2-Lattice Polyhedra: Duality

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Abstract

This is the first in a series of papers that explores a class of polyhedra we call 2-lattice polyhedra. 2-Lattice polyhedra are a special class of lattice polyhedra that include network flow polyhedra, fractional matching polyhedra, matroid intersection polyhedra, the intersection of two polymatroids, etc. In this paper we show that the maximum sum of components of a vector in a 2-lattice polyhderon is equal to the minimum capacity of a cover for the polyhedron. For special classes of 2-lattice polyhedra, called matching 2-lattice polyhedra, that include all of the mentioned special cases except the intersection of two polymatroids, we characterize the largest member in the family of minimum covers in terms of the maximum "cardinality" vectors in the polyhedron. In fact, we show that this same characterization arises from considering only the extreme maximum cardinality vectors. This characterization is at the heart of our extreme point algorithm [3] for finding a maximum cardinality vector in a matching 2-lattice polyhedron.

1 Introduction

This is the first in a series of papers that explores a class of polyhedra called 2-lattice polyhedra. Vande Vate [30] first introduced 2-lattice polyhedra as a natural linear relaxation of the matroid matching problem and showed that the relationship between matroid matching and its linear relaxation via 2-lattice polyhedra is in many ways analogous to the relationship between graphic matching and its linear relaxation via fractional matching.

When the matroid matching problem is in fact a graphic matching problem, its relaxation via a 2-lattice polyhedron is the corresponding fractional matching polytope. Further, just as the graphic matching polytope and the fractional matching polytope coincide when the underlying graph is bipartite, the matroid matching polytope and its relaxation via a 2-lattice polyhedron coincide when the matroid matching problem is in fact a matroid intersection problem. More generally, the extreme points of a 2-lattice polyhedron are half-integral and can be characterized by the extreme points of fractional matching polyhedra.

These close analogies suggest the possibility that, just as the graphic matching polytope can be obtained by adding "rank 1" inequalities to the constraints defining the fractional matching polytope, the matroid matching polytope might be obtained by adding rank 1 inequalities to the constraints defining the corresponding 2-lattice polyhedron. The demonstrated intractability of the matching problem in general matroids [25, 21] suggests that this is not possible in general and Vande Vate [30] showed that a natural 2-lattice relaxation need not have this rank 1 property even for representable matroids. Nevertheless, it remains an open question whether there is a 2-lattice relaxation of the matching polytope for a representable matroid that enjoys this rank 1 property. Thus, one motivation for further investigation into 2-lattice polyhedra is to resolve this important question.

Although there are a number of polynomial algorithms for finding a maximum cardinality matching in a representable matroid [25, 27, 13], there can be no efficient algorithm for the problem in general matroids that relies on an oracle to determine ranks [25, 21]. The distinction in the tractability of these two problems does not carry over to their natural relaxations via 2-lattice polyhedra. The third paper in this series presents an extreme point algorithm for finding a vector with maximum sum of components in a 2-lattice relaxation of the general matroid matching problem. This algorithm is efficient if there is an efficient procedure for recognizing the extreme points of the 2-lattice polyhedron. The second paper [2] presents an efficient procedure for this problem. Thus, another motivation for further investigation into 2-lattice polyhedra is to sharpen the boundary between tractable and intractable problems.

2-Lattice polyhedra belong to a class of polyhedra, called lattice polyhedra, originally introduced by Hoffman and Schwartz [20]. Many classic polyhedra including polymatroid polyhedra, network flow polyhedra and submodular flow polyhedra fall into this class. Although there are efficient algorithms for optimizing a linear function over all of the special cases mentioned, there is to date no polynomial time algorithm for optimizing a linear function over a general lattice polyhedron. The optimization problem is equivalent to the separation problem [17] and separation over a lattice polyhedron is equivalent to submodular function minimization. At present, however, algorithms for minimizing a submodular function [6] assume the underlying lattice has join and meet defined by set union and set intersection. 2-Lattice polyhedra represent a computationally tractable class of lattice polyhedra for which the underlying lattice has more general meet and join. Thus, another motivation for further investigation into 2-lattice polyhedra is to extend the practical scope of submodularity.

In this, the first paper in the series, we explore duality relationships for the problem of finding a vector in a 2-lattice polyhedron with maximum sum of components. This problem generalizes the problems of finding a maximum cardinality matching in a bipartite graph, finding a maximum cardinality intersection in two polymatroids, and other related problems. Thus, it is not surprising that our min-max characterization, which states that the maximum "cardinality" of a vector in a 2-lattice polyhedron is the minimum capacity of a cover, generalizes such classic special cases as König's Theorem [23], Menger's Theorem [26], Dilworth's Theorem [7] and Edmonds' theorems for cardinality matroid intersection [8] and polymatroid intersection [11]. In fact, the methods used to prove this result are by now rather standard.

There are, however, more intimate duality relationships for these problems. For example, Shapley and Shubik [29] showed that the collection of optimal dual solutions to a bipartite matching problem forms a lattice and this result extends to the cardinality matroid intersection problem. This paper shows that the collection of minimum covers for a 2-lattice polytope contains an upper semi-lattice. It remains an open question whether these covers in fact form a lattice.

Understanding the structure of certain optimal dual solutions often provides insight into the structure of all optimal primal solutions. For example, König's Theorem [23] can be refined to characterize all maximum cardinality matchings in terms of a specific minimum cover. This result extends to the cardinality matroid intersection problem and, in fact, to non-bipartite matching. The Gallai-Edmonds Theorem [10, 14], which characterizes all maximum cardinality matchings in a non-bipartite graph in terms of a specific minimum odd set cover, is at the heart of non-bipartite matching algorithms. This paper extends König's characterization to those 2-lattice polyhedra, called matching 2-lattice polyhedra, that arise as linear relaxations of matroid matching problems. In particular, we characterize a minimum capacity cover, called the dominant cover, of a matching 2-lattice polyhedron in terms of the collection of all maximum cardinality vectors. This characterization is at the heart of our algorithm [3] for finding a maximum cardinality vector in a matching 2-lattice polyhedron.

The second paper in this series takes up the problem of characterizing and recognizing extreme points of matching 2-lattice polyhedra. We show that the problem of determining whether a given half-integral vector is an extreme point of a matching 2-lattice polyhedron is equivalent to finding a maximum word in a greedoid on a possibly infinite alphabet. This problem raises an interesting issue in complexity analysis: Matching 2-lattice polyhedra can be defined on a lattice over an infinite ground set, eg., the lattice of subspaces of a vector space, but this ground set is not explicitly part of the problem data. Our algorithms assume there are oracles that provide information about the lattice and require only polynomially many calls to these oracles. The third paper in this series develops these results into an efficient extreme point algorithm for finding a maximum cardinality vector in a matching 2-lattice polyhedron. This algorithm generalizes augmenting path algorithms for finding a maximum cardinality intersection in two matroids, although the possibility of half-integral components makes it more complicated. It also provides an extreme point method for finding a maximum cardinality vector in a fractional matching polytope.

Section 3 gives notation and preliminaries. In Section 4 we prove our min-max theorem for 2-lattice polyhedra and show that the family of minimum covers of a 2-lattice polyhedron contains an upper semi-lattice. In Section 5, we characterize the largest member in the family of nested minimum covers for matching 2-lattice polyhedra in terms of maximum cardinality matching 2-lattice vectors. This last result is at the heart of our algorithm [3] for solving (2.2) and (2.3).

2 2-Lattice Polyhedra

Let L be a finite set of elements (called lines) and let Γ be a finite lattice with partial order $(\Gamma, \preceq)$, which induces meet operation $\wedge$ and join operation $\vee$. Let $\beta : \Gamma \mapsto Z$ be submodular and, for each element $\ell \in L$, let $\alpha_\ell : \Gamma \mapsto Z$ be supermodular. Given $S \in \Gamma$ and $x \in \mathbb{R}_+^{|L|}$, let $\alpha(S)x = \sum(\alpha_\ell(S)x(\ell) : \ell \in L)$. Then

$$\{x \in \mathbb{R}_+^{|L|} : \alpha(S)x \leq \beta(S) \text{ for each } S \in \Gamma\} \tag{2.1}$$

is a *lattice polyhedron*. Lattice polyhedra were introduced by Hoffman and Schwartz [20] and independently by Johnson [22], and further studied by Hoffman [18], and Gröflin and Hoffman [15] (We use the term “lattice polyhedron” somewhat differently than its coiners, who further restrict α to be $0, \pm 1$ valued).

