A "bucket brigade" assembly line is capable of complex new dynamics if workers differ in the speeds at which they walk back to get more work. In this extended model the line is self-balancing if each worker takes work from a colleague who is more slowed by the act of work than he. Under alternative schemes of sharing work the behavior of an assembly line can be chaotic and so effectively indistinguishable from randomness, even though our model is completely deterministic.

Key words: bucket brigades, dynamical systems, chaos, self-organization, assembly line, work sharing

1. Bucket Brigade Assembly Lines

“Bucket brigades” are a way to coordinate the efforts of workers along an assembly line. The idea is to allow the workers to move where needed, thereby avoiding the imbalance typical of static assignments. The movement of the workers is coordinated by a simple decentralized rule: Each worker carries work forward, from work station to work station, until he either completes an item or it is taken by a downstream colleague; then he walks back to get more work, either from an upstream colleague or from a buffer at the start of the line. Bucket brigades have been used most notably in distribution centers where assembly lines of workers pick items for customer orders. They have also been used in assembly of apparel, tractors, televisions, automotive electrical harnesses and other items, as documented in Bartholdi and Eisenstein (1996b), Bartholdi and Eisenstein (1996a), Bartholdi et al. (2001), Villalobos et al. (1999b), and Villalobos et al. (1999a).

The simplest model to capture the essential behavior of bucket brigades is the so-called Normative Model, which represents a sort of ideal of bucket brigades and so has been the basis for most implementations, such as those reported in Bartholdi and Eisenstein (1996a,b). The Normative Model assumes that the work-content of the product is deterministic and is continuously and evenly distributed along the assembly line (rather than being clumped at work stations). This model of work was justified at length in Bartholdi and Eisenstein (1996b) and used in Bartholdi et al. (1999).

The Normative Model also assumes that each worker $i$ is characterized by an arbitrary but constant work velocity $v_i$ and that the walk-back velocity $w_i = \infty$. In addition, the original bucket brigade rules required the workers to start and remain sequenced from slowest to fastest in the direction of material flow.

Under these assumptions, a bucket brigade assembly line is self-balancing: If workers are sequenced from slowest to fastest in the direction of material flow then, independent of the starting
positions of the workers, the positions of the hand-offs will quickly converge to a stable, *de facto* partition of the work-content so that, if the work-content is scaled to 1, worker \( i \) will always hand off work at position

\[
\frac{\sum_{j=1}^{i} v_j}{\sum_{j=1}^{n} v_j}.
\]

The predictions of the model have been confirmed in both experiment and industrial application, such as those described in Bartholdi and Eisenstein (1996a,b).

Here we extend the Normative Model to a new class of applications in which each worker \( i \) walks back, not instantaneously, but at a constant velocity \( w_i \) of arbitrary value. In this extended model, workers are no longer directly comparable as slower or faster and so it is not clear whether the dynamics of self-balance can be preserved. The main result of this paper is to present a new condition sufficient to guarantee self-balance. Furthermore, we show that if this condition is not enforced then the output of the bucket brigade line can be chaotic and so effectively indistinguishable from randomness.

2. An Extended Model of Bucket Brigades

In most previous models, such as those described in Bartholdi et al. (1999) and Bartholdi et al. (2001), each worker \( i \) has a distinct, constant forward work velocity \( v_i \). Others, such as Armbruster and Gel (2006), have assumed that forward velocities are piecewise constant, while Bartholdi and Eisenstein (1996b) assumed more generally that forward velocities are continuous almost everywhere. Most have also assumed that the time required for a worker to walk back upstream and get more work is insignificant compared to the time to work forward; and so workers have been modeled as having a common walk-back velocity \( w_i = \infty \). One exception is Bartholdi and Eisenstein (2005), which describes a case study wherein the time for worker \( i \) to walk back to get more work from worker \( i - 1 \) is a constant that does not depend on the progress of the item of either worker. Another exception is Bratcu and Dolgui (2005), in which each worker shares the same constant walk back velocity. Our model generalizes this by assuming that each worker \( i \) is characterized by both an arbitrary but constant velocity \( v_i \) in the forward direction and an arbitrary but constant velocity \( w_i \) in the backward direction.

Our generalization applies in some new application contexts, such as McMaster-Carr, an industrial supply house that carries over 350,000 stockkeeping units, and from whom a typical customer order requests fewer than three. To provide a high level of service, orders must be picked within 30 minutes of receipt, and so workers typically pick few orders per trip. Therefore, the time to pick a customer order is composed mostly of walking and so the time to walk forward picking an order is comparable to the time required to walk back to get the next order. Urbanfetch.com, Peapod.com and other high-service distributors are similar in that they must pick small orders soon after receipt.

Let there be \( n \) workers in the bucket brigade, indexed from 1 to \( n \). Workers 1, \ldots, \( i - 1 \) are the *predecessors* of worker \( i \) and workers \( i + 1, \ldots, n \) are his *successors* and each worker must be able to distinguish his predecessors from his successors. Each worker follows the (extended) Bucket Brigade Rules given in Table 1.

