seller’s revenue from decreasing. Hence, the revenue of the seller is not made any worse by eliminating the lowest price step from the markdown. Using these arguments repeatedly, we can eliminate all price steps (strictly) smaller than \( p_l \).

(iii) By contradiction, suppose \( p_m \in [\bar{v}_j+1, \bar{v}_j] \) for some \( j < N \). By setting \( p = \bar{v}_j \), the seller does not decrease the demand at any price step, hence, the competition at each price step remains the same. However, due to the increase in \( p_m \), some customers who were previously bidding at \( p_m \) may now choose to bid at higher price steps, potentially increasing the seller’s revenues. □

**Proof of Observation 10:** Since \( p_c \in [\bar{v}_N, \bar{v}_N] \), from Observation 8, there is at most one price, \( p_1 \), exceeding \( \bar{v}_N \). By the definition of an INT markdown, only one price can be chosen from \([\bar{v}_N, \bar{v}_N] \). □
Proof of Proposition 9: If customer 1 bids \( D_1 \) at \( p_1 \), his surplus is \( \Pi_{11} = (v_1 - p_1)D_1 \). Alternatively, if he bids \( D_1 \) at \( p_2 \) his expected surplus is \( \Pi_{12} = (v_1 - p_2) \left\{ F_2(p_2)D_1 + [1 - F_2(p_2)] \left( \frac{D_1 + K - D_2}{2} \right) \right\} \). Hence, customer 1 bids at \( p_1 \) if and only if \( \Pi_{11} \geq \Pi_{12} \). Rearranging terms, we get Equations (4) and (5). \( \square \)

Characterization of Feasible TS and PS Markdowns

To identify the range of \( p_2 \) values for which TS or PS markdowns are feasible, first we find the \( p_2 \) values which satisfy \( \hat{p}_{IV}(v_1, p_2) = \tilde{v}_2 \) (the lower bound on allowable prices for \( p_1 \) under an INT markdown), and find two solutions, namely, \( \tilde{v}_2 \) and \( p_2(v_1) \) where

\[
p_2(v_1) = v_1 - \frac{2D_1}{D_1 + D_2 - K}(\tilde{v}_2 - v_2)
\]

(Note that when customer valuations are uniformly distributed, the threshold step 1 price, \( \hat{p}_{IV}(v_1, p_2) \), is a quadratic convex function of \( p_2 \). To induce some customer 1 types to bid at \( p_1 \), we need \( p_2 \leq p_2(\tilde{v}_1) \). In addition, \( p_2 < \tilde{v}_2 \) from A4-IV, hence, we need \( p_2 \leq \min\{\tilde{v}_2, p_2(\tilde{v}_1)\} \). Combining this with the result of Corollary 12, we get the following condition for a feasible PS markdown.

**Observation 12.** A feasible PS markdown satisfies the following:

\[
\tilde{v}_2 \leq p_2 < \min\{\tilde{v}_2, p_2(\tilde{v}_1)\} \quad \text{and} \quad \max\{\hat{p}_{IV}(v_1, p_2), \tilde{v}_2\} < p_1 \leq \hat{p}_{IV}(v_1, p_2)
\]

Similarly, we can identify conditions for the existence of a TS markdown.

**Observation 13.** A feasible TS markdown satisfies the following:

\[
\tilde{v}_2 \leq p_2 < \min\{\tilde{v}_2, p_2(\tilde{v}_1)\} \quad \text{and} \quad \tilde{v}_2 < p_1 \leq \hat{p}_{IV}(v_1, p_2)
\]

Note that since \( \hat{p}_{IV}(\tilde{v}_1, p_2) > \hat{p}_{IV}(v_1, p_2) \), a PS markdown exists whenever a TS markdown exists. Results 14 and 15 spell out conditions (7) and (8) (presented in Observations 12 and 13) for uniformly distributed valuations, and are used in the proofs of some of the theorems that follow.

**Result 14.** When the valuations are uniformly distributed, there exists a PS markdown with price \( p_2 \) if and only if one of the following conditions holds:

i. for \( p_2 \in (\tilde{v}_2, p_2(\tilde{v}_1)) \) if \( \frac{\tilde{v}_2 - v_2}{\tilde{v}_1 - v_2} \leq \frac{D_1 + D_2 - K}{2D_1} < \frac{\tilde{v}_2 - v_2}{\tilde{v}_1 - \tilde{v}_2} \) or equivalently if \( v_2 \leq p_2(\tilde{v}_1) < \tilde{v}_2 \)

ii. for all \( p_2 \in [\tilde{v}_2, \tilde{v}_1] \) if \( \frac{\tilde{v}_2 - v_2}{\tilde{v}_1 - v_2} \leq \frac{D_1 + D_2 - K}{2D_1} \) or equivalently if \( \tilde{v}_2 \leq p_2(\tilde{v}_1) \)

The proof of Result 14 follows immediately by inserting the appropriate values for the uniform distribution into the conditions (7) of Observation 12. We have a similar result for the TS markdown.