Here we consider those lattice polyhedra in which we allow Γ to be infinite, but require a finite bound on the length of chains in Γ. This ensures that Γ is a complete lattice and includes, for example, the lattice of linear subspaces of a finite dimensional vector space. We further require that $\beta : \Gamma \mapsto Z_+$ and for each $\ell \in L$, α_ℓ is not only supermodular, but also non-decreasing and maps Γ into $\{0, 1, 2\}$. The set

$$P(\alpha, \beta) = \{x \in \mathbb{R}_+^{|L|} : \alpha(S)x \leq \beta(S) \text{ for each } S \in \Gamma\},$$

is called a 2-lattice polyhedron and each vector $x \in P(\alpha, \beta)$ is called a *2-lattice vector*. Examples of 2-lattice polyhedra include bipartite matching polyhedra [19, 24], the intersection of two integral polymatroids [11], and the perfectly matchable subgraph polytope of a bipartite graph [1].

In this paper we consider the relationships between the problem of finding a 2-lattice vector with maximum sum of components:

$$\begin{aligned} \max \sum_{\ell \in L} & x(\ell) \\ \text{s.t } \alpha(S)x & \leq \beta(S) \text{ for each } S \in \Gamma \\ x & \geq 0 \end{aligned} \tag{2.2}$$

and the dual problem:

$$\begin{aligned} \min \sum_{S \in \Gamma} & y(S)\beta(S) \\ \text{s.t. } \sum_{S \in \Gamma} y(S)\alpha_\ell(S) & \geq 1 \quad \text{for each } \ell \in L \\ y & \geq 0 \end{aligned} \tag{2.3}$$

This paper focuses on the relationships between the the linear programs (2.2) and (2.3), not on the integrality of extreme solutions to (2.2). We refer to $\sum_{\ell \in L} x(\ell)$ as the "cardinality" of a vector x even though x may not be integral.

We show that the maximum cardinality of a 2-lattice vector is the minimum capacity of a "cover". Special cases of this result include König's Theorem [23], Menger's Theorem [26], Dilworth's Theorem [7], and Edmonds' Theorems for cardinality matroid intersection and polymatroid intersection [11].

The primary purpose of this paper is to establish the more detailed duality relationships between the linear programs (2.2) and (2.3) upon which we develop an efficient combinatorial algorithm for solving them. In fact, we show that the collection of minimum covers of a 2-lattice polyhedron contains an upper semi-lattice and, when the 2-lattice polyhedron is of a special class called matching 2-lattice polyhedra, we characterize the largest member of this semi-lattice in terms of the collection of maximum cardinality 2-lattice vectors. This characterization generalizes analogous results for the matroid intersection problem and is at the heart of our algorithm for solving (2.2) and (2.3).

2.1 Matching 2-lattice Polyhedra

All the classic examples of 2-lattice polyhedra relate α and β in some way. We capture these relationships with the following general conditions. First, let $\mathcal{E}$ be a (possibly infinite) set, and let L be a finite subset of $2^{\mathcal{E}}$ (generally chosen to

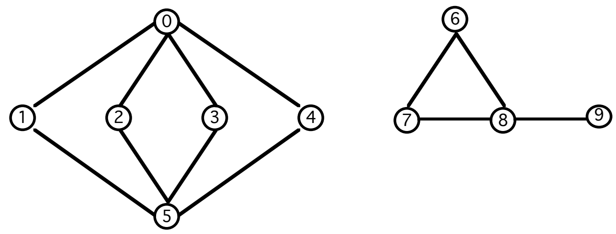

Figure 2.1: Example

be a collection of pairs from $\mathcal{E}$). We also require that Γ include $\mathcal{E}$, the empty set and be closed under intersections. In this way, we may associate with each set $S \subseteq \mathcal{E}$ the smallest member, $\sigma(S)$, of Γ containing S. We further require that β be normalized, i.e., $\beta(\emptyset) = 0$, increasing and satisfy $\beta(\sigma\{e\}) = 1$ for each $e \in \mathcal{E}$, and $\beta(\sigma(\ell)) = 2$ for each $\ell \in L$. Finally, we model the relationship between α and β via the condition $\alpha_\ell(S) = \beta(\sigma(\ell) \wedge S)$ for each $\ell \in L$ and $S \in \Gamma$. It is easy to see that α_ℓ is normalized and non-decreasing. It is also straightforward to prove (see [30]) that α_ℓ is supermodular. We call the resulting 2-lattice polyhedra matching 2-lattice polyhdera.

To avoid awkward notation, we extend the meet and join operations of Γ to all subsets of $\mathcal{E}$ so that for S and $T \subseteq \mathcal{E}$, $S \wedge T = \sigma(S) \cap \sigma(T)$ and $S \vee T = \sigma(S) \vee \sigma(T)$. We also extend the range of α_ℓ and β to $2^{\mathcal{E}}$ as follows. For $S \subset \mathcal{E}$, let $\beta(S) = \beta(\sigma(S))$ and let $\alpha_\ell(S) = \beta(S \wedge \ell)$. We must exercise some care in employing this extension: while $\alpha_\ell : \Gamma \rightarrow \{0, 1, 2\}$ is supermodular, its extension to $2^{\mathcal{E}}$ may not be.

Note that when $\mathcal{E}$ is finite, Γ is the collection of flats and β is the rank function of a matroid. We belabor these definitions, however, because of our interest in those cases in which $\mathcal{E}$ is infinite. The following examples illustrate a hierarchy of matching 2-lattice polyhedra and motivate our interest in those problems in which $\mathcal{E}$ is infinite.

When Γ is the collection of all subsets of a finite set $\mathcal{E}$ and L is a partition of $\mathcal{E}$ into pairs we refer to the matching 2-lattice polyhedra $P(\alpha, \beta)$ as an *incidence 2-lattice* polyhedron (note that in this setting , $\alpha_\ell : \Gamma \rightarrow \{0, 1, 2\}$ is defined by $\alpha_\ell(S) = |S \cap \ell|$). Integral incidence 2-lattice polyhedra include bipartite matching polytopes [19, 24], network flow polyhedra [12], and the intersection of two matroids [11]. Incidence 2-lattice polyhedra have also been studied in the context of non-bipartite matching [28].

Example 1 Let $\beta : 2^{\mathcal{E}} \rightarrow Z_+$ be the rank function of the cycle matroid for the graph shown in Figure 2.1 and let $L = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$, where $\ell_1 = \{(0,1),(1,5)\}$, $\ell_2 = \{(0,2),(2,5)\}$, $\ell_3 = \{(0,3),(3,5)\}$, $\ell_4 = \{(0,4),(4,5)\}$, $\ell_5 = \{(6,7),(6,8)\}$, $\ell_6 = \{(7,8),(8,9)\}$. Then, $P(\alpha, \beta)$ is the set of $x \in \mathbb{R}^6_+$

satisfying

$$\begin{aligned}
2x_i &\le 2 && \text{for } i = 1, 2, \ldots, 6 \\
2x_i + 2x_j &\le 3 && \text{for } i, j \in \{1, \ldots, 4\}, i \ne j \\
2x_i + 2x_j + 2x_k &\le 4 && \text{for } i, j, k \in \{1, \ldots, 4\}, i \ne j \ne k \\
2x_1 + 2x_2 + 2x_3 + 2x_4 &\le 5 \\
2x_5 + x_6 &\le 2
\end{aligned}$$

Frequently, we are interested in the convex hull of the integer solutions to a given system of inequalities. Despite the significant successes to date, the formulation via an incidence 2-lattice polyhedron is not always the best available. We can, for instance, improve the incidence formulation in Example 1 via the following matroid formulation. When β is the rank function of a matroid $\mathbf{M}$ defined on $\mathcal{E}$, L is a partition of $\mathcal{E}$ into pairs, and Γ is the lattice of flats or closed subsets in $\mathbf{M}$, we refer to $P(\alpha, \beta)$ as a *matroid 2-lattice* polyhedron.

Example 2 Let $\beta : \Gamma \to \mathbb{R}_+$ be the rank function and let Γ be the flats of the cycle matroid of the graph in Figure 2.1. Under the matroid formulation $P(\alpha, \beta)$ is the set of $x \in \mathbb{R}^6_+$ satisfying

$$\begin{aligned}
2x_i &\le 2 && \text{for } i = 1, 2, \ldots, 6 \\
2x_i + 2x_j &\le 3 && \text{for } i, j \in \{1, \ldots, 4\}, i \ne j \\
2x_i + 2x_j + 2x_k &\le 4 && \text{for } i, j, k \in \{1, \ldots, 4\}, i \ne j \ne k \\
2x_1 + 2x_2 + 2x_3 + 2x_4 &\le 5 \\
2x_5 + 2x_6 &\le 2
\end{aligned}$$

Note that this formulation has the same integral solutions as that of Example 1, but has cut off all extreme points with $x_5 = \frac{1}{2}$ and $x_6 = 1$. For example, it has cut off the extreme points $(0, 0, 0, 0, \frac{1}{2}, 1)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$.

When the matroid is linear and a representation is available, we can do still better than the matroid formulation via the following linear formulation. Let A be a rational matrix and let V denote the linear subspace spanned by the columns of A. We refer to $P(\alpha, \beta)$ as a *linear 2-lattice* polyhedron when L is a collection of pairs of columns of A, Γ is the lattice of linear subspaces of V and, for each $S \in \Gamma$, $\beta(S)$ denotes the linear rank of S.