Under this new version of the Bucket Brigade Rules several types of instantaneous events are possible. First, there are the events familiar from previous models of bucket brigades:

- **Starts**, in which a worker begins new work at position 0, the start of the assembly line.
- **Completions**, in which a worker finishes work at position 1, the end of the assembly line.
- **Hand-offs**, in which a successor who is walking back takes over the work from a predecessor who is working forward.
Table 1 Each team member independently follows these extended Bucket Brigade Rules.

<table>
<thead>
<tr>
<th>Forward Rule: Work forward with your item until</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. your item is taken by a successor; or</td>
</tr>
<tr>
<td>2. you complete your item;</td>
</tr>
<tr>
<td>then follow the Backward Rule.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Backward Rule: Walk back to get more work:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. if you encounter a predecessor working forward then take over his item;</td>
</tr>
<tr>
<td>2. otherwise, begin a new item at the start of the line;</td>
</tr>
<tr>
<td>then follow the Forward Rule.</td>
</tr>
</tbody>
</table>

Unlike the Normative Model, any worker may start or complete an item, and a worker $i$ can hand off his item to any successor $j > i$. There are also two new behaviors:

- **Overtaking**, in which one worker catches up to and passes another as both walk back or as both work forward.

- **Passing**, in which a worker going back to get more work walks past a successor who is working forward. (They must pass because a worker may not take work from a successor. On the other hand he cannot pass any predecessor because he is required to take work from him.)

Starts occur at location 0, completions at location 1, and overtaking and passing occur in the open interval $(0,1)$. In case two or more workers meet simultaneously at either 0 or 1, we give precedence to the higher indexed workers, and thus hand-offs occur in the interval $(0, 1]$.

Overtaking has been forbidden in previous models and in most implementations of bucket brigades because of the special requirements of the applications. For example, overtaking has been forbidden in high-density order-picking to ensure that customer orders remain in sequence, either to satisfy technological constraints such as imposed by pick-to-light systems; or so orders can flow directly onto outbound trucks in reverse sequence of delivery (Bartholdi and Eisenstein (1996b), Bartholdi et al. (2001)).

Under this extended model of bucket brigades, even familiar events such as hand-offs appear in more complicated patterns. For example, because of finite velocities of walk-back, hand-offs are no longer contemporaneous. Furthermore, there can be multiple completions before the next hand-off or before the next start. It is even possible that there are no hand-offs at all, as when workers with velocities $v_1 = 2, w_1 = 1, v_2 = 1, w_2 = 2$ start together at the origin. In this case, the intended bucket brigade disintegrates into the uncoordinated efforts of individual workers.

Because we now allow overtaking, it is no longer possible for one worker to block the movement of another. Therefore the production rate is as large as possible, regardless of how the workers are sequenced. In the long run each worker must travel as far forward as he does backward and so worker $i$ has an effective production rate of

$$
\psi_i = (1/v_i + 1/w_i)^{-1}. 
$$

Therefore the long-run average production rate of a given set of workers is $\sum_{i=1}^n \psi_i$, and this is independent of the starting positions of the workers.

Expression 1 arises as follows. The term $1/v_i$ represents the encumbered transit time of worker $i$; that is, the time for worker $i$ to accomplish one unit of work-content by himself. Similarly, $1/w_i$ represents the unencumbered transit time, which is the time for worker $i$ to walk back past one unit of work-content. Therefore $1/v_i + 1/w_i$ is the total time required for worker $i$ to assemble one item.

Even though production rate is not an issue in this extended model, the allocation of work-content remains important in practice because workers will learn more quickly and so be more productive if they are able to repeat the same interval of work-content on successive items. Muñoz
and Villalobos (2002) documented the advantages of bucket brigades under a model of learning; and learning was vital to the application described by Bartholdi and Eisenstein (2005). Furthermore, a balanced allocation of work means that the assembly line will consume raw materials and produce finished product at a steady rate, reducing required safety stocks both upstream and downstream of the bucket brigade.

The main result of this paper is to present a condition sufficient to guarantee self-balance (emergence of a stable partition of work), when walk-back velocities have arbitrary but constant values and both overtaking and passing are allowed. It seems natural to guess that the condition, which must generalize slowest-to-fastest, is to index the workers by their effective production rates $\psi_i$, but this is not so:

**Convergence Condition:** The workers on the bucket brigade assembly line should be indexed so that

$$\frac{1}{v_1} - \frac{1}{w_1} > \frac{1}{v_2} - \frac{1}{w_2} > \ldots > \frac{1}{v_n} - \frac{1}{w_n};$$

or, in other words, from most-slowed to least-slowed.