**Result 15.** When the valuations are uniformly distributed, there exists a TS markdown with price \( p_2 \) if and only if one of the following conditions holds:

i. for \( p_2 \in (\tilde{v}_2, p_2(\tilde{v}_1)) \) if \( \frac{\tilde{v}_2 - v_2}{\tilde{v}_1 - v_2} \leq \frac{D_1 + D_2 - K}{2D_1} < \frac{\tilde{v}_2 - v_1}{\tilde{v}_1 - \tilde{v}_2} \) or equivalently if \( v_2 \leq p_2(\tilde{v}_1) < \tilde{v}_2 \)

ii. for all \( p_2 \in [\tilde{v}_2, \tilde{v}_1] \) if \( \frac{\tilde{v}_2 - v_2}{\tilde{v}_1 - v_2} \leq \frac{D_1 + D_2 - K}{2D_1} \) or equivalently if \( \tilde{v}_2 \leq p_2(\tilde{v}_1) \)
In the following, whenever there is a strict inequality \( a < b \) as a constraint, this should be perceived as \( a \leq b - \epsilon \), where \( \epsilon \approx 0 \). To simplify the notation, we use \( a < b \). \( \epsilon \) can be thought of as the minimum possible price increment.

The seller can determine the optimal PS markdown by maximizing \( \Pi_S^{PS} \) subject to (7), where:

\[
\Pi_S^{PS} = [1 - F_1(\hat{v}_1(p_1, p_2))]p_1D_1 + F_1(\hat{v}_1(p_1, p_2))p_2D_1 + (K - D_1)[1 - F_2(p_2)]p_2
\]

(9)

Similarly, the seller can determine the optimal TS markdown by maximizing \( \Pi_S^{TS} \) subject to (8), where:

\[
\Pi_S^{TS} = p_1D_1 + (K - D_1)[1 - F_2(p_2)]p_2
\]

(10)

Define \( P_{PS} \) and \( P_{TS} \) to be the set of \( p_2 \) values for which a PS and a TS markdown exists, respectively. Note that since \( P_{TS} \subseteq P_{PS} \), we have \( P_{TS} \cap P_{PS} = P_{TS} \). In what follows, we characterize the optimal PS and TS markdowns. Based on those results, we are able to derive sufficient conditions for a PS or a TS markdown to be optimal. Define \( p_1^M(p_2) \) to be the optimal first step price given a second price step of \( p_2 \) under markdown type \( M = TS, PS \).

**Characterization of the Optimal TS Markdown**

To characterize the optimal TS markdown, we find \( p_1^{TS}(p_2) \) and then demonstrate some properties of the seller’s revenue function.

**Theorem 16.** The optimal TS markdown has the following properties:

(i) For a given second step price \( p_2 \), the optimal first step price is \( \hat{p}_{IV}(\bar{v}_1, p_2) \).

(ii) \( \Pi^{TS}_S(\hat{p}_{IV}(\bar{v}_1, p_2), p_2) \) is convex in \( p_2 \) if \( K < D_1 + D_2/3 \), and it is concave in \( p_2 \) if \( K \geq D_1 + D_2/3 \).

(iii) If \( \Pi^{TS}_S(\hat{p}_{IV}(\bar{v}_1, p_2), p_2) \) is concave in \( p_2 \), then the optimal step 2 price is:

\[
p_2^{TS} = \begin{cases} 
\frac{\bar{v}_2}{p_2^{TS}} & \text{if } p_2^{TS} \leq \bar{v}_2 \\
\frac{\bar{v}_2}{p_2^{TS}} & \text{if } \bar{v}_2 < p_2^{TS} \leq \min\{p_2(\bar{v}_1), \bar{v}_2\} \\
p_2(\bar{v}_1) & \text{if } p_2(\bar{v}_1) < \min\{p_2^{TS}, \bar{v}_2\} \\
\bar{v}_2 - \epsilon & \text{if } \bar{v}_2 \leq \min\{p_2^{TS}, p_2(\bar{v}_1)\}
\end{cases}
\]

where \( p_2^{TS} = \frac{\bar{v}_2(3K - D_1 - D_2) - 2D_1 \bar{v}_2 - (D_1 + D_2 - K)\bar{v}_1}{2(3K - 3D_1 - D_2)} \).

(iv) If \( \Pi^{TS}_S(\hat{p}_{IV}(\bar{v}_1, p_2), p_2) \) is convex in \( p_2 \), then the optimal step 2 price is:

\[
p_2^{TS} = \begin{cases} 
\frac{\bar{v}_2}{p_2(\bar{v}_1)} & \text{if } \frac{D_1 + D_2 - K}{D_1} \geq \frac{D_1 \bar{v}_2 - K \bar{v}_1}{D_1(\bar{v}_1 - \bar{v}_2)} + \frac{(K - D_1)(\bar{v}_2 - p_2(\bar{v}_1))}{(\bar{v}_2 - \bar{v}_2)(\bar{v}_1 - \bar{v}_2)}p_2(\bar{v}_1) \\
\bar{v}_2 & \text{otherwise}
\end{cases}
\]

**Proof of Theorem 16:**

(i) The seller’s profit is increasing in \( p_1 \) (since \( \frac{\partial \Pi^{TS}_S}{\partial p_1} = D_1 > 0 \)). Therefore, for a given \( p_2 \), the seller would prefer to set \( p_1 \) to its upper bound, which is \( \hat{p}_{IV}(\bar{v}_1, p_2) \).