Example 3 Let A be the node-edge incidence matrix of a directed version of the graph in Figure 2.1. Under the linear 2-lattice formulation, $P(\alpha, \beta)$ is given by the set of $x \in \mathbb{R}^6_+$ satisfying

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &\le 1 \\
x_5 + x_6 &\le 1
\end{aligned}$$

Notice that this is in fact the convex hull of integral solutions to the polyhedron defined in Example 1.

The differences between linear 2-lattice polyhedra and the corresponding matroid 2-lattice polyhedra motivated us to consider those cases in which $\mathcal{E}$ is infinite. We conjecture, for example, that linear 2-lattice polyhedra do have Chvátal rank 1.

3 Preliminaries

For ease of argument and presentation, we append a new smallest element, $*$, to Γ to form a complete lattice Γ^* and define $\beta(*) = 0$ and $\alpha_\ell(*) = 0$ for each $\ell \in L$. Note that as a smallest element in Γ^*, $S \wedge * = *$ and $S \vee * = S$ for each $S \in \Gamma$. Further, since $\alpha_\ell(*) = \beta(*) = 0$ it is clear that α_ℓ is supermodular and β is submodular on Γ^*.

Given S and T in Γ, $\beta(S/T)$ is defined by

$$\beta(S/T) = \beta(S \vee T) - \beta(T).$$

For $x \in \mathbb{R}_+^{|L|}$ and $S \subseteq L$, we define $x_S \in \mathbb{R}_+^{|L|}$ by

$$x_S(\ell) = \begin{cases} x(\ell) & \text{if } \ell \in S \\ 0 & \text{otherwise} \end{cases}$$

and we denote the support of x by *supp*(x). We refer to the members in Γ as *flats* and denote by $\Gamma(x) = \{S \in \Gamma : \alpha(S)x = \beta(S)\}$ the set of flats tight with respect to a 2-lattice vector x. The following lemma shows that $\Gamma(x)$ is a sublattice of Γ.

Lemma 3.1 *Let x be a 2-lattice vector and suppose S and S' are in $\Gamma(x)$, then $S \vee S'$ and $S \wedge S'$ are in $\Gamma(x)$.*

Proof.

$$\begin{aligned} \beta(S \vee S') + \beta(S \wedge S') &\geq \alpha(S \vee S')x + \alpha(S \wedge S')x \\ &\geq \alpha(S)x + \alpha(S')x \\ &= \beta(S) + \beta(S') \\ &\geq \beta(S \vee S') + \beta(S \wedge S') \end{aligned}$$

□

Since $\Gamma(x)$ is a sublattice of the complete lattice Γ, it has a largest member, which we denote by *cl*(x).

The following lemma is an immediate consequence of Lemma 3.1 and will prove useful in arguing that certain vectors $x \in \mathbb{R}_+^{|L|}$ are 2-lattice vectors.

Lemma 3.2 *Let $x \in I\!R_+^{|L|}$ and suppose Z and Z' are flats such that*

$$\begin{array}{rcl} \alpha(Z)x & > & \beta(Z), \\ \alpha(Z')x & = & \beta(Z') \text{ and} \\ \alpha(Z \wedge Z')x & \leq & \beta(Z \wedge Z'), \end{array}$$

then $\alpha(Z \vee Z')x > \beta(Z \vee Z')$.

Proof. By assumption,

$$\begin{array}{rcl} \alpha(Z \vee Z')x + \alpha(Z \wedge Z')z & \geq & \alpha(Z)x + \alpha(Z')x \\ & > & \beta(Z) + \beta(Z') \\ & \geq & \beta(Z \vee Z') + \beta(Z \wedge Z') \\ & \geq & \beta(Z \vee Z') + \alpha(Z \wedge Z')x. \end{array}$$

Hence, $\alpha(Z \vee Z')x > \beta(Z \vee Z')$. □

Theorem 3.3, is a very slight generalization of Theorem 4.2 in [30] and provides a mechanism for describing extreme 2-lattice vectors in terms of perfect fractional matchings of graphs. Given a graph $G = (V, E)$ and an integer vector $b \in \mathbb{R}^{|V|}$, the *perfect fractional b-matching polytope* of G, denoted $FP(G, b)$, is:

$$\{x \in \mathbb{R}_+^{|E|} : \sum(d_e(v)x(e) : e \in E) = b(v) \text{ for each } v \in V\}.$$

Here, $d_e(v)$ is the degree of edge e at node v. As the graph G may have loops, $d_e(v) \in \{0, 1, 2\}$ and as the graph G may have spurs (i.e., edges with only one end), $\sum(d_e(v) : v \in V) \in \{1, 2\}$. Letting D be the $|V| \times |E|$ matrix with elements $d_e(v)$, $FP(G, b)$ may be written as:

$$FP(G, b) = \{x \in \mathbb{R}_+^{|E|} : Dx = b\}$$

Each vector $x \in FP(G, b)$ is a *perfect fractional b-matching* (or, more briefly, a fractional matching) of G.

A subset T of edges in a graph G is a *bloom* if the subgraph induced by the edges in T is connected, contains exactly one cycle and that cycle has an odd number of edges. A subset T of columns is a base of the node-edge incidence matrix of a graph G if and only if the corresponding set of edges is a maximal set with the property that each component of the subgraph (V, T) is either a tree or a bloom. (If G has spurs, we add a distinguished node, called the *root*, incident to each spur edge. In this case, the component containing the root must be a tree). When a set of columns is a base of the node-edge incidence matrix of a graph G, we also refer to the corresponding set of edges as a *base* of G.

Gröflin and Hoffman [15] showed that each extreme 2-lattice vector x^* is defined by a subset N of L and a family $\mathcal{S} = \{S_i : i \in [1, \ldots, t]\}$ of flats with $S_1 \prec S_2 \prec \ldots \prec S_t$. The pair $(\mathcal{S}, N)$ induces a graph, denoted $G(\mathcal{S}, L \setminus N)$, defined as follows. For each $S_i \in \mathcal{S}$, there is a node S_i in $G(\mathcal{S}, L \setminus N)$ and for each line $\ell \in L \setminus N$ there is an edge ℓ in $G(\mathcal{S}, L \setminus N)$. Let $S_0 = *$. The edge ℓ is incident to node S_i if $\alpha_\ell(S_i) - \alpha_\ell(S_{i-1}) = 1$ and is a loop at node S_i if $\alpha_\ell(S_i) - \alpha_\ell(S_{i-1}) = 2$.

Theorem 3.3 ([30]) *A 2-lattice vector x^* is extreme if and only if there is a subset N of L and a family $\mathcal{S} = \{S_i : i \in [1, \ldots, t]\}$ of flats with $S_1 \prec S_2 \prec \ldots \prec S_t$ such that:*

1. *$x^*(\ell) = 0$ for each $\ell \in N$,*
2. *$L \setminus N$ is a base of $G(\mathcal{S}, L \setminus N)$, and*
3. *The projection of x^* onto the components indexed by lines in $L \setminus N$ is the unique, perfect fractional b-matching in $G(\mathcal{S}, L \setminus N)$, where $b(S_i) = \beta(S_i) - \beta(S_{i-1})$ for each $i \in [1, \ldots, t]$.*

Corollary 3.4 *Each extreme 2-lattice vector is half-integral.*

4 A Min-Max Formula

Theorem 4.1 develops a min-max formula for the maximum cardinality of a 2-lattice vector, which generalizes König's Theorem [23] and Edmonds' Theorems for cardinality matroid and polymatroid intersection [11]. It provides a "good" characterization of the maximum cardinality of a 2-lattice vector in terms of the minimum capacity of a cover. Here, a *cover* is a pair (S, T) of (possibly identical) members of Γ^* such that

$$\alpha_\ell(S) + \alpha_\ell(T) \geq 2 \text{ for each } \ell \in L,$$

and the capacity of a cover (S, T), denoted $\beta(S, T)$, is

$$1/2[\beta(S) + \beta(T)].$$

Theorem 4.1 *The maximum cardinality of a 2-lattice vector is the minimum capacity of a cover.*

Proof. To see that the maximum cardinality of a 2-lattice vector is at most the minimum capacity of a cover, observe that for any cover (S, T), the solution $y(S) = y(T) = 1/2$ is dual feasible and has objective value $\beta(S, T)$.

To prove that the maximum cardinality of a 2-lattice vector equals the minimum capacity of a cover, we show that there is an optimum solution y^* to the dual problem such that:

1. $supp(y^*)$ forms a chain in $(\Gamma, \preceq)$.
2. y^* is half-integral,
3. $y^*(S) > 0$ for at most two flats S.