This condition is somewhat surprising, because it may require a worker who is slower in both directions to be the one who sets the pace for the bucket brigade. For example, consider two workers described by $v_1 = 10, w_1 = 40, v_2 = 9, w_2 = 20$. Worker 2 is slower; but the Convergence Condition requires him to be worker of higher index.

The Convergence Condition may be interpreted as follows: The term $1/v_i - 1/w_i$ represents the difference in the encumbered and the unencumbered transit times of worker $i$ and so gives the extent to which he is slowed by work. Therefore the workers should be indexed according to the extent to which each is slowed by work. The worker who is least slowed by work is assigned index $n$ and may take over the work of any colleague, but no one can take over his work. The worker who is most slowed by work has index 1 and cannot take work from anyone; he can only retrieve items at the beginning of the assembly line.

In the remainder of this paper we explore the Convergence Condition in several ways. We first show that, independent of the initial positions and directions of movement of the workers, behavior develops according to this pattern: After a transient period in which the workers spontaneously sort themselves by index, there follows the period of self-balancing when workers move so that work is reallocated to approach perfect balance. An example of this behavior appears in Figure 1.

Finally we completely analyze asymptotic behavior for almost all 2-worker bucket brigades and discover that they can behave chaotically if the Convergence Condition does not hold.

### 3. System Dynamics

#### 3.1. The “Dual” of a Bucket Brigade

Every bucket brigade contains a sort of dual “disassembly” line that is moving in the opposite direction. In this interpretation, the work content to assemble an item is identical to the work content to disassemble it. Imagine that, when worker $i$ completes an item, he immediately begins disassembling it at rate $w_i$. Similarly when worker $i$ has completely disassembled an item, he immediately begins to re-assemble it at rate $v_i$. When worker $i < j$ working forward meets worker $j$ working back, worker $i$ exchanges his item being assembled for the item being disassembled by worker $j$. At all times there are $n$ items in process—some being assembled and some disassembled.

If the $i$-th worker in the bucket brigade has forward velocity $v_i$ and backward velocity $w_i$ then in the dual bucket brigade the $i$-th worker has forward velocity $w_{n-i+1}$ and backward velocity $v_{n-i+1}$.

This change in perspective is useful because the bucket brigade and its dual are equivalent in some important ways. For example, any position $x$ in one bucket brigade corresponds to the
position $1-x$ in the other. If one is balanced then the other is as well. (Indeed, the Convergence Condition is invariant under this transformation.) This equivalence greatly simplifies some of the analysis, for any statement about one line translates directly to a statement about the other.

### 3.2. Self-balance

A bucket brigade that satisfies the Convergence Condition goes through two phases of behavior: Sortation—in which the workers sort themselves according to their index; and Convergence—in which the system converges to a unique balance point.

Under bucket brigades workers share work-content by handing off items to successors. The locations at which hand-offs occur determine how the work is shared. The bucket brigade assembly line is balanced if each worker invests the same clock time and repeats the same interval of work content for each item produced, and, moreover, those intervals are non-overlapping. This implies that, at balance, each worker hands off his current item at a fixed position. Let the balance point at which worker $i$ hands off work, given as a fraction of work-content completed, be $x_i^*$ and let $x^* = (x_1^*, x_2^*, \ldots, x_{n-1}^*)$. The following result explicitly identifies $x^*$ and proves that it is unique.

**Theorem 1.** The fixed point

$$x_i^* = \frac{\sum_{j=1}^{i} \psi_j}{\sum_{j=1}^{n} \psi_j} \text{ for } i=1, \ldots, n-1.$$  \hspace{1cm} (3)
Proof. At the balance point there is no passing or overtaking and each worker $i$ repeats a simple loop for each item produced, retrieving work from worker $i-1$ at point $x_{i-1}^*$ and relinquishing his work to worker $i+1$ at point $x_i^*$ (where for convenience we define $x_0^* = 0$ and $x_n^* = 1$).

Since each worker must repeat his portion of work-content in a common cycle time for each item produced, the balance point is the unique solution to the $n-1$ equations

$$(x_i^* - x_{i-1}^*)/\psi_i = (x_{i+1}^* - x_i^*)/\psi_{i+1} \quad \text{for each } i = 1, \ldots, n-1. \tag{4}$$

It is worth remarking that the uniqueness of the balance point depends on our requirement that the work be partitioned among the workers. Consider again the bucket brigade line with workers $v_1 = 2$, $w_1 = 1$ and $v_2 = 1$, $w_2 = 2$. If these workers start at $1/2$ in opposite directions of travel then $1/2$ is a fixed point of balance; but if they start together at $1$, each worker repeats the same (entire) interval of work content and $1$ is a fixed point—but the line is not balanced because the work is not partitioned.

Theorem 1 tells us that once the assembly line is balanced then, in the absence of perturbations, it will remain balanced. But for the balance point to be useful in practice, it is not enough that it be “merely” a fixed point, but that it be an attracting fixed point, so that if the system starts sufficiently close to balance, it will spontaneously return.