(ii) The first derivative of seller’s revenue with respect to \( p_2 \) after substituting \( p_1 = \hat{p}_{IV}(\bar{v}_1, p_2) \) and uniform CDF and pdf for \( F_i(p_1) \) and \( f_i(p_1) \) is as follows:

\[
\frac{\partial \Pi^{TS}_S(\hat{p}_{IV}(\bar{v}_1, p_2), p_2)}{\partial p_2} = D_1 - \frac{\bar{v}_1}{(\bar{v}_2 - \bar{v}_2)} \frac{(D_1 + D_2 - K)}{2} + \frac{(\bar{v}_2 - 2p_2)}{(\bar{v}_2 - \bar{v}_2)} \frac{(3K - 3D_1 - D_2)}{2}
\]

Taking the second derivative with respect to \( p_2 \) yields:

\[
\frac{\partial^2 \Pi^{TS}_S(\hat{p}_{IV}(\bar{v}_1, p_2), p_2)}{\partial p_2^2} = (3D_1 + D_2 - 3K) \frac{1}{(\bar{v}_2 - \bar{v}_2)}
\]
Since $\bar{v}_2 > v_2$ we obtain a sufficient and necessary condition for the function to be concave (convex) in $p_2$ as $K \geq D_1 + D_2/3$ ($K < D_1 + D_2/3$).

(iii) When the revenue function is concave in $p_2$, its maximizer is $p^{TS}_2$. If $p^{TS}_2 \in \{v_1, \min\{p_2(v_1), \bar{v}_2\}\}$, then it is the optimal step 2 price. Otherwise, the optimal $p_2$ is one of the boundary values of the feasible $p_2$ range.

(iv) When the revenue is a convex function of $p_2$, one of the boundary points will be the optimal step 2 price. First we show that if a TS markdown exists for all $p_2 \in [\bar{v}_2, \bar{v}_2]$, then $p^{TS}_2 = v_2$. For $v_2$ to be the optimal step 2 price, revenue at this price should be higher than the revenue at $p_2 = \bar{v}_2$, i.e.,

$$\Pi^{TS}_S(\hat{p}_{IV}(v_1, v_2), v_2) \geq \Pi^{TS}_S(\hat{p}_{IV}(v_1, \bar{v}_2), \bar{v}_2)$$

$$\rightarrow K\bar{v}_2 + (v_1 - \bar{v}_2) \frac{D_1 + D_2 - K}{2} \geq D_1 \bar{v}_2 \rightarrow \frac{D_1 + D_2 - K}{2} \geq \frac{D_1 \bar{v}_2 - K\bar{v}_2}{v_1 - \bar{v}_2}$$

From Result 15(ii), we have $\frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} \leq \frac{D_1 + D_2 - K}{2D_1}$ as the condition for a TS markdown to exist for all $p_2$, hence we conclude that $v_2$ is always optimal provided that a TS markdown exists for all $p_2$ as a result of the following series of inequalities,

$$\frac{D_1 + D_2 - K}{2D_1} \geq \frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} > \frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} > \frac{\bar{v}_2 - v_2 - K - D_1 \frac{v_2}{D_1} - v_2}{\bar{v}_1 - \bar{v}_2} = \frac{D_1 \bar{v}_2 - K\bar{v}_2}{D_1(v_1 - \bar{v}_2)}$$

If a TS markdown exists for only $p_2 \in [\bar{v}_2, p_2(v_1)]$, where $p_2(v_1) < \bar{v}_2$, then either $p^{TS}_2 = p_2(v_1)$ or $p^{TS}_2 = \bar{v}_2$. We compare the corresponding revenues to find conditions under which either one is optimal. For $p^{TS}_2 = \bar{v}_2$, we need $\Pi^{TS}_S(\hat{p}_{IV}(v_1, v_2), v_2) = \Pi^{TS}_S(\hat{p}_{IV}(v_1, p_2(v_1)), p_2(v_1)) = \Pi^{TS}_S(\bar{v}_2, p_2(v_1), p_2(v_1))$, i.e.,

$$\frac{D_1 + D_2 - K}{2D_1} \geq \frac{D_1 \bar{v}_2 - K\bar{v}_2}{D_1(v_1 - \bar{v}_2)} + \frac{(K - D_1) (\bar{v}_2 - p_2(v_1))}{D_1(v_1 - \bar{v}_2)} p_2(v_1)$$

From Result 15 we know that a TS markdown exists for $p_2 \in [\bar{v}_2, p_2(v_1)]$ when $\frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} < \frac{D_1 + D_2 - K}{2D_1}$.

Hence $p^{TS}_2 = \bar{v}_2$ if $\frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} > \frac{D_1 + D_2 - K}{2D_1} > \frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} > \frac{D_1 \bar{v}_2 - K\bar{v}_2}{D_1(v_1 - \bar{v}_2)} + \frac{(K - D_1) (\bar{v}_2 - p_2(v_1))}{D_1(v_1 - \bar{v}_2)} p_2(v_1)$.

Combining the ranges that yield the same $p^{TS}_2$ we get the desired conditions. $\square$
Characterization of the Optimal PS Markdown

To characterize the optimal PS markdown, we first show some properties of the seller’s revenue function and then derive $p^*_1(p_2)$ for a given $p_2$.

**Theorem 17.** The optimal PS markdown has the following properties

(i) $\Pi^{pS}_S$ is concave in $p_1$ and $p^*_1(p_2) = p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{4D_1}$ maximizes the unconstrained $\Pi^{pS}_S$.