First, to see that there is an optimum solution y^* to the dual problem satisfying (1) we employ an "uncrossing" argument similar to that of Hoffman and Schwartz [20], but modified to accommodate an infinite lattice Γ. Consider an optimal dual solution $\bar{y}$ with finite support (e.g. each extreme point optimal solution has finite support). If $supp(\bar{y})$ forms a chain in $(\Gamma, \preceq)$, we are done. Otherwise, define a complete order $\prec'$ on $supp(\bar{y})$ that is consistent with the partial order $\preceq$. We argue that $\bar{y}$ can be converted into a dual solution y^* such that $supp(y^*)$ forms a chain in $(\Gamma, \preceq)$ as follows.

Let $S_0 = *$ and index the elements of $supp(\bar{y})$ so that

$$S_0 \prec' S_1 \prec' S_2 \prec' \cdots \prec' S_t.$$

Define $i = i_{\bar{y}}$ to be the smallest index such that $S_{i-1} \not\preceq S_i$ and $j = j_{\bar{y}}$ to be the smallest index such that $S_j \not\preceq S_i$. Consider the dual solution $\tilde{y}$ such that

$$\tilde{y}(S) = \begin{cases} \bar{y}(S) - \epsilon & \text{if } S \in \{S_i, S_j\} \\ \bar{y}(S) + \epsilon & \text{if } S \in \{S_i \vee S_j, S_i \wedge S_j\} \\ \bar{y}(S) & \text{otherwise,} \end{cases}$$

where $\epsilon = \min\{\bar{y}(S_i), \bar{y}(S_j)\}$. Since

$$\alpha_\ell(S_i \vee S_j) + \alpha_\ell(S_i \wedge S_j) \geq \alpha_\ell(S_i) + \alpha_\ell(S_j)$$

for each line $\ell \in L$, $\tilde{y}$ is dual feasible. Further, since

$$\beta(S_i \vee S_j) + \beta(S_i \wedge S_j) \leq \beta(S_i) + \beta(S_j),$$

$$\sum_{S \in \Gamma} \tilde{y}(S)\beta(S) \leq \sum_{S \in \Gamma} \bar{y}(S)\beta(S).$$

So, the (dual) objective value of $\tilde{y}$ is no worse than that of $\bar{y}$.

Note that the chain $S_0 \preceq S_1 \preceq \cdots \preceq (S_i \wedge S_j) \preceq S_j \preceq \cdots \preceq S_{i-1}$ in $(\Gamma, \preceq)$ grows with each successive revision of this kind. Since there is a finite upper bound on the length of any chain in Γ, this process must ultimately terminate with a dual solution y^* such that $supp(y^*)$ is a chain in $(\Gamma, \preceq)$.

Now, to see that y^* satisfies (2), let $\mathcal{S} = \{S_i : i = 1, \ldots, t\}$ be a nested family of flats and N a subset of L such that y^* is the unique solution to the system:

$$\sum_{S_i \in \mathcal{S}} y(S_i)\alpha_\ell(S_i) = 1 \text{ for each } \ell \in L \setminus N \tag{4.4}$$

Let y' be the unique solution to the system:

$$\sum_{S_i \in \mathcal{S}} y(S_i)(\alpha_\ell(S_i) - \alpha_\ell(S_{i-1})) = 1 \text{ for each } \ell \in L \setminus N \tag{4.5}$$

Then y' is the unique solution to the system $yA = \mathbf{1}$, where A is the node-edge incidence matrix of the basis graph $G(\mathcal{S}, L \setminus N)$, i.e.,

$$y'(S_i) = \begin{cases} 1/2 & \text{if there is no path in } G(\mathcal{S}, L \setminus N) \text{ from the root to } S_i, \\ 1 & \text{if there are an odd number of edges on the path in} \\ & G(\mathcal{S}, L \setminus N) \text{ from the root to } S_i, \text{ and} \\ 0 & \text{if there are an even number of edges on the path in} \\ & G(\mathcal{S}, L \setminus N) \text{ from the root to } S_i. \end{cases}$$

And, we may compute y^* as follows:

$$\begin{aligned} y^*(S_i) &= y'(S_i) - y'(S_{i+1}) \text{ for } i = 1, \ldots, t-1, \text{ and} \\ y^*(S_t) &= y'(S_t). \end{aligned}$$

It follows immediately that y^* is half-integral.

Finally, to see that y^* has at most two non-zero components observe that since y^* is dual feasible, it is non-negative. Thus, the corresponding vector y' must be of the form

$$y'(S_i) = \begin{cases} 1 & \text{for } i = 1, \ldots i_1 \\ 1/2 & \text{for } i = i_1 + 1, \ldots, i_2 \\ 0 & \text{for } i = i_2 + 1, \ldots, t. \end{cases}$$

It follows that y^* has at most two non-zero components and $\sum_{S_i \in \mathcal{S}} y^*(S_i) \in \{1/2, 1\}$. If y^* has exactly two non-zero components S and T, then $y^*(S) = y^*(T) = 1/2$ and (S, T) is a minimum cover. If y^* has only one non-zero component S, then either $y^*(S) = 1$, in which case (S, S) is a minimum cover, or $y^*(S) = 1/2$ in which case $(*, S)$ is a minimum cover. □

A cover (S, T) with $S \preceq T$ is called a *nested cover*. The following lemma shows that we may associate a nested cover with each minimum cover and hence that there is always a nested minimum cover.

Lemma 4.2 *If (S, T) is a minimum cover, then $(S \wedge T, S \vee T)$ is a nested minimum cover.*

Proof. For each $\ell \in L$, $\alpha_\ell(S \wedge T) + \alpha_\ell(S \vee T) \geq \alpha_\ell(S) + \alpha_\ell(T) \geq 2$. Therefore, $(S \wedge T, S \vee T)$ is a cover. Since $\beta(S \vee T) + \beta(S \wedge T) \leq \beta(S) + \beta(T)$, it follows that $(S \wedge T, S \vee T)$ is a minimum cover. □

Corollary 4.3 *The maximum cardinality of a 2-lattice vector is the minimum capacity of a nested cover.*

We present Edmond's duality theorem for cardinality matroid intersection as a special case of Theorem 4.1.

Corollary 4.4 *Let* $\mathbf{M}_1$ *be a matroid with rank function* r_1 *and let* $\mathbf{M}_2$ *be a matroid with rank function* r_2 *both defined on the same ground set* E*. Then the maximum cardinality of an intersection in* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *is*

$$\min_{S \subseteq E} \{r_1(S) + r_2(E \setminus S)\}.$$

Proof. The matroid intersection polyhedron

$$P = \{x \in \mathbb{R}_+^E : x(S) \leq r_1(S) \text{ and } x(S) \leq r_2(S) \text{ for each } S \subseteq E\}$$

is equivalent to the 2-lattice polyhedron

$$\{x \in \mathbb{R}_+^L : \alpha(S)x \leq \beta(S) \text{ for each } S \subseteq \mathcal{E}\}$$

where

- $\mathcal{E}$ consists of two copies E and E' of E,
- L consists of the lines $\{e, e'\}$ with an element from E and its copy in E',
- For each $\ell \in L$ and $S \subseteq \mathcal{E}$, $\alpha_\ell(S) = |\ell \cap S|$, and
- For each $S \in \mathcal{E}$, $\beta(S) = r_1(S \cap E) + r_2(S \cap E')$.

Thus, Corollary 4.3 implies that the maximum cardinality of an intersection in $\mathbf{M}_1$ and $\mathbf{M}_2$ is the minimum capacity of a nested cover. Let (S, T) be a minimum capacity nested cover. Define $S_1 = S \cap E$ and $S_2 = S \cap E'$. Similarly, let $T_1 = T \cap E$ and $T_2 = T \cap E'$. Since (S, T) is a nested cover, if $e \notin S_1$, then $e' \in T_2$ and, if $e' \notin S_2$ then $e \in T_1$. Thus,

$$r_1(S_1) + r_2(E \setminus S_1) \leq r_1(S_1) + r_2(T_2)$$

and

$$r_1(E \setminus S_2) + r_2(S_2) \leq r_1(T_1) + r_2(S_2).$$

It is easy to establish that for each $x \in P$ and $S \subseteq E$

$$\sum_{e \in E} x(e) \leq r_1(S) + r_2(E \setminus S).$$

Since each maximum cardinality $x \in P$ satisfies

$$\sum_{e \in E} x(e) = \beta(S, T) = 1/2[r_1(S_1) + r_2(T_2) + r_1(T_1) + r_2(S_2)],$$

It follows that $\sum_{e \in E} x(e) = r_1(S_1) + r_2(E \setminus S_1) = r_1(E \setminus S_2) + r_2(S_2)$ and hence that the maximum cardinality of an intersection in $\mathbf{M}_1$ and $\mathbf{M}_2$ is equal to $\min_{S \subseteq E}\{r_1(S) + r_2(E \setminus S)\}$. □

When the 2-lattice polyhedron is known to have integral extreme points, we may restrict attention to integer 2-lattice vectors in Theorem 4.1. In the case of matroid intersection, it is easy to verify that for each family $\mathcal{S} = \{S_i : i \in [1, \ldots, t]\}$ of flats with $S_1 \prec S_2 \prec \ldots \prec S_t$, $G(\mathcal{S}, L)$ is bipartite and hence, as is well known, the extreme points of the matroid intersection polyhedron are integral.

