Solutions to Homework # 1

1) \( T = \{1, 2, ..., N\} \)
\( S = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \)
\( A_S = \{0, 1, \ldots, M - K\} \)

\( p_k(j | s, a) = \begin{cases} 0 & \text{if } j > s + a \\ p(s | s, j) & \text{if } j \leq s + a \end{cases} \)

\( r_k(s, a) = -I(a > 0) (K + c(a)) - h(s, a) I(s + a > 0) \)
\(- b(s, a) I(s + a \leq 0) + \sum_{j=0}^{s+a} \int p(s | s, j) p(j) d j + f(a) I(s < 0) I(s + a < 0) \)

with the convention that summation over an empty set is 0 and \( f(0) = 0 \).

2) As we have shown in class, there exists a monotone optimal policy for this problem. Then using monotone backward induction, you can show that

\( d_1^*(s) = \begin{cases} 0 & \text{if } s \leq 1 \\ 1 & \text{if } s > 1 \end{cases} \)
\( d_2^*(s) = \begin{cases} 0 & \text{if } s \leq 1 \\ 1 & \text{if } s > 1 \end{cases} \)
3) Let $Z = f(X_1, X_2, X_3) = \mathbb{E} \mathbb{E} y | X_1, X_2, X_3$. We need to show that
\[
\mathbb{E} Z | X_1, X_2 = \mathbb{E} \mathbb{E} y | X_1, X_2.
\] (1)

We have
\[
\mathbb{E} Z | X_1 = a_1, X_2 = a_2 = \mathbb{E} f(X_1, X_2, X_3) | X_1 = a_1, X_2 = a_2 = \sum_{a_3} f(a_1, a_2, a_3) \mathbb{P}(X_3 = a_3 | X_1 = a_1, X_2 = a_2)
\]
and
\[
f(a_1, a_2, a_3) = \frac{\sum b \mathbb{P}(Y = b | X_1 = a_1, X_2 = a_2, X_3 = a_3)}{b}.
\]

Putting these two together, we have
\[
\mathbb{E} Z | X_1 = a_1, X_2 = a_2 = \sum_{a_3} \mathbb{P}(Y = b | X_1 = a_1, X_2 = a_2, X_3 = a_3) \mathbb{P}(X_3 = a_3 | X_1 = a_1, X_2 = a_2) = \mathbb{E} \mathbb{E} y | X_1 = a_1, X_2 = a_2, (2)
\]
since
\[
\sum_{a_3} \mathbb{P}(Y = b | X_1 = a_1, X_2 = a_2, X_3 = a_3) \mathbb{P}(X_3 = a_3 | X_1 = a_1, X_2 = a_2)
\]
\[
= \sum_{a_3} \mathbb{P}(Y = b, X_3 = a_3 | X_1 = a_1, X_2 = a_2) = \mathbb{P}(Y = b | X_1 = a_1, X_2 = a_2).
\]

Noting that (2) is the same as (1) completes the proof.