(ii) The optimal step 1 price is the following:

$$p^*_1(p_2) = \begin{cases} p^1_1(p_2) & \text{if } \bar{v}_1 < \frac{\bar{v}_1 + p_2}{2} \text{ and } p_2 < \bar{p}_2 = \bar{v}_1 - \frac{4D_1}{D_1 + D_2 - K}(\bar{v}_2 - \bar{v}_3) \\ \max\{\hat{p}_{IV}(\bar{v}_1, p_2), \bar{v}_2\} + \epsilon & \text{otherwise} \end{cases}$$

**Proof of Theorem 17:**

(i) First we show that the revenue function is concave in $p_1$ from the second order condition, and find the maximizer from the first order condition. The first and second order partial derivatives of $\Pi^{pS}_S(p_1, p_2)$ in Equation (9) with respect to $p_1$ are:

$$\frac{\partial \Pi^{pS}_S}{\partial p_1} = [1 - F_1(\hat{v}_1(p_1, p_2))] D_1 - \frac{\partial \hat{v}_1(p_1, p_2)}{\partial p_1} f_1(\hat{v}_1(p_1, p_2))(p_1 - p_2) D_1$$

$$\frac{\partial^2 \Pi^{pS}_S}{\partial p_1^2} = -2f_1(\hat{v}_1(p_1, p_2)) \frac{\partial(\hat{v}_1(p_1, p_2))}{\partial p_1} D_1$$

Since $\frac{\partial(\hat{v}_1(p_1, p_2))}{\partial p_1} = \frac{1}{[1 - F_2(p_2)] \frac{2D_1}{D_1 + D_2 - K}}$, and $f_1(.)$ are both positive and independent of $p_1$ under uniform distribution, the revenue function is concave in $p_1$ for all $p_2$.

Substituting $\hat{v}_1(p_1, p_2)$ from Equation (5), $\frac{\partial(\hat{v}_1(p_1, p_2))}{\partial p_1} = \frac{1}{[1 - F_2(p_2)] \frac{2D_1}{D_1 + D_2 - K}}$, and uniform CDF and pdf for $F_1(.)$ and $f_1(.)$ in the first derivative and setting equal to zero, we get the first order condition:

$$\frac{\bar{v}_1 - p_2}{\hat{v}_1 - \bar{v}_1} D_1 - \frac{1}{\hat{v}_1 - \bar{v}_1} \frac{(p_1 - p_2)}{[1 - F_2(p_2)] \frac{4D_1^2}{D_1 + D_2 - K}} = 0$$

We solve for the $p_1$ value that satisfies the first order condition and get $p^*_1(p_2) = p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{4D_1}$.

(ii) To find out the optimal $p_1$ for a given $p_2$, we need to understand the conditions under which $p^*_1(p_2) \in (\max\{\hat{p}_{IV}(\bar{v}_1, p_2), \bar{v}_2\}, \hat{p}_{IV}(\bar{v}_1, p_2))$ as stated in constraint (7). It is easy to see that $p^*_1(p_2) < \hat{p}_{IV}(\bar{v}_1, p_2)$ always, hence, we only need to check the lower bounds. Next, we show that $p^*_1(p_2) \in (\max\{\hat{p}_{IV}(\bar{v}_1, p_2), \bar{v}_2\}, \hat{p}_{IV}(\bar{v}_1, p_2))$ if and only if $\bar{v}_1 < \frac{\bar{v}_1 + p_2}{2}$ and $p_2 < \bar{p}_2$.

(ii.a) $p^*_1(p_2) > \hat{p}_{IV}(\bar{v}_1, p_2)$ if and only if $\bar{v}_1 < \frac{\bar{v}_1 + p_2}{2}$.

We substitute $v_1 = \bar{v}_1$ in Equation (4), and compare with $p^*_1(p_2)$.

$$p^1_1(p_2) = p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{4D_1} > p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{2D_1}$$

Simplifying the expression, we get $\bar{v}_1 < \frac{\bar{v}_1 + p_2}{2}$ as the equivalent condition.

(ii.b) $p^*_1(p_2) > \bar{v}_2$ if and only if $p_2 < \bar{p}_2 = \bar{v}_1 - \frac{4D_1}{D_1 + D_2 - K}(\bar{v}_2 - \bar{v}_3)$. 

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In order for $p_1^{PS}(p_2)$ to exceed $\bar{v}_2$, we get:

$$p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)]\frac{D_1 + D_2 - K}{4D_1} > \bar{v}_2$$

Substituting uniform CDF for $F_2$ and rearranging we get $p_2 < \bar{p}_2$ as the equivalent condition. □

Plugging $p_1 = p_1^{PS}(p_2)$ in Equation (9), now we can determine the optimal step 2 price, $p_2^{PS}$, corresponding to each case.

**Result 18.** When $p_1^{PS}(p_2) = p_1^{PS}(p_2)$, the optimal step 2 price for a PS markdown is:

$$p_2^{PS} = \begin{cases} 
\frac{v_2}{p_2^{PS}} & \text{if } p_2^{PS} < \frac{v_2}{p_2^{PS}} < \min\{\bar{v}_2, p_2(\bar{v}_1)\} \\
p_2(\bar{v}_1) & \text{if } p_2(\bar{v}_1) \leq p_2^{PS} \text{ and } p_2(\bar{v}_1) < \bar{v}_2 \\
\bar{v}_2 - \epsilon & \text{otherwise}
\end{cases}$$

**Proof of Result 18:** Substituting $p_1 = p_1^{PS}(p_2)$ into Equation (9), we get:

$$\Pi_S^{PS}(p_1^{PS}(p_2), p_2) = p_2D_1 + [1 - F_2(p_2)]\left(\frac{(\bar{v}_1 - p_2)^2(D_1 + D_2 - K)}{8(\bar{v}_1 - \bar{v}_1)} + (K - D_1)p_2\right)$$