We can use our linear programming formulation to further characterize minimum capacity covers.

Corollary 4.5 *For each minimum cover (S, T) and maximum 2-lattice vector x,*

- $\alpha(S)x = \beta(S)$
- $\alpha(T)x = \beta(T)$ *and,*
- *if* $\alpha_\ell(S) + \alpha_\ell(T) > 2$, *then* $x(\ell) = 0$

Proof. By complementary slackness. □

Given a 2-lattice polyhedron, let Ω be the collection of all maximum cardinality 2-lattice vectors and Ω_{ext} be the collection of all extreme maximum cardinality 2-lattice vectors.

Corollary 4.6 *For each minimum cover (S, T),*

$$S, T \preceq \wedge(cl(x) : x \in \Omega) \preceq \wedge(cl(x) : x \in \Omega_{ext})$$

Shapley and Shubik [29] showed that the collection of optimal dual solutions to a bipartite matching problem forms a lattice. The same result holds for cardinality matroid intersection. In particular, if $(S, E \setminus S)$ and $(S', E \setminus S')$ are dual solutions in the sense of Corollary 4.4 to a matroid intersection problem, then so are $(S \cap S', E \setminus (S \cap S'))$ and $(S \cup S', E \setminus (S \cup S'))$. We show that the set of nested minimum covers for a 2-lattice polytope forms an upper semi-lattice.

Lemma 4.7 *If (S_1, T_1) and (S_2, T_2) are nested minimum covers, then $(S_1 \wedge S_2, T_1 \vee T_2)$ and $(S_1 \vee S_2, T_1 \wedge T_2)$ are minimum covers.*

Proof. We first show that $(S_1 \wedge S_2, T_1 \vee T_2)$ and $(S_1 \vee S_2, T_1 \wedge T_2)$ are covers. Since (S_1, T_1) and (S_2, T_2) are covers and α_ℓ is supermodular for each $\ell \in L$,

$$\begin{aligned} \alpha_\ell(S_1 \wedge S_2) + \alpha_\ell(S_1 \vee S_2) + \alpha_\ell(T_1 \wedge T_2) + \alpha_\ell(T_1 \vee T_2) &\geq \\ \alpha_\ell(S_1) + \alpha_\ell(S_2) + \alpha_\ell(T_1) + \alpha_\ell(T_2) &\geq 4. \end{aligned}$$

And so, we need only consider the cases in which $\alpha_\ell(S_1 \vee S_2) + \alpha_\ell(T_1 \wedge T_2)$ or $\alpha_\ell(S_1 \wedge S_2) + \alpha_\ell(T_1 \vee T_2)$ is strictly greater than 2.

Case 1. If $\alpha_\ell(S_1 \vee S_2) + \alpha_\ell(T_1 \wedge T_2) > 2$, either $\alpha_\ell(S_1 \vee S_2)$ or $\alpha_\ell(T_1 \wedge T_2) = 2$. However, since (S_1, T_1) and (S_2, T_2) are nested,

$$(S_1 \wedge S_2) \preceq (S_1 \vee S_2) \preceq (T_1 \vee T_2)$$

and

$$(S_1 \wedge S_2) \preceq (T_1 \wedge T_2) \preceq (T_1 \vee T_2).$$

So, $\alpha_\ell(T_1 \vee T_2) = 2$; proving that $\alpha_\ell(S_1 \wedge S_2) + \alpha_\ell(T_1 \vee T_2) \geq 2$.

Case 2. If $\alpha_\ell(S_1 \wedge S_2) + \alpha_\ell(T_1 \vee T_2) > 2$, then $\alpha_\ell(S_1 \wedge S_2) \geq 1$ and so,

$$1 \leq \alpha_\ell(S_1 \wedge S_2) \leq \alpha_\ell(S_1 \vee S_2).$$

Similarly,

$$1 \leq \alpha_\ell(S_1 \wedge S_2) \leq \alpha_\ell(T_1 \wedge T_2);$$

proving that $\alpha_\ell(S_1 \vee S_2) + \alpha_\ell(T_1 \wedge T_2) \geq 2$.

Thus, $\alpha_\ell(S_1 \vee S_2) + \alpha_\ell(T_1 \wedge T_2) \geq 2$ and $\alpha_\ell(S_1 \wedge S_2) + \alpha_\ell(T_1 \vee T_2) \geq 2$ for each $\ell \in L$, i.e., $(S_1 \wedge S_2, T_1 \vee T_2)$ and $(S_1 \vee S_2, T_1 \wedge T_2)$ are covers.

Since $(S_1 \wedge S_2, T_1 \vee T_2)$ and $(S_1 \vee S_2, T_1 \wedge T_2)$ are covers and (S_1, T_1) and (S_2, T_2) are minimum covers

$$\beta(S_1 \wedge S_2) + \beta(T_1 \vee T_2) \geq \beta(S_1) + \beta(T_1)$$

and

$$\beta(S_1 \vee S_2) + \beta(T_1 \wedge T_2) \geq \beta(S_2) + \beta(T_2).$$

But, since β is submodular,

$$\beta(S_1 \wedge S_2) + \beta(S_1 \vee S_2) + \beta(T_1 \wedge T_2) + \beta(T_1 \vee T_2) \leq \beta(S_1) + \beta(S_2) + \beta(T_1) + \beta(T_2).$$

Thus, we must have equality throughout. □

Let $\mathcal{C}$ be the collection of all nested minimum covers. We show that $\mathcal{C}$ is an upper semi-lattice with partial order defined by $(S, T) \preceq (S', T')$ if

- $T \preceq T'$ and
- $S' \preceq S$.

In fact, we show that the binary operation $\vee_c$ on $\mathcal{C}$ defined by

$$(S, T) \vee_c (S', T') = (S \wedge S', T \vee T')$$

is the join operation in $\mathcal{C}$.

Lemma 4.8 *$\mathcal{C}$ is an upper semi-lattice.*

Proof. By Lemma 4.2 and Lemma 4.7, $(S \wedge S', T \vee T')$ is a nested minimum cover. It is easy to verify that this is also the least upper bound of (S, T) and (S', T'). Thus, $\mathcal{C}$ is an upper semi-lattice. $\Box$

The following example shows that $\mathcal{C}$ need not be a lattice. Consider the incidence 2-lattice polyhedra on $\mathcal{E} = \{e, f\}$ with the single line $\ell = \{e, f\}$ and $\beta(S)$ defined by $|S|$. The nested minimum covers are $(\emptyset, \mathcal{E})$, $(\{e\}, \{e\})$ and $(\{f\}, \{f\})$. Clearly $(\emptyset, \mathcal{E})$ is the least upper bound of $(\{e\}, \{e\})$ and $(\{f\}, \{f\})$, but these two nested covers do not have a common lower bound in $\mathcal{C}$.

The following corollary shows that there is a largest cover in $\mathcal{C}$ and in some sense this cover dominates all others.

Corollary 4.9 *There is a nested minimum cover (S^*, T^*), such that $T \preceq T^*$ and $S^* \preceq S$ for each minimum cover (S, T).*

Proof. Let (S^*, T^*) be any nested minimum cover with the property that no nested minimum cover (S, T) has $T^* \prec T$ or $S \prec S^*$ (since there is a finite bound on the length of any chain in Γ, such a cover exists). Suppose that (S, T) is a minimum cover with $T \not\preceq T^*$ or $S^* \not\preceq S$. By Lemma 4.2 $(S \wedge T, S \vee T)$ is a nested minimum cover. So, by Lemma 4.7, $(S \wedge T \wedge S^*, S \vee T \vee T^*)$ is a nested minimum cover and $T^* \prec S \vee T \vee T^*$ or $S \wedge T \wedge S^* \prec S^*$ contradicting the choice of (S^*, T^*). $\Box$

We refer to the nested minimum cover of Corollary 4.9 as the *dominant cover.*

5 The Dominant Cover

In the case of matching 2-lattice polyhedra, the relationship between α and β enables us to characterize the dominant cover in a manner analogous to the Gallai-Edmonds characterization of a minimum odd set cover. This characterization is at the heart of our algorithm [3] for finding a maximum cardinality matching 2-lattice vector.

Lemma 5.1 shows that given one component of a nested minimum cover, we can characterize the other. To facilitate this characterization, for $T \in \Gamma$ we denote by $L(T)$ those lines $\ell \in L$ with one point in T, i.e., $L(T) = \{\ell \in L : \alpha_\ell(T) = 1\}$.

Lemma 5.1 *If (S, T) is a nested minimum cover then $S = \sigma(\{\ell \wedge T : \ell \in L(T)\})$ and $T = S \vee \sigma(\{\ell \in L : \alpha_\ell(S) = 0\})$.*

Proof. Since (S, T) is a nested cover, $S' = \sigma(\{\ell \wedge T : \ell \in L(T)\}) \subseteq S$. Further, since (S', T) is a cover,

$$\beta(S) + \beta(T) \leq \beta(S') + \beta(T).$$

It follows that $S' = S$.