Theorem 2. If workers are sequenced on the assembly line from most-slowed to least-slowed (the Convergence Condition) then $x^*$ is an attractor.

Proof. See Appendix, Section A.

The proof of Theorem 2 shows that when passing and overtaking has ceased the system will converge to the balance point. We can formally show that for two workers, passing and overtaking ceases quickly, and thus the system converges to the balance point when the Convergence Condition holds. We believe this to be true for $n$ workers as well, so that the attracting fixed point is a global attractor. A myriad of simulations confirm this, but we have been unable to prove the result in general. The difficulty seems to be an exponentially growing number of rather intricate sub-cases to consider. This complexity is illustrated even within the following proof for the $n = 2$ worker case.

Theorem 3. If $n = 2$ workers are sequenced according to the Convergence Condition, then they will sort themselves by index and all overtaking and passing will cease within time $\max_i (1/v_i + 1/w_i)$.

Proof. See Appendix, Section B.

In fact, we can completely categorize all the dynamics of 2-worker bucket brigades. In Figure 2 any 2-worker bucket brigade is specified by giving values for the velocities of the first worker $v_1, w_1$ and of the second worker $v_2, w_2$ and then plotting the point $(1/v_2 - 1/v_1, 1/w_2 - 1/w_1)$. Note, however, that this mapping is not one-to-one, as a point will, in general, correspond to many different possible bucket brigades, all sharing the same qualitative behavior.

In building this classification, two points of special interest emerge:

- If the Convergence Condition holds (the unshaded regions of Figure 2) then balance is a global attractor. Furthermore, in converging to balance, the bucket brigade assembly line passes through two distinct phases: First a transient phase in which the workers sort themselves according to index; and then a period of convergence during which all overtaking and all passing have ceased.
- If the Convergence Condition does not hold (the shaded regions of Figure 2) then chaotic behavior is possible, as we will establish shortly.
Similarly, variability of item starts creates problems upstream. Furthermore, variability of intercompletion times complicates behavior. These systems are often called chaotic systems to reflect the fact that their behavior seems to be deeply connected to randomness. (See Alligood et al. (1996), Devaney (1992), or Martelli (1999) for discussions.)

The existence of chaotic behavior can have practical implications for assembly lines because variability often degrades performance of manufacturing systems (see Chapter 9 of Hopp and Spearman (2000)). For example, the variation in processing times of task primitives at a work station may disrupt balance and create bottlenecks, increase work-in-process, and increase the average flow time of product through the line. Furthermore, variability of intercompletion times creates problems downstream, where coordination is complicated by erratic product completions. Similarly, variability of item starts creates problems upstream.

The words “chaos” and “chaotic” have been used informally to suggest complexity, but these words also have precise meanings and it is to these meanings we appeal. To our knowledge, only one other model of a manufacturing system has been formally shown to exhibit chaotic dynamics and that is the switched arrival system studied in Chase et al. (1993). In this model a single switching server distributes work over n parallel machines. The amount of work in the buffer in front of each machine is assumed to be a continuous variable and the processing rate of each machine is assumed to be constant. The server continues to fill the current buffer until some other buffer empties. The rate at which the server fills a buffer is equal to the sum of the processing rates of all machines.
Figure 3  The dynamics map of a chaotic bucket brigade.

Note. The dynamics map of the hand-off positions $x^k$ of the bucket brigade with $v_1 = 1$, $w_1 = 1/3$; $v_2 = 1$, $w_2 = 1$. This map is a reflection of the chaotic Bernoulli map $x^{k+1} = 2x^k \mod 1$. It is an expanding map with discontinuities at $1/2$ and 1, and repelling fixed points at $1/3$ and $2/3$.

When the system is sampled at the instants when a buffer empties, the dynamics of the system can be represented by a function that maps the unit interval into itself. Chase et al. (1993) showed that this function, which describes the amount of work in the buffers, can be chaotic. Others have extended the model in various ways, as may be found in Armbruster (2004), Peters et al. (2004) and citations therein.

4.1. Chaos in bucket brigades

If configured inappropriately, a bucket brigade described by our extended model can be capable of chaotic behavior:

**Theorem 4.** There exists a 2-worker bucket brigade in which the sequence of hand-off positions, and therefore intercompletion times of items, are chaotic.

Consider the bucket brigade composed of workers with the following velocities: $v_1 = 1$, $w_1 = 1/3$; $v_2 = 1$, $w_2 = 1$. This bucket brigade fails to satisfy the Convergence Condition and it is straightforward to verify that the dynamics function relating the positions of successive hand-offs is given by the following, where $x^k$ denotes the location of the $k$-th hand-off.

$$x^{k+1} = 1 - (2x^k \mod 1).$$ (5)

As illustrated in Figure 3, this is an expanding map; that is, it has slope of absolute value strictly greater than 1, where defined. Furthermore, it has discontinuities at 1/2 and 1, and repelling fixed points at 1/3 and 2/3.

