1. The sum of the initial probabilities of being in each state must be equal to one so
\( \mathbf{a} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \). Similarly, the sum of each row in the transition matrix must be equal to one so
\[
\mathbf{P} = \begin{bmatrix}
0.2 & 0.4 & 0.4 \\
0 & 0.4 & 0.6 \\
0.3 & 0.1 & 0.6
\end{bmatrix}.
\]

1.a. From the transition matrix \( \mathbf{P} \) we see that
\[ P\{X_1 = 0|X_0 = 1\} = 0. \]

1.b The row vector \( \mathbf{aP} = (0.175, 0.325, 0.500) \) describes the distribution of \( X_1 \).

1.c The row vector \( \mathbf{aP}^{15} = (0.2093, 0.2326, 0.5581) \) describes the distribution of \( X_{15} \).

1.d. Yes, the Markov chain is irreducible. Clearly we can move from state 0 to state 1 and then from state 1 to state 2 and finally from state 2 back to state 0 so that all states communicate with one another.

1.e The Markov chain is aperiodic since we can always go from state 0 back to itself and the fact that we also we know from part c that all states communicate with one another.

1.f. To compute that stationary distribution for \( \pi \) we solve the following system of equations
\[
\begin{align*}
0.2\pi_0 + 0.1\pi_1 + 0.3\pi_2 &= \pi_0 \\
0.4\pi_0 + 0.4\pi_1 + 0.0\pi_2 &= \pi_1 \\
0.4\pi_0 + 0.6\pi_1 + 0.6\pi_2 &= \pi_2 \\
\pi_0 + \pi_1 + \pi_2 &= 1,
\end{align*}
\]
and find the solution to be \( \pi = (0.2093, 0.2326, 0.5581) \).

1.g Yes, the Markov chain is positive recurrent since it is irreducible and aperiodic and a stationary distribution also exists.

1.h To compute the long run average award per step we simply compute the expected reward under the stationary distribution. Formally, we have that the long run average reward per step equals
\[
-10 \times 0.2093 + 15 \times 0.2326 + 20 \times 0.5581 = 12.5580.
\]

2. Yes, they both may be modeled by discrete-time Markov chains (since the future is independent of the past).

For \( \{X_n, n \geq 1\} \) we have the following.
i. \( S = (1, 2, 3, 4, 5, 6) \) and

\[
P = \begin{bmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

ii. \( X_1 \) will simply be the number rolled in the first roll and so the distribution of \( X_1 \) is \((\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})\).

iii. No, the Markov chain is not irreducible since we can only move up in states. For instance, once we are in state three we will never return to states one or two.

iv. Yes, a steady-state distribution exists. It is easy to see that we will eventually end up in state six and once we are there we will never leave. The steady-state distribution is therefore \((0, 0, 0, 0, 0, 1)\). This may be formally found by solving \( \pi = \pi P \) subject to \( \pi e = 1 \) where \( e = (1, 1, 1, 1, 1, 1)' \). Another alternative would be to take \( P \) to a very high power, say \( P^{1000} \), and look at the corresponding values in the matrix.

For \( \{Y_n, n \geq 1\} \) we have the following.

i. \( S = (1, 2, 3, ...) \) and we may define the entries of \( P \) in the following way

\[
P_{ij} = \begin{cases} 
\frac{5}{6} & \text{if } j = i, \\
\frac{1}{6} & \text{if } j = i + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

ii. \( X_1 \) is simply be the number of sixes rolled in the first roll and so the distribution of \( X_1 \) is \((\frac{5}{6}, \frac{1}{6}, 0, 0, 0, ...)\).

iii. No, the Markov chain is not irreducible since again we can only move up in states.

iv. No, a steady state distribution does not exist. The state of the Markov chain will continue to increase as time moves on and so there is no steady state probability or long run average of it being in any particular state.

3.a. The Markov chain is irreducible since we may move from state two to state one to state three and we may also move in the reverse direction from state three to state one to state two. The stationary distribution is \((.2857, .0714, .6429)\). Since state three is aperiodic every state must be aperiodic because the Markov chain is irreducible.

3.b The Markov chain is not irreducible since state one only communicates with itself. We may move from any state to state one and since we never leave state one once we enter it the stationary distribution must be \((1, 0, 0)\). States one and two are aperiodic while the period of state three is essentially infinity.

3.c. The Markov chain is irreducible since we may move from state one to state two to state three and then back to state one again. The stationary distribution is \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). The period of each state is two.
4. We will adopt the second approach. Let’s let $N = 3$ and note that the transition matrix

$$P = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}$$

is doubly stochastic, irreducible, and aperiodic. The stationary distribution of $P$ is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We will now guess that in general if there are $N$ states then the stationary distribution will be $(\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N})$. To verify our guess we note that by letting $e$ be the $n$-dimensional row vector with all 1’s for its entries we have that

$$\pi P = \left( \frac{1}{N} e \right) P = \frac{1}{N} (e P) = \frac{1}{N} e = \pi$$

where $e = eP$ follows from the fact that $P$ is doubly stochastic. Our guess has therefore been verified.

5. Yes, $\{X_n, n \geq 0\}$ is a Markov Chain. The state space is $S = \{0, 1, 2, 3, \ldots\}$, the initial distribution is $a = (1, 0, 0, 0, \ldots)$, and the transition matrix is

$$P_{ij} = \begin{cases}
\frac{2}{100} & \text{if } j = 0, \\
\frac{98}{100} & \text{if } j = i + 1, \\
0 & \text{otherwise.}
\end{cases}$$

5.a. Yes, the Markov chain is irreducible. From any state we may drop to zero and from zero we may move to any state $n$ will positive probability.

5.b. The Markov chain is aperiodic since it is irreducible and state zero is aperiodic.

5.c. The stationary distribution is $\pi = (\frac{2}{100}, \frac{2}{100}, \frac{98}{100}, \frac{2}{100}, (\frac{98}{100})^2, \ldots)$.

5.d Yes, the Markov chain is positive recurrent since it is aperiodic, irreducible, and it has a stationary distribution.