Similarly, since (S,T) is a nested cover, $T' = S \vee \sigma(\{\ell \in L : \alpha_\ell(S) = 0\}) \subseteq T$ and since (S,T') is a cover,

$$\beta(S) + \beta(T) \leq \beta(S) + \beta(T').$$

It follows that $T' = T$. □

Lemma 5.2 establishes a relationship between nested minimum covers and an induced matroid intersection problem. In particular it shows that if (S,T) is the dominant cover, then the maximum cardinality of an intersection in the induced matroids is $\beta(S)$. After some technical preliminaries, Lemma 5.6 shows how to construct a maximum matching 2-lattice vector from a $\beta(S)$-intersection. Together, these observations lead to our characterization in Theorem 5.8 of the dominant cover in terms of maximum matching 2-lattice vectors.

The close analogies between our characterization of the dominant cover via an induced matroid intersection problem and the algorithm of Orlin and Vande Vate [27] for finding a maximum cardinality matching in a representable matroid via a sequence of induced intersection problems highlight the similarities between the two problems and suggest the possibility of a polyhedral interpretation for their procedure. Our methods for handling non-integral components and our reliance on the graph structures associated with extreme matching 2-lattice vectors in [3] highlight the differences between the problems.

Let (S,T) be a nested cover and for each $\ell \in L(T)$, let $e(\ell) \in \ell \wedge T$. Define the matroid $\mathbf{M}_1(S,T)$ with rank function r_1 on $L(T)$ as follows. A set X of lines in $L(T)$ is independent in $\mathbf{M}_1(S,T)$ if $\beta(\{e(\ell) : \ell \in X\}) = |X|$.

Define the matroid $\mathbf{M}_2(S,T;e)$ with rank function r_2 on $L(T)$ as follows. A set X of lines in $L(T)$ is independent in $\mathbf{M}_2(S,T;e)$ if $\beta(X/T \vee \{e\})$ is equal to the number of lines in X. We henceforth use the less cumbersome $T \vee e$ in place of the more correct $T \vee \{e\}$.

Since β is normalized, non-decreasing (on $2^{\mathcal{E}}$) and submodular, the fact that $\beta(\{e(\ell)\}) = 1$ for each $\ell \in L(T)$ ensures that $\mathbf{M}_1(S,T)$ is in fact a matroid. To see that $\mathbf{M}_2(S,T;e)$ is a matroid, it is enough to observe that for each line $\ell \in L(T)$, $\beta(\ell/T \vee e) = \beta(\ell \vee T \vee e) - \beta(T \vee e) \leq \beta(\ell) - \beta(\ell \wedge (T \vee e)) \leq 1$.

Lemma 5.2 shows that if the maximum cardinality of an intersection in $\mathbf{M}_1$ and $\mathbf{M}_2$ is $\beta(S) - 1$, there is a cover (S',T') with $T \vee e \subseteq T'$.

Lemma 5.2 *If (S,T) is a nested minimum cover and $e \notin T$, then the maximum cardinality of an intersection in $\mathbf{M}_1(S,T)$ and $\mathbf{M}_2(S,T;e)$ is either $\beta(S)$ or $\beta(S) - 1$. Furthermore, if the maximum cardinality of an intersection in $\mathbf{M}_1$ and $\mathbf{M}_2$ is $\beta(S) - 1$ then there is a minimum cover (S',T') such that $T \vee e \subseteq T'$.*

Proof. The maximum cardinality of an intersection in $\mathbf{M}_1$ and $\mathbf{M}_2$ is bounded by $\beta(S)$. Suppose the maximum cardinality of an intersection in $\mathbf{M}_1$ and $\mathbf{M}_2$

is less than or equal to $\beta(S)-1$, then there is a minimum rank cover (X_1, X_2) of $L(T)$ for the matroid intersection problem such that

$$r_1(X_1) + r_2(X_2) \le \beta(S) - 1,$$

that is,

$$\beta(\{e(\ell) : \ell \in X_1\}) + \beta(X_2/(T \vee e)) \le \beta(S) - 1$$

and so

$$\beta(\{e(\ell) : \ell \in X_1\}) + \beta(X_2 \vee T \vee e) \le \beta(S) + \beta(T \vee e) - 1 = \beta(S) + \beta(T).$$

Let $S' = \sigma(\{e(\ell) : \ell \in X_1\})$ and $T' = X_2 \vee T \vee e$. Then (S', T') is a cover of L with $T \vee e \subseteq T'$ and $\beta(S', T') \le \beta(S, T)$. Since (S, T) is a minimum cover, it follows that (S', T') is a minimum cover and the size of a maximum intersection must be at least $\beta(S) - 1$. □

Corollary 5.3 *If (S^*, T^*) is the dominant cover and $e \notin T^*$, then the maximum cardinality of an intersection in $\mathbf{M}_1(S^*, T^*)$ and $\mathbf{M}_2(S^*, T^*; e)$ is $\beta(S^*)$.*

The following two lemmas identify special properties of maximum matching 2-lattice vectors and show conditions under which we may combine portions of two matching 2-lattice vectors to form a third. We exploit these conditions to construct maximum matching 2-lattice vectors from $\beta(S)$-intersections in $\mathbf{M}_1(S^*, T^*)$ and $\mathbf{M}_2(S^*, T^*;\, e)$.

Lemma 5.4 *Let x be a maximum matching 2-lattice vector and let (S, T) be a nested minimum cover. Then $x_{L\setminus L(T)}$ satisfies*

1. *$\alpha(T)x_{L\setminus L(T)} = \beta(T/S)$ and*

2. *for $T' \subseteq T$, $\alpha(T')x_{L\setminus L(T)} \le \beta(T'/S)$*

and $x_{L(T)}$ satisfies

3. *$\alpha(T)x_{L(T)} = \beta(S)$,*

4. *for $T' \subseteq T$, $\alpha(T')x_{L(T)} \le \beta(T' \wedge S)$.*

Proof. First, observe that for each line $\ell \in L \setminus L(T)$, $\alpha_\ell(T) = 2$. So, if $\alpha_\ell(S) > 0$, $\alpha_\ell(S) + \alpha_\ell(T) > 2$ and, by Corollary 4.5, $x(\ell) = 0$. Thus, $\alpha(S)x_{L\setminus L(T)} = 0$. Since $\alpha(S)x = \beta(S)$, it follows that $\alpha(S)x_{L(T)} = \beta(S)$.

To see (3), observe that for each $\ell \in L(T)$, $\alpha_\ell(T) = \alpha_\ell(S) = 1$. So,

$$\alpha(T)x_{L(T)} = \alpha(S)x_{L(T)} = \beta(S).$$

To see (1), observe that since

$$\alpha(T)x = \beta(T) = \beta(T \vee S) \text{ and } \alpha(T)x_{L(T)} = \beta(S)$$

it follows that

$$\alpha(T)x_{L\setminus L(T)} = \beta(T/S).$$

To see (2), observe that for $T' \subseteq T$,

$$\alpha(T' \vee S)x \leq \beta(T' \vee S) \text{ and } \alpha(T' \vee S)x_{L(T)} = \beta(S).$$

Thus,

$$\begin{aligned} \alpha(T')x_{L\setminus L(T)} &\leq \alpha(T' \vee S)x_{L\setminus L(T)} \\ &= \alpha(T' \vee S)x - \alpha(T' \vee S)x_{L(T)} \\ &\leq \beta(T' \vee S) - \beta(S) \\ &= \beta(T'/S). \end{aligned}$$

To see (4), note that for $\ell \in L(T)$, $\alpha_\ell(S) = \alpha_\ell(T)$. So,

$$\alpha(T')x_{L(T)} = \alpha(T' \wedge S)x_{L(T)} \leq \beta(T' \wedge S).$$

□

Lemma 5.5 *Let x and $\tilde{x}$ be matching 2-lattice vectors and let (S,T) be a nested minimum cover. If x satisfies (1) and (2) of Lemma 5.4, $\tilde{x}$ satisfies (3) and (4) of Lemma 5.4, and*

a. $\beta(T/cl(\tilde{x}_{L(T)})) = \beta(T/S)$,

b. $\alpha(cl(\tilde{x}_{L(T)}))x_{L\setminus L(T)} = 0$, *and*

c. $supp(\tilde{x}_{L(T)}) \subseteq cl(\tilde{x}_{L(T)})$.

then $x' = \tilde{x}_{L(T)} + x_{L\setminus L(T)}$ is a matching 2-lattice vector.

Proof. Suppose x' is not a matching 2-lattice vector, then there is a flat $Z \in \Gamma$ such that $\alpha(Z)x' > \beta(Z)$. We first show that we may choose Z to contain $cl(\tilde{x}_{L(T)})$.