This dynamics function is a reflection of the Bernoulli map (also known as the shift map, the doubling map, or the baker’s map):

$$x^{k+1} = 2x^k \mod 1.$$

Martelli (1999) calls the Bernoulli map “one of the most quoted examples of chaotic behavior” and proves it is chaotic in the sense that there exists $x^0$ such that the orbit $O(x^0) = \{x^0, x^1, \ldots\}$ is both dense and unstable in $[0, 1]$. Devaney (1989) agrees that the map is chaotic but uses a slightly
different definition of chaos (sensitive dependence on initial conditions, topological transitivity, and density of periodic points).

The reflected Bernoulli map (5) is chaotic under either definition. First note that analysis of both maps is simplified by considering the values of the $x^k$ to be represented by their binary expansions. Then each iteration of either map simply shifts digits leftward one position and then drops any integer part. The reflected Bernoulli map then complements each bit. A consequence is that the two-fold composition of the Bernoulli map is identical to the two-fold composition of the reflected Bernoulli map (except at $0, 1/4, 1/2, 3/4$, and 1).

It is now easy to see that long term behavior of our bucket brigade depends sensitively on initial conditions. If $x^0$ is given to an accuracy of $n$ binary digits, then after $n$ iterations all information will have evaporated. Thus, when the Convergence Condition fails to hold, it is impossible to predict the distant future state of a bucket brigade due to unavoidable inaccuracy in the measurement of initial conditions. Similarly, the difference between starting at $x^0$ and $x^0 + \epsilon$ grows as $2^n \epsilon$. In fact, both the Bernoulli map and its reflection are expansive, which means that the orbits of all nearby points eventually separate (Devaney (1989)).

If $x^0$ is the binary representation of a rational number, then the orbit $O(x^0)$ is periodic because the binary expansion repeats. Therefore, the starting points that lead to periodic orbits are dense in the unit interval. Moreover, there are a countably infinite number of periodic orbits having arbitrarily large period. If the rational number $x^0$ has a finite binary expansion then the orbit converges to the period-1 cycle, $x = 1$. (This should be taken as a reminder to be suspicious of computer simulations of bucket brigades using fixed precision.)

If, on the other hand, $x^0$ is the binary representation of an irrational number, then the orbit $O(x^0)$ is non-periodic. Therefore, there are uncountably many starting points (the irrational numbers) for which the resultant orbit of hand-offs never repeats. In fact, there exist orbits that are dense; that is, there exists an irrational $x^0$ for which the resultant orbit approaches every number in $[0, 1]$ arbitrarily closely. The binary expansion of such a starting point can be constructed as follows. At the $j$-th step, append two copies of each of the $2^j$ sequences of $j$ binary digits, so that the first digits of $x^0$ would be

$$0.0 0 1 1 00 00 01 01 10 10 11 11 \ldots,$$

and thus any number will appear, in successively more accurate approximations, within the binary expansion of $x^0$. Consequently, within $2^{j+2}$ iterations, the reflected Bernoulli map will be within $1/2^j$ of that number.

In addition, orbits that do not map to either point of discontinuity, $x = 1/2$ or $x = 1$, have Lyapunov exponent greater than 0: Letting $f$ represent the reflected Bernoulli map, the Lyapunov exponent is given by

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln |f'(x^k)| = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \ln 2 = \ln 2 > 0.$$

Therefore there are an uncountably infinite number of orbits that are chaotic in the sense of Alligood et al. (1996) (not asymptotically periodic and of Lyapunov exponent greater than 0).

The example just given shows a specific instance of chaos. In fact, complex behavior may be found more generally among those 2-worker bucket brigades for which the Convergence Condition fails. Theorem 1 of Li and Yorke (1978) (and restated in Theorem 6.15 of Alligood et al. (1996)) shows that any function has an invariant measure if it maps a unit interval into itself, is piecewise smooth, and is piecewise expanding. This property is associated with the existence of a chaotic attractor. Lim (2005) shows that the dynamics function for 2-worker bucket brigades always satisfies the first two criteria and satisfies the third when the Convergence Condition fails.
4.2. Manifestation of Chaos

Figure 4 shows the behavior of two similar bucket brigades after both workers start at position 0. The left graph shows the positions of successive hand-offs when the Convergence Condition holds. The hand-offs quickly converge to the fixed point as predicted by Theorem 1. The right graph plots successive hand-offs of a companion bucket brigade that fails the Convergence Condition. Note that the fixed point is surrounded by a white band, a region in which there are few or no hand-offs. This reflects the fact that the fixed point has become a repeller instead of an attractor, so that the assembly line positively avoids balance. Furthermore, anyone waiting for completed products would find it difficult to predict when the next will arrive.