The first derivative with respect to $p_2$ yields:

$$\frac{\partial \Pi_S^{PS}}{\partial p_2} = D_1 - [1 - F_2(p_2)]\left(\frac{(\bar{v}_1 - p_2)(D_1 + D_2 - K)}{4(\bar{v}_1 - \bar{v}_1)} + (K - D_1)\right)$$

$$- f_2(p_2)\left(\frac{(\bar{v}_1 - p_2)^2(D_1 + D_2 - K)}{8(\bar{v}_1 - \bar{v}_1)} + (K - D_1)p_2\right)$$

The second derivative with respect to $p_2$ after substituting uniform distribution for $F_2$ yields:

$$\frac{\partial^2 \Pi_S^{PS}}{\partial p_2^2} = \frac{(\bar{v}_1 - p_2)(D_1 + D_2 - K)}{2(\bar{v}_1 - \bar{v}_1)(\bar{v}_2 - \bar{v}_2)} - \frac{2(K - D_1)}{(\bar{v}_2 - \bar{v}_2)} + \frac{(\bar{v}_2 - p_2)(D_1 + D_2 - K)}{4(\bar{v}_1 - \bar{v}_1)(\bar{v}_2 - \bar{v}_2)}$$

Solving for $p_2$ from the first order condition $\frac{\partial \Pi_S^{PS}}{\partial p_2} = 0$, we get $p_2 = \frac{A + \sqrt{A^2 - 4B}}{2C}$, where $A = (\bar{v}_2 + 2\bar{v}_1)(D_1 + D_2 - K) - 8(K - D_1)(\bar{v}_1 - \bar{v}_1)$, $B = 8(\bar{v}_2 D_1 - \bar{v}_2 K)(\bar{v}_1 - \bar{v}_1) + \bar{v}_1(\bar{v}_1 + 2\bar{v}_2)(D_1 + D_2 - K)$ and $C = 3(D_1 + D_2 - K)$.

The second order condition required for $p_2$ to be a maximizer yields $p_2 > \frac{A}{2}$, hence we get $p_2^{PS} = \frac{A + \sqrt{A^2 - 4B}}{2C}$ as the unique maximizer of the unconstrained revenue function. By comparing $p_2^{PS}$ with boundary points of the feasible $p_2$ range, $[\bar{v}_2, \min\{p_2(\bar{v}_1), \bar{v}_2\}]$, we determine the optimal step 2 price. □

In order not to clutter the presentation in the proofs in this document, we will drop $\epsilon$ when substituting arguments with $\epsilon$ in other expressions and use $\lim_{\epsilon \to 0}$ in expressions that involve $\epsilon$.

**Result 19.** When $p_1^{PS}(p_2) = \bar{v}_2 + \epsilon$, then in a PS markdown we have:

(i) $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is convex in $p_2$ if and only if $\frac{(K - D_1)}{\bar{v}_2 - \bar{v}_1} < \frac{D_1}{\bar{v}_1 - \bar{v}_1}$, and it is concave in $p_2$ otherwise.

(ii) If $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is concave in $p_2$, then the optimal step 2 price is:
\[ p_{2}^{PS}(\tilde{v}_2) = \begin{cases} \frac{v_2}{p_2^{PS}(\tilde{v}_2)} & \text{if } p_{2}^{PS}(\tilde{v}_2) \leq \frac{v_2}{\bar{v}} \\ \frac{p_2(\bar{v}_1)}{p_2(\tilde{v}_1)} & \text{if } \frac{v_2}{p_2(\bar{v}_1)} \leq p_{2}^{PS}(\tilde{v}_2) \\ \bar{v}_1 & \text{if } p_2(\bar{v}_1) < p_{2}^{PS}(\tilde{v}_2) \end{cases} \]

where \( p_{2}^{PS}(\tilde{v}_2) \) be the value of \( p_2 \) that solves \( \frac{\partial \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2)}{\partial p_2} = 0 \).

(iii) If \( \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2) \) is convex in \( p_2 \), then the optimal step 2 price is:

\[ p_{2}^{PS}(\tilde{v}_2) = \begin{cases} \bar{v}_2 - \epsilon & \text{if } p_2(\bar{v}_1) \geq \left( \frac{1}{D_1(v_2 - v_2)} \right) \frac{v_2}{\bar{v}_2} \text{ and } p_2(\bar{v}_1) > \bar{v}_2 \\ p_2(\bar{v}_1) & \text{if } p_2(\bar{v}_1) \left( D_1 + K - D_1 \frac{v_1 - v_2}{v_2} \right) \geq \Pi_S^{PS}(\tilde{v}_2 + \epsilon, \bar{v}_2) \text{ and } p_2(\bar{v}_1) < \bar{v}_2 \\ \bar{v}_1 & \text{otherwise} \end{cases} \]

where \( \lim_{\epsilon \to 0} \Pi_S^{PS}(\tilde{v}_2 + \epsilon, \bar{v}_2) = K\bar{v}_2 + D_1(\bar{v}_2 - v_2) \left( \frac{v_1 - v_2}{v_1 - \bar{v}_1} - \frac{v_2 - \bar{v}_2}{v_1 - \bar{v}_1} \frac{2D_1}{D_1 + D_2 - K} \right) \).