By condition (b), $\alpha(Z \wedge cl(\tilde{x}_{L(T)}))x' = \alpha(Z \wedge cl(\tilde{x}_{L(T)}))\tilde{x}_{L(T)}$, and since $\tilde{x}_{L(T)}$ is feasible $\alpha(Z \wedge cl(\tilde{x}_{L(T)}))\tilde{x}_{L(T)} \leq \beta(Z \wedge cl(\tilde{x}_{L(T)}))$. It follows by Lemma 3.2 that

$$\alpha(Z \vee cl(\tilde{x}_{L(T)}))x' > \beta(Z \vee cl(\tilde{x}_{L(T)})).$$

Thus, if x' is not a matching 2-lattice vector, there is a flat $Z \in \Gamma$ with $cl(\tilde{x}_{L(T)}) \subseteq Z$ such that $\alpha(Z)x' > \beta(Z)$.

We next show that we may also assume $T \subseteq Z$.

By conditions (3) and (1)

$$\alpha(T)x' = \alpha(T)\tilde{x}_{L(T)} + \alpha(T)x_{L\setminus L(T)} = \beta(S) + \beta(T/S) = \beta(T).$$

Further, by conditions (2) and (4)

$$\alpha(Z\wedge T)x' = \alpha(Z\wedge T)\tilde{x}_{L(T)} + \alpha(Z\wedge T)x_{L\setminus L(T)} \le \beta(Z\wedge T\wedge S) + \beta((Z\wedge T)/S).$$

By the submodularity of β, $\alpha(Z\wedge T)x' \le \beta(Z\wedge T)$, and so it follows by Lemma 3.2 that $\alpha(Z\vee T)x' > \beta(Z\vee T)$. But,

$$\begin{aligned}
\alpha(Z\vee T)x' &= \alpha(Z\vee T)\tilde{x}_{L(T)} + \alpha(Z\vee T)x_{L\setminus L(T)} \\
&= \beta(cl(\tilde{x}_{L(T)})) + \alpha(Z\vee T)x_{L\setminus L(T)} && \text{since } supp(\tilde{x}_{L(T)}) \subseteq cl(\tilde{x}_{L(T)}) \subseteq Z \\
&= \beta(cl(\tilde{x}_{L(T)})) + \alpha(T)x_{L\setminus L(T)} && \text{since } supp(x_{L\setminus L(T)}) \subseteq T \\
&= \beta(cl(\tilde{x}_{L(T)})) + \beta(T/S) && \text{by (1)} \\
&= \beta(cl(\tilde{x}_{L(T)})) + \beta(T/cl(\tilde{x}_{L(T)})) && \text{by (a)} \\
&= \beta(cl(\tilde{x}_{L(T)}) \vee T) \\
&\le \beta(Z\vee T) && \text{since } cl(\tilde{x}_{L(T)}) \subseteq Z.
\end{aligned}$$

This contradicts the existence of Z and proves that x' is a matching 2-lattice vector. □

Lemma 5.6 shows that if (S,T) is a nested minimum cover and $e \notin T$, then each $\beta(S)$-intersection in $\mathbf{M}_1(S,T)$ and $\mathbf{M}_2(S,T;e)$ gives rise to a maximum matching 2-lattice vector x with $e \notin cl(x)$.

Lemma 5.6 *Let x be a maximum matching 2-lattice vector, (S,T) be a nested minimum cover and $e \notin T$. If X is a $\beta(S)$-intersection in $\mathbf{M}_1(S,T)$ and $\mathbf{M}_2(S,T;e)$, then x' defined by*

$$x'(\ell) = \begin{cases} 1 & \text{if } \ell \in X \\ 0 & \text{if } \ell \in L(T)\setminus X \\ x(\ell) & \text{otherwise} \end{cases}$$

is a maximum matching 2-lattice vector and $e \notin cl(x')$.

Proof. First, since X is independent in $\mathbf{M}_2(S,T;e)$ and $|X| = \beta(S)$,

$$\beta(X/T\vee e) = |X| = \beta(S).$$

Second, since X is independent in $\mathbf{M}_1(S,T)$ and $|X| = \beta(S)$, $\beta(X\wedge(T\vee e)) \ge \beta(\{e(\ell) : \ell\in X\}) = \beta(S)$. By the submodularity of β,

$$\beta(X) + \beta(T\vee e) \ge \beta(X\vee T\vee e) + \beta(X\wedge(T\vee e)).$$

So,

$$\beta(X) \ge \beta(S) + \beta(X/(T\vee e)) = 2\beta(S) = 2|X|,$$

and $x'_{L(T)}$ is a matching 2-lattice vector.

We see that $x'_{L(T)}$ satisfies (3) of Lemma 5.4 as follows. Since $\alpha_\ell(T) = 1$ for each $\ell \in X$,

$$\alpha(T)x'_{L(T)} = |X| = \beta(S).$$

We see that $x'_{L(T)}$ satisfies (4) of Lemma 5.4 as follows. Since $\alpha_\ell(S) = \alpha_\ell(T) = 1$ for each $\ell \in X$, if $T' \subseteq T$,

$$\alpha(T')x'_{L(T)} = \alpha(T' \wedge S)x'_{L(T)} \leq \beta(T' \wedge S).$$

Since x is a maximum matching 2-lattice vector, $x_{L\setminus L(T)}$ satisfies conditions (1) and (2) of Lemma 5.4. Thus, to show that x' is a matching 2-lattice vector, we need only show that $x'_{L(T)}$ and $x_{L\setminus L(T)}$ satisfy conditions (a) and (b) of Lemma 5.5.

We see that $x'_{L(T)}$ satisfies (a) of Lemma 5.5 as follows. Since

$$cl(x'_{L(T)}) \subseteq \sigma(supp(x'_{L(T)})) = \sigma(X)$$

and

$$\alpha(\sigma(X))x'_{L(T)} = 2|X| = \beta(\sigma(X)),$$

it follows that $cl(x'_{L(T)}) = \sigma(X)$. Therefore,

$$\beta(T/cl(x'_{L(T)})) = \beta(T/X) = |B| = \beta(T/\sigma(\{e(\ell) : \ell \in X\})) = \beta(T/S).$$

We see that $x'_{L(T)}$ and $x_{L\setminus L(T)}$ satisfy condition (b) of Lemma 5.5 as follows. Since $supp(x'_{L\setminus L(T)}) \subseteq T$ and $cl(x'_{L(T)}) = \sigma(X)$,

$$\alpha(cl(x'_{L(T)}))x_{L\setminus L(T)} = \alpha(X \wedge T)x_{L\setminus L(T)}.$$

But $X \wedge T = S$ so

$$\alpha(cl(x'_{L(T)}))x_{L\setminus L(T)} = \alpha(S)x_{L\setminus L(T)} = 0.$$

Thus, by Lemma 5.5, x' is a matching 2-lattice vector.

Since $e \notin \sigma(X \cup T)$ and $cl(x') \subseteq \sigma(supp(x')) \subseteq \sigma(X \cup T)$, it follows that $e \notin cl(x')$.

To see that x' is a maximum matching 2-lattice vector, observe that

$$\begin{aligned}
\sum_{\ell \in L} x'(\ell) &= \sum_{\ell \in L(T)} x'(\ell) + \sum_{\ell \in L\setminus L(T)} x(\ell) \\
&= \beta(S) + \sum_{\ell \in L} x(\ell) - \sum_{\ell \in L(T)} x(\ell) \\
&= \beta(S) + \beta(S,T) - \sum_{\ell \in L(T)} x(\ell) \\
&\geq \beta(S) + \beta(S,T) - \beta(S) \\
&= \beta(S,T).
\end{aligned}$$

□

Corollary 5.7 *If* (S^*, T^*) *is the dominant cover, then* $T^* \supseteq \cap(cl(x) : x \in \Omega)$.

Proof. By Corollary 5.3, if $e \notin T^*$, then the maximum cardinality of an intersection in $\mathbf{M}_1(S^*, T^*)$ and $\mathbf{M}_2(S^*, T^*; e)$ is $\beta(S^*)$. By Lemma 5.6, there is $x \in \Omega$ such that $e \notin cl(x)$, hence, $e \notin \cap(cl(x) : x \in \Omega)$. Therefore, $T^* \supseteq \cap(cl(x) : x \in \Omega)$. □

Combining Corollary 4.6, Corollary 5.7 and Lemma 5.1, we have the following characterization of the dominant cover in terms of maximum matching 2-lattice vectors.

Theorem 5.8 *Let* $T^* = \cap(cl(x) : x \in \Omega)$ *and* $S^* = \sigma(\{\ell \wedge T^* : \ell \in L(T^*)\})$. *Then* (S^*, T^*) *is the dominant cover.*

The following results refine Lemma 5.6 to extreme maximum matching 2-lattice vectors. In particular, we show that $T^* = \cap(cl(x) : x \in \Omega) = \cap(cl(x) : x \in \Omega_{ext})$. and so the minimum cover can be characterized in terms of the extreme maximum matching 2-lattice vectors. While it is clear that $\cap(cl(x) : x \in \Omega) \subseteq \cap(cl(x) : x \in \Omega_{ext})$, the fact that $\cap(cl(x) : x \in \Omega) \supseteq \cap(cl(x) : x \in \Omega_{ext})$ is rather surprising and in an essential way underlies both our algorithm for solving (2.2) and (2.3) and classic algorithms for special cases.