Figure 5 gives another view of chaos. In this example, the bucket brigade initially satisfies the Convergence Condition, with workers of velocities $v_1 = 0.1$, $w_1 = 1$, $v_2 = 3$, and $w_2 = 2$. But as the parameter $v_1$ increases, the bucket brigade makes a transition from stability to chaos. We constructed this graph by stepping through values of $v_1 \in [0.1, 10]$, computing the positions of hand-offs through 10,000 iterations (presumably long enough for transients to fade away), and then plotted the positions of the next 1,000 hand-offs. For $v_1 < 6/5$ the Convergence Condition holds (this is within Region 3a of Figure 2), and as expected, all hand-offs occurred at a fixed point, the value of which increases with $v_1$ as predicted by Theorem 1. At the threshold of chaos, $v_1 = 6/5$, the Convergence Condition fails to hold, and the formerly attracting fixed point appears to become explosively repelling as the system moves into the shaded Region 3b. Here behavior appears to be nearly periodic; but on closer examination, each thin branch may be seen to be composed of two still thinner branches, and so on through ever finer levels of detail. In this region of chaos, the asymptotic sets corresponding to each value of $v_1$ appear Cantor-like (Alligood et al. (1996), Devaney (1992), Martelli (1999)). Another regime of behavior occurs as $v_1 > 3$ and the system moves into Region 4. Lim (2005) explains much of the fine structure, including “gaps”, “shadows”, and “threads”.

5. Conclusions

5.1. Convergence Conditions

We have generalized bucket brigades to model workers with arbitrary but constant walk-back velocities and to allow overtaking and passing. Then if workers follow the bucket brigade rule and
if the Convergence Condition holds, balance will assert itself independently of the initial locations and directions of travel of the workers. System behavior will typically consist of two distinct phases: An initial, transient phase in which workers sort themselves by index (that is, from most-slowed to least-slowed); and then a phase in which the hand-offs converge to fixed locations.

When walk-back velocities are arbitrary but constant then workers must be sequenced from most-slowed to least-slowed \((1/v_i - 1/w_{i-1} > 1/v_i - 1/w_i)\). This may open new applications for bucket brigades, such as assembling one product when moving to the right and a different product when moving to the left.

In the applications described at Bartholdi and Eisenstein (1996a), the condition “most-slowed to least-slowed” reduces to “slowest-to-fastest” because one of the following two conditions holds. Significant burden of work: For each worker, forward velocity is much less than backward velocity \((v_i \ll w_i\) for all \(i\)).

Differences in skill: Workers differ much more in forward work velocity \(v_i\) than in backward velocity \(w_i\) \((w_i \approx w_j\) for all \(i, j\)).

The first condition holds in many manufacturing environments and in the high-volume distribution centers where bucket brigades have been most successful. Both conditions hold for the ant species *Messor barbarus*, which has been observed to convey seeds from source to nest via bucket brigades (reported in Reyes and Fernández-Haegar (1999); analyzed in Anderson et al. (2002)). All ants
travel at about the same velocity when unladen; and all are slowed when laden, but smaller ants are slowed more than larger ants.

5.2. Implications of chaos

Our extended model of bucket brigades, though fully deterministic, is capable of chaotic behavior if the Convergence Condition fails to hold; which means that it can be, in effect, indistinguishable from randomness.

Chaotic behavior of an assembly line would have costs not only within the assembly line, but also upstream and downstream of the line. Most immediately, the apparently random locations of hand-offs would dilute any learning effect because workers would not experience a stable assignment of work. And because hand-offs could occur almost anywhere on the assembly line, the upstream worker must be prepared to be interrupted within any interval of work content, no matter how small and no matter where located in the sequence of assembly. This renders uneconomical the re-engineering of work to make hand-offs more efficient. In contrast, such improvements are possible when hand-off positions are known in advance, even if only approximately, as for traditional assembly lines; or for bucket brigades in which the workers have been indexed to satisfy the Convergence Condition. But it is hard to improve the process when work is passed without pattern.

Another difficulty is that apparently random locations of hand-offs is manifest in similarly random completion times of products at the end of the assembly line. Therefore, downstream processes such as checking, packing, and shipping would see arrivals that appeared at random, even though the bucket brigade line was perfectly deterministic. Similarly, consumption of parts to support assembly would be apparently random. This will work against any attempt to achieve just-in-time production and will inflate requirements for safety stock.

More generally, the possibility of chaotic behavior sounds a cautionary note for the modeling of manufacturing systems. A central goal of manufacturing systems control is the reduction of variability (Hopp and Spearman (2000)); but malformed bucket brigades provide a fully deterministic model of an assembly line for which the timing of output appears to be irreducibly random. This randomness seems inherent in the system and quite independent of the usual sources of variability, such as machine breakdowns, vagaries in the positioning of work and in task execution, human inconsistency, and so on.