Proof of Result 19: (i) We substitute \( p_1 = \bar{v}_2 + \epsilon \) in the seller’s revenue given in (9) and uniform CDF for \( F_i(.) \).

\[ \lim_{\epsilon \to 0} \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2) = \left( \frac{\bar{v}_1 - p_2}{\bar{v}_1 - v_1} - \frac{\bar{v}_2 - v_2}{(\bar{v}_1 - v_1)} \left( \frac{2D_1}{D_1 + D_2 - K} \right) \right) \bar{v}_2 D_1 + \left( \frac{-\bar{v}_1 - p_2}{\bar{v}_1 - v_1} + \frac{\bar{v}_2 - v_2}{(\bar{v}_1 - v_1)} \left( \frac{2D_1}{D_1 + D_2 - K} \right) \right) p_2 D_1 + (K - D_1)p_2 \frac{\bar{v}_2 - v_2}{\bar{v}_2 - v_2} \]

The first derivative with respect to \( p_2 \) is:

\[ \frac{\partial \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2)}{\partial p_2} = -\frac{\bar{v}_3 D_1}{\bar{v}_1 - v_1} + p_2 D_1 + \left( -\frac{\bar{v}_1 - p_2}{\bar{v}_1 - v_1} + \frac{\bar{v}_2 - v_2}{(\bar{v}_1 - v_1)} \left( \frac{2D_1}{D_1 + D_2 - K} \right) \right) D_1 + (K - D_1) \frac{\bar{v}_2 - v_2}{\bar{v}_2 - v_2} \]

Taking the second derivative with respect to \( p_2 \), we get:

\[ \frac{\partial^2 \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2)}{\partial p_2^2} = \frac{2D_1}{\bar{v}_1 - v_1} - \frac{2(K - D_1)}{\bar{v}_2 - v_2} \]

Hence the function is concave if \( \frac{(K - D_1)}{\bar{v}_2 - v_2} \geq \frac{D_1}{\bar{v}_1 - v_1} \) and concave otherwise.

(ii) Given that \( \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_1) \) is concave in \( p_2 \), the optimal step 2 price is \( p_{2}^{PS}(\tilde{v}_2) = \frac{\bar{v}_2}{2} + \frac{p_2(\bar{v}_1) D_1}{\bar{v}_1 - \bar{v}_2} \), provided that \( p_{2}^{PS}(\tilde{v}_2) \) is in the \( p_2 \) range for which a PS markdown exists.

We first show that \( p_{2}^{PS}(\tilde{v}_2) \) is less than \( \bar{v}_2 \). Since \( \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2) \) is concave, the denominator of the second term in \( p_{2}^{PS}(\tilde{v}_2) \) expression above is negative, hence \( p_{2}^{PS}(\tilde{v}_2) < \frac{\bar{v}_2}{2} < \bar{v}_2 \).

If \( p_{2}^{PS}(\tilde{v}_2) < \frac{\bar{v}_2}{2} \), we conclude that the revenue function is decreasing in \( p_2 \) over the entire range and \( \frac{\bar{v}_2}{2} \) is optimal. If \( p_{2}^{PS}(\tilde{v}_2) > \frac{\bar{v}_2}{2} \), then the revenue function is increasing in \( p_2 \) at \( p_2 = \frac{\bar{v}_2}{2} \), and we have to compare \( p_{2}^{PS}(\tilde{v}_2) \) with the maximum \( p_2 \) for which a PS markdown exists in order to find the optimal step 2 price. From Observation 12 we know that a PS markdown exists for all \( p_2 < \min\{p_2(\bar{v}_1), \bar{v}_2\} \). If \( p_{2}^{PS}(\tilde{v}_2) < p_2(\bar{v}_1) \), then the markdown with prices \( (\bar{v}_2, p_{2}^{PS}(\tilde{v}_2)) \) is optimal. If \( p_{2}^{PS}(\tilde{v}_2) \geq p_2(\bar{v}_1) \), then \( p_2(\bar{v}_1) \) is optimal.

(iii) When \( \Pi_S^{PS}(\tilde{v}_2 + \epsilon, p_2) \) is convex in \( p_2 \), one of the endpoints of the \( p_2 \) range will be the optimal step 2 price. If \( p_2(\bar{v}_1) < \bar{v}_2 \), then the \( p_2 \) range for which a PS markdown exists is \([\bar{v}_2, \bar{v}_2] \). Otherwise, the \( p_2 \) range is \([\bar{v}_2, p_2(\bar{v}_1)] \).
In each case, we identify which endpoint maximizes the seller’s revenue by evaluating \( \Pi^\text{PS}(\bar{v}_2 + \epsilon, p_2) \) at each endpoint and comparing the resulting revenues.

For \( p_2 \in [\bar{v}_2, \tilde{v}_2] \), we show that \( p^\text{PS}_2(\bar{v}_2) = \bar{v}_2 - \epsilon \) if \( \Pi^\text{PS}_S(\bar{v}_2 + \epsilon, \bar{v}_2) \leq \Pi^\text{PS}_S(\bar{v}_2 + \epsilon, \tilde{v}_2 - \epsilon) \). By substituting the \( p_1 \) and \( p_2 \) values in (9), we get the condition for \( p^\text{PS}_2(\bar{v}_2) = \bar{v}_2 - \epsilon \):

\[
\lim_{\epsilon \to 0} \Pi^\text{PS}_S(\bar{v}_2 + \epsilon, \tilde{v}_2 - \epsilon) = \bar{v}_2 D_1, \text{ hence we require } p_2(\tilde{v}_1) \left( D_1 + (K - D_1)\frac{\bar{v}_1 - \tilde{v}_1}{\bar{v}_2 - \tilde{v}_2} \right) \geq \bar{v}_2 D_1. \]

Rearranging terms and simplifying we get, \( p_2(\tilde{v}_1) \geq \left( 1 - \frac{(K-D_1)(\bar{v}_1 - \tilde{v}_1)}{D_1(\bar{v}_2 - \tilde{v}_2)} \right) \bar{v}_2 \).