Lemma 5.9 *Let* (S^*, T^*) *be the dominant cover and* $x \in \Omega_{ext}$. *Then*

1. *for each* $\ell \in L(T^*)$, $x(\ell) \in \{0, 1\}$,
2. $\beta(T^*/cl(x_{L(T^*)})) = \beta(T^*/S^*)$, *and*
3. $T^* \wedge cl(x_{L(T^*)}) = S^*$

Proof. For each $x^* \in \Omega_{ext}$, there is a complementary dual solution y^*. Let $\mathcal{S} = \{S_i : i = 1, \ldots, t\}$ be a nested family of flats in Γ and N a subset of L such that x^* is the unique solution to the system:

$$\begin{array}{rcll} \alpha(S_i)x & = & \beta(S_i) & \text{for each } S_i \in \mathcal{S} \\ x(\ell) & = & 0 & \text{for each } \ell \in N \end{array}$$

and y^* is the unique solution to the system:

$$\sum_{S_i \in \mathcal{S}} y(S_i)\alpha_\ell(S_i) \quad = \quad 1 \qquad \text{for each } \ell \in L \setminus N$$

By arguments similar to those used in the proof of Theorem 4.1, there are two indexes i_1 and i_2, $i_1 \leq i_2$, $i_1, i_2 \in \{0, 1, \ldots, t\}$ such that

- $S_1, \ldots, S_{i_1}$ correspond to the nodes in $G(\mathcal{S}, L \setminus N)$ that have an odd number of edges in the unique path from S_i to the root;

- $S_{i_1+1}, \ldots, S_{i_2}$ correspond to the nodes in $G(\mathcal{S}, L \setminus N)$ that have no path from S_i to the root;
- $S_{i_2+1}, \ldots, S_t$ correspond to the nodes in $G(\mathcal{S}, L \setminus N)$ that have an even number of edges in the unique path from S_i to the root; and
- (S_{i_1}, S_{i_2}) forms a minimum cover.

Since $S_{i_2} \subseteq T^*$, if $\alpha_\ell(T^*) = 1$, then $\alpha_\ell(S_{i_2}) = 1$. Clearly, if $\ell \in N$, then $x^*(\ell) = 0$. If $\ell \notin N$ and $\alpha_\ell(T^*) = 1$ then ℓ must correspond to an edge in a tree component of $G(\mathcal{S}, L \setminus N)$. Therefore, $x^*(\ell) \in \{0, 1\}$ if $\alpha_\ell(T^*) = 1$.

To see (2), observe that by Corollary 4.5, $\alpha_\ell(S^*)x(\ell) = 0$ for each $\ell \in L \setminus L(T^*)$ and $\alpha_\ell(S^*) = 1$ for each $\ell \in L(T^*)$. It follows that

$$\alpha_\ell(S^*)x = \sum_{\ell \in L(T^*)} x(\ell) = \beta(S^*).$$

Further, since $x_{L(T^*)}$ is integral, $cl(x_{L(T^*)}) = \sigma(supp(x_{L(T^*)}))$ and so

$$\alpha(cl(x_{L(T^*)}))x = 2 \sum_{\ell \in L(T^*)} x(\ell) = 2\beta(S^*) = \beta(cl(x_{L(T^*)})) \tag{5.6}$$

and

$$\alpha(T^* \vee cl(x_{L(T^*)}))x = 2 \sum_{\ell \in L} x(\ell) = \beta(T^*) + \beta(S^*) = \beta(T^* \vee cl(x_{L(T^*)})). \tag{5.7}$$

Combining (5.6) and (5.7) we see that $\beta(T^*/cl(x_{L(T^*)}) = \beta(T^*/S^*)$.

Finally, to see (3), observe that $S^* \subseteq T^* \wedge cl(x_{L(T^*)})$, but since

$$\beta(T^* \vee cl(x_{L(T^*)})) + \beta(T^* \wedge cl(x_{L(T^*)})) \leq \beta(T^*) + \beta(cl(x_{L(T^*)})),$$

it follows that $\beta(T^* \wedge cl(x_{L(T^*)})) \leq \beta(S^*)$. □

Corollary 5.10 *Let x be an extreme maximum matching 2-lattice vector, (S^*, T^*) be the dominant cover and $e \notin T^*$. If X is a $\beta(S^*)$ intersection in $\mathbf{M}_1(S^*, T^*)$ and $\mathbf{M}_2(S^*, T^*; e)$, then x' defined by*

$$x'(\ell) = \begin{cases} 1 & \text{if } \ell \in X \\ 0 & \text{if } \ell \in L(T^*) \setminus X \\ x(\ell) & \text{otherwise} \end{cases}$$

is an extreme maximum matching 2-lattice vector with $e \notin cl(x')$.

Proof. In Lemma 5.6, we showed that $x' \in \Omega$. If x' is not extreme, there is a subset $\{x^1, x^2, \ldots, x^k\}$ of distinct vectors in Ω_{ext}, such that

$$x' = \lambda_1 x^1 + \lambda_2 x^2 \ldots + \lambda_k x^k$$

for some $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) > 0$ with $\sum \lambda_i = 1$. We show that $z^i = x_{L(T^*)} + x^i_{L \setminus L(T^*)}$ is in Ω for each $i \in \{1, \dots, k\}$ as follows.

Since $x \in \Omega$, $x_{L(T^*)}$ satisfies conditions (3) and (4) of Lemma 5.4. Similarly, since $x^i \in \Omega$, $x^i_{L \setminus L(T^*)}$ satisfies conditions (1) and (2) of Lemma 5.4 for $i = 1, 2, \dots, k$. Thus, it remains to show that $x_{L(T^*)}$ and $x^i_{L \setminus L(T^*)}$ satisfy conditions (a) and (b) of Lemma 5.5.

By (2) of Lemma 5.9, $x_{L(T^*)}$ satisfies (a) of Lemma 5.5. Further, since $supp(x^i_{L \setminus L(T^*)}) \subseteq T^*$,

$$\alpha(cl(x_{L(T^*)}))x^i_{L \setminus L(T^*)} = \alpha(cl(x_{L(T^*)}) \wedge T^*)x^i_{L \setminus L(T^*)}.$$

But $cl(x_{L(T^*)}) \wedge T^* = S^*$ so

$$\alpha(cl(x_{L(T^*)}))x^i_{L \setminus L(T^*)} = \alpha(S)x^i_{L \setminus L(T^*)} = 0;$$

proving that $x_{L(T^*)}$ and $x^i_{L(T^*)}$ satisfy condition (b) of Lemma 5.5.

Thus, by Lemma 5.5, z^i is a matching 2-lattice vector for each $i \in \{1, \dots, k\}$.

Since

$$x_{L \setminus L(T^*)} = x'_{L \setminus L(T^*)} = \lambda_1 x^1_{L \setminus L(T^*)} + \lambda_2 x^2_{L \setminus L(T^*)} \dots + \lambda_k x^k_{L \setminus L(T^*)}$$

it follows that

$$x = \lambda_1 z^1 + \lambda_2 z^2 \dots + \lambda_k z^k.$$

Further, since $x'_{L(T^*)} \in \{0, 1\}$, $x^i_{L(T^*)} = x'_{L(T^*)}$ for $i = 1, \dots, k$. Hence, the members of $\{x^i_{L \setminus L(T^*)} : i \in [1, \dots, k]\}$ are distinct and therefore so are the members of $\{z^i : i \in [1, \dots, k]\}$. This contradicts the assumption that x is extreme. □

Corollary 5.11 *Let (S^*, T^*) be the dominant cover, then*

$$T^* = \cap(cl(x) : x \in \Omega) = \cap(cl(x) : x \in \Omega_{ext}).$$

In the case of matroid intersection, we have the following characterization.

Corollary 5.12 *Let $\mathbf{M}_1$ be a matroid with rank function r_1 and closure operator σ_1 and let $\mathbf{M}_2$ be a matroid with rank function r_2 and closure operator σ_2 both defined on the same ground set E and let Ω_{ext} be the collection of all maximum cardinality intersections in $\mathbf{M}_1$ and $\mathbf{M}_2$. Then for each $I \in \Omega_{ext}$,*

$$|I| = r_1(T_1) + r_2(E \setminus T_1) = r_1(E \setminus T_2) + r_2(T_2),$$

where

$$\begin{aligned} T_1 &= \cap(\sigma_1(I) : I \in \Omega_{ext}), \text{ and} \\ T_2 &= \cap(\sigma_2(I) : I \in \Omega_{ext}). \end{aligned}$$

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