On the other hand, chaotic behavior might be useful in some contexts. For example, military sentries patrolling a perimeter might avoid regular, easily predictable movements if they adopted different speeds of travel in each direction so that their meeting points would appear without obvious pattern.
Appendix A: Theorem 2: Local Convergence of n-Worker Bucket Brigades

Proof. Iteration $t$ follows the hand-off points of the $t$-th disassembled item from the end of the line to the start. We let $x_i^t$ be the hand-off point where worker $i$ receives the $t$-th disassembled item from worker $i + 1$.

From the hand-off at $x_i^t$, worker $i + 1$ moves forward to hand-off point $x_{i+1}^t$ and then back to hand-off point $x_i^{t+1}$; and at the same time, worker $i$ moves back to hand-off point $x_i^t$ and then forward to hand-off point $x_i^{t+1}$. The following equates these movements of worker $i$ and $i + 1$ from one iteration to the next:

\[
\frac{x_i^t - x_i^{t-1}}{w_i} + \frac{x_i^{t+1} - x_i^t}{v_i} = \frac{x_{i+1}^t - x_i^{t+1}}{w_{i+1}} + \frac{x_{i+1}^{t+1} - x_{i+1}^t}{v_{i+1}}, \quad \text{for each } i = 1, \ldots, n - 1,
\]

where we define $x_0^t = 0$ and $x_n^t = 1$ for all $t$.

Rewriting yields:

\[
x_{i+1}^t = \left( \frac{1/v_i + 1/w_i}{1/v_i + 1/w_{i+1}} \right) x_i^t - \left( \frac{1/v_i + 1/w_i}{1/v_i + 1/w_{i+1}} \right) x_i^t + \left( \frac{1/v_{i+1} + 1/w_{i+1}}{1/v_i + 1/w_{i+1}} \right) x_{i+1}^t. \tag{7}
\]

Or we can write

\[
x_i^{t+1} = (1 + \alpha_i)\gamma_i x_{i-1}^t - \alpha_i x_i^t + (1 + \alpha_i)(1 - \gamma_i)x_{i+1}^{t+1}, \tag{8}
\]

where

\[
\alpha_i = \frac{1/v_i + 1/w_i}{1/v_i + 1/w_{i+1}}, \tag{9}
\]

and $0 < \alpha_i < 1$ corresponds to our Convergence Condition; and where

\[
\gamma_i = \frac{1/v_i + 1/w_i}{1/v_i + 1/w_i + 1/v_{i+1} + 1/w_{i+1}}.
\]

We show that the dynamics of the system can be described by an affine linear mapping,

\[
y^{t+1} = Ay^t + b, \tag{10}
\]

where $y^t = (x_1^t, x_2^t, \ldots, x_{n-2}^t, x_{n-1}^{t+1})^T$. (Our vector $y^t$ indeed holds the last $n - 2$ disassembly hand-offs of item $t$, followed by the first disassembly hand-off of item $t + 1$—as we will see, this is to accommodate the vector $b$, which affects only the updating of $x_{n-1}^{t+1}$.)

We now factor the matrix $A = A_{n-1}A_1A_2\ldots A_{n-2}$, where each matrix $A_i$ updates $x_i$ according to Equation (8), and we have

\[
b = (0, 0, \ldots, 0, (1 + \alpha_{n-1})(1 - \gamma_{n-1}))^T.
\]

In this way we first update $x_{n-2}^t$, then $x_{n-3}^t$, and so forth until $x_1^t$, and then finally $x_{n-1}^{t+1}$ which utilizes the last component of $b$.

Each $A_i$ is the identity matrix except for row $i$. Each $A_2, A_3, \ldots, A_{n-2}$ has three non-zero terms in row $i$ that sum to one, with values $(1 + \alpha_i)\gamma_i$, $-\alpha_i$, and $(1 + \alpha_i)(1 - \gamma_i)$ in columns $i - 1$, $i$, and $i + 1$ respectively. For $A_1$ the first term $(1 + \alpha_1)\gamma_1 > 0$ is omitted from row 1, and thus the first row sum has absolute value less than one. And for $A_{n-1}$ the last term $(1 + \alpha_{n-1})(1 - \gamma_{n-1}) > 0$ is omitted from row $n - 1$, so the last row sum has absolute value less than one.

For the full transition matrix $A$, all the eigenvalues have modulus less than one. In short, this follows since each $A_2, \ldots, A_{n-2}$ can be replaced with a stochastic matrix, while both $A_1$ and $A_{n-1}$ can be replaced with a strictly sub-stochastic matrix. And since all states communicate, the system tends to the zero matrix. Thus the orbit $y^t, y^{t+1}, \ldots$ converges to the unique fixed point $y^*$ of hand-off positions. (For dynamics of affine systems, see, for example, Martelli (1999)).
Appendix B: Theorem 3: Passing and overtaking are transient for a 2-worker bucket brigade

Proof. There are three cases to consider:
Region 1: $v_1 > v_2$; $w_1 > w_2$.
Region 2: $v_1 \leq v_2$; $w_1 \geq w_2$.
Region 3: $v_1 < v_2$; $w_1 < w_2$.
(The natural fourth case, $v_1 \geq v_2$; $w_1 \leq w_2$, cannot occur if the Convergence Condition holds and may therefore be ignored.)