Similarly, for \( p_2 \in [\bar{v}_2, p_2(\tilde{v}_1)] \), we show that \( p^\text{PS}_2(\bar{v}_2) = p_2(\tilde{v}_1) \) if \( \Pi^\text{PS}_S(\bar{v}_2 + \epsilon, \bar{v}_2) \leq \Pi^\text{PS}_S(\bar{v}_2 + \epsilon, \tilde{v}_2) \).

\[
\lim_{\epsilon \to 0} \Pi^\text{PS}_S(\bar{v}_2 + \epsilon, \tilde{v}_2) = p_2(\tilde{v}_1) \left( D_1 + (K - D_1)\frac{\tilde{v}_1 - \bar{v}_1}{\tilde{v}_2 - \bar{v}_2} \right), \text{ and }
\lim_{\epsilon \to 0} \Pi^\text{PS}_S(\tilde{v}_2 + \epsilon, \bar{v}_2) = K \bar{v}_2 + D_1(\bar{v}_2 - \bar{v}_2) \left( \frac{\bar{v}_1 - \tilde{v}_1}{\bar{v}_2 - \tilde{v}_2} - \frac{\bar{v}_2 - \bar{v}_1}{\bar{v}_2 - \tilde{v}_2} \frac{2D_1}{D_1 + D_2 - K} \right). \]

Hence, we have the following as the condition for \( p^\text{PS}_2(\bar{v}_2) = p_2(\tilde{v}_1) \):

\[
p_2(\tilde{v}_1) \left( D_1 + (K - D_1)\frac{\bar{v}_1 - \tilde{v}_1}{\bar{v}_2 - \tilde{v}_2} \right) \geq K \bar{v}_2 + D_1(\bar{v}_2 - \bar{v}_2) \left( \frac{\bar{v}_1 - \tilde{v}_1}{\bar{v}_2 - \tilde{v}_2} - \frac{\bar{v}_2 - \bar{v}_1}{\bar{v}_2 - \tilde{v}_2} \frac{2D_1}{D_1 + D_2 - K} \right).
\]

For both \( p_2 \) ranges, the other alternative optimal is to have \( p_2 \) at the lower bound, which is \( \bar{v}_2 \), hence we get the conditions given in the result. \( \square \)

**Comparing PS and TS Markdowns**

**Proposition 20.** When \( \mathcal{P}_T \neq \emptyset \),

(i) the optimal markdown is PS if \( \bar{v}_1 < \frac{\bar{v}_1 + \tilde{v}_2}{2} \).

(ii) the optimal markdown is TS if \( \bar{v}_1 \geq \frac{\bar{v}_1 + \tilde{v}_2}{2} \).

Proposition 20(i) implies that it is optimal for the seller to only partially separate the high types when the range of the high types is large relative to the low valuation range. Proposition 20(ii) implies that the seller is best served by inducing all of the high types to purchase at the first price step when the ‘high’ and ‘low’ types are fairly far apart.

**Proof of Proposition 20:** (i) From Theorem 17 we know that for a given \( p_2 \), the PS revenue function is maximized at \( p^*_1(\tilde{v}_2) = p^*_1(\bar{v}_2) \) if \( \bar{v}_1 < \frac{\bar{v}_1 + \tilde{v}_2}{2} \) (i.e., \( p^*_1(\bar{v}_2) > \tilde{p}_1(\bar{v}_1, p_2) \)) and \( p_2 < \bar{v}_2 \) (i.e., \( p^*_1(\bar{v}_2) > \tilde{v}_2 \)). Since Proposition 20 is stated only for the case when TS is feasible, from Equation 8, we have \( \bar{v}_2 < \tilde{p}_1(\bar{v}_1, p_2) \). Hence, the condition \( \bar{v}_1 < \frac{\bar{v}_1 + \tilde{v}_2}{2} \) combined with Equation 8 automatically implies that \( p^*_1(\tilde{v}_2) > \bar{v}_2 \) and therefore \( p^*_1(\tilde{v}_2) \) is optimal, i.e., the PS markdown’s optimal step 1 price is not at the boundaries. Since the PS revenue is concave (and the boundaries approach the TS revenue), the PS markdown dominates the TS markdown.

(ii) From Theorem 17, we know that for a given \( p_2 \), the PS revenue function is maximized at \( p^*_1(\tilde{v}_2) = \max\{\tilde{p}_1(\bar{v}_1, p_2), \bar{v}_2\} \) + \( \epsilon \) if \( \bar{v}_1 \geq \frac{\bar{v}_1 + \tilde{v}_2}{2} \). As \( \epsilon \to 0 \), the markdown converges to a TS markdown, and hence the TS markdown revenue exceeds the optimal PS markdown revenue in this case. If \( \bar{v}_1 \geq \frac{\bar{v}_1 + \tilde{v}_2}{2} \), then \( \bar{v}_1 \geq \frac{\bar{v}_1 + \tilde{v}_2}{2} \) for all \( p_2 \in [\bar{v}_2, \tilde{v}_2] \) since the right-hand-side is maximized at \( p_2 = \tilde{v}_2 \). \( \square \)