B.0.1. Region 1: Transient behavior when $v_1 > v_2$ and $w_1 > w_2$ We first show that in this case all hand-offs are soon confined to the right side of the line.

Lemma 1. After the first completion by worker 2, all subsequent hand-offs must occur within the interval

$$\left[\frac{1}{w_2} - \frac{1}{w_1}, 1\right].$$

Proof. Consider any hand-off after the first. How close to the start of the line can that hand-off occur? That hand-off must have been directly preceded by a completion by worker 2. After the most recent such completion, worker 2 must encounter worker 1 for a hand-off at position $h$ within time $1/w_1 + h/v_1$ and so

$$\frac{1}{w_1} + \frac{h}{v_1} \geq 1 - \frac{h}{w_2},$$

from which the result follows. □

For worker 1 to overtake worker 2 in the forward direction, the preceding hand-off must occur close enough to the start of the line that worker 1 has time to return for new work and still catch worker 2 before the end of the line:

Lemma 2. If a hand-off occurs in

$$\left[\frac{1}{v_2} - \frac{1}{v_1}, 1\right]$$

then worker 1 cannot overtake worker 2 before the next completion.

Proof. Let a hand-off occur at position $h$. For worker 1 to overtake worker 2 before the next completion requires

$$\frac{h}{w_1} + \frac{1}{v_1} < \frac{1 - h}{v_2},$$

so that

$$h < \frac{1/v_2 - 1/v_1}{1/v_2 + 1/w_1}.$$  □

Lemma 3. When the Convergence Condition holds, in Region 1 all overtaking and passing ceases after worker 2 first completes work.

Proof. After worker 2 first completes work he will never again be upstream of worker 1 because

- Worker 2 cannot overtake worker 1 in either direction because he is too slow ($v_1 > v_2$ and $w_1 > w_2$).
- Worker 1 cannot overtake worker 2 in the forward direction. This follows since by Lemma 1 all hand-offs subsequent to the first completion by worker 2 must occur to the right of

$$\frac{1/v_2 - 1/v_1}{1/v_2 + 1/w_1}.$$
but by Lemma 2, worker 1 does not have time to overtake worker 2 after such a hand-off because we claim that

\[
\frac{1/v_2 - 1/v_1}{1/v_2 + 1/w_1} \leq \frac{1/w_2 - 1/w_1}{1/v_1 + 1/w_2}.
\]

(11)

We let

\[ k = \frac{1}{v_1} - \frac{1}{w_1} - \frac{1}{v_2} + \frac{1}{w_2} \]

and note that \(k > 0\) follows from the Convergence Condition. Now letting \(a = 1/w_2 - 1/w_1\) and \(b = 1/v_1 + 1/w_2\) we rewrite our claim (11) as

\[
\frac{a-k}{b-k} \leq \frac{a}{b}.
\]

This claim holds since

\[
\frac{a}{b} - \frac{a-k}{b-k} = \frac{k(b-a)}{b(b-k)};
\]

and the last term is positive, because \(b > k > 0\) and \(b > a\).

Because worker 1 can never subsequently be to the right of worker 2, he is never in position to overtake or to pass worker 2 while walking back. Similarly, worker 2 is never in a position to pass worker 1 while moving forward; and, by the bucket brigade rules, worker 2 can never pass worker 1 while walking back.

Therefore, after worker 2 first completes work, which will be within time \(1/v_2 + 1/w_2\), all overtaking and passing must cease.

\[\Box\]

**B.0.2. Region 2: Transient behavior when \(v_1 \leq v_2\) and \(w_1 \geq w_2\)**

By the bucket brigade rules, worker 1 can never take work from worker 2; therefore worker 2 will reach the end of the line (position 1) within time \(1/v_2 + 1/w_2\). Once worker 2 is at the end of the line he can never be upstream of worker 1 because:

- Worker 2 can never overtake worker 1 in the backward direction \((w_1 \geq w_2)\).
- Worker 1 can never overtake worker 2 in the forward direction \((v_1 \leq v_2)\).
- By the bucket brigade rules worker 2 can never pass worker 1 while walking back.

Finally, because the workers maintain in sequence, worker 1 will never be in a position to pass worker 2 while walking back. Therefore, after worker 2 first completes work, which will be within time \(1/v_2 + 1/w_2\), there can be no more overtaking or passing.

**B.0.3. Region 3: Transient behavior when \(v_1 < v_2\) and \(w_1 < w_2\)**

This case follows immediately since this case is simply the dual bucket brigade from Region 1.
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