**Tabulated Results for the Numerical Examples in Section 4.3**

Each table corresponds to one parameter being changed while keeping everything else the same. First column of the header identifies the parameter that is altered and the original value. Each row corresponds to a different instance and the new value of the parameter is given in the first column in that row. The last table considers two simultaneous parameter changes, and the new value of
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textbf{$D_1 = 3$} & \multicolumn{3}{c|}{\textbf{Single Price}} & \multicolumn{2}{c|}{\textbf{TS Markdown}} & \multicolumn{2}{c|}{\textbf{PS Markdown}} \\
\hline
& $p^*$ & Revenue & Type & $p_1^{TS}$ & $p_2^{TS}$ & Revenue & $p_1^{PS}$ & $p_2^{PS}$ & Revenue \\
\hline
2 & 2.67 & 42.67 & SP2 & none exists & 5.00 & 2.5 & 45.42 \\
3 & 2.76 & 43.31 & SP2 & 5.17 & 2.41 & 50.88 & 5.17+$\epsilon$ & 2.41 \\
4 & 2.88 & 44.08 & SP2 & 5.58 & 2.29 & 55.42 & 5.58+$\epsilon$ & 2.29 \\
5 & 12.00 & 60.00 & SP1 & 5.89 & 2.15 & 60.10 & 5.89+$\epsilon$ & 2.15 \\
6 & 12.00 & 72.00 & SP1 & 6.17 & 2.00 & 65.00 & 6.17+$\epsilon$ & 2.00 \\
7 & 12.00 & 84.00 & SP1 & 6.29 & 2.00 & 70.00 & 6.29+$\epsilon$ & 2.00 \\
8 & 12.00 & 96.00 & SP1 & 6.38 & 2.00 & 75.00 & 6.38+$\epsilon$ & 2.00 \\
9 & 12.00 & 108.00 & SP1 & 6.44 & 2.00 & 80.00 & 6.44+$\epsilon$ & 2.00 \\
10 & 12.00 & 120.00 & SP1 & 6.50 & 2.00 & 85.00 & 6.50+$\epsilon$ & 2.00 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
20 & 12.00 & 240.00 & SP1 & 6.75 & 2.00 & 135.00 & 6.75+$\epsilon$ & 2.00 \\
\hline
\end{tabular}
\caption{Varying $D_1$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\textbf{$D_2 = 19$} & \multicolumn{3}{c|}{\textbf{Single Price}} & \multicolumn{2}{c|}{\textbf{TS Markdown}} & \multicolumn{2}{c|}{\textbf{PS Markdown}} \\
\hline
& $p^*$ & Revenue & Type & $p_1^{TS}$ & $p_2^{TS}$ & Revenue & $p_1^{PS}$ & $p_2^{PS}$ & Revenue \\
\hline
18 & 2.76 & 43.31 & SP2 & none exists & none exists & none exists & none exists \\
19 & 2.76 & 43.31 & SP2 & 5.17 & 2.41 & 50.88 & 5.17+$\epsilon$ & 2.41 \\
20 & 2.76 & 43.31 & SP2 & 6.76 & 2.21 & 55.23 & 6.76+$\epsilon$ & 2.21 \\
\hline
\end{tabular}
\caption{Varying $D_2$.}
\end{table}

$v_1$ is presented at the top left corner and first column in each row has the value of $D_1$ used in that row. Whenever the PS markdown converges to a TS markdown, $\epsilon$ is used to differentiate step 1 prices of PS and TS markdowns. In this case, the corresponding PS markdown revenue is slightly less than the TS markdown revenue, hence the corresponding cell has been left blank. $\theta$ is similar to $\epsilon$, and it is used to assure that customer 2 cannot bid at $p_1$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textbf{$K = 20$} & \multicolumn{3}{c|}{\textbf{Single Price}} & \multicolumn{2}{c|}{\textbf{TS Markdown}} & \multicolumn{2}{c|}{\textbf{PS Markdown}} \\
\hline
& $p^*$ & Revenue & Type & $p_1^{TS}$ & $p_2^{TS}$ & Revenue & $p_1^{PS}$ & $p_2^{PS}$ & Revenue \\
\hline
19 & 2.78 & 41.26 & SP2 & 6.19 & 2.78 & 53.17 & 6.19+$\epsilon$ & 2.78 \\
20 & 2.76 & 43.31 & SP2 & 5.17 & 2.41 & 50.88 & 5.17+$\epsilon$ & 2.41 \\
21 & 2.75 & 45.38 & SP2 & none exists & none exists & none exists & none exists \\
\hline
\end{tabular}
\caption{Varying $K$.}
\end{table}
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<th>PS Markdown</th>
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<td>2.76</td>
<td>43.31</td>
<td>SP2</td>
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<tr>
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<td>60.00</td>
<td>SP2</td>
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<td>80.00</td>
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Table 8: Varying $v_2$.

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<th>PS Markdown</th>
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<td>52.81</td>
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Table 9: Varying $v_2$.

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<th>PS Markdown</th>
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<td>Type</td>
<td>$p_1^{TS}$</td>
</tr>
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<td>2.76</td>
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<td>SP2</td>
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Table 10: Varying $v_1$. 

12
<table>
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<th>$\bar{v}_1 = 18$</th>
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<tr>
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<td>$p_1^{TS}$</td>
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<td>2.76</td>
<td>43.31</td>
<td>SP2</td>
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Table 11: Varying $\bar{v}_1$. 