1. (a) Let $\mu_i$ denote the service rate at server $i$; hence, $\mu_1 = 1/2$ and $\mu_2 = 1/4$, and $\Pr\{T_A > 5\} = e^{-\mu_1} = e^{-5/2} \approx 0.082$.

(b) $E[T_A] = 2$ minutes.

(c) 

\[
E[T_A \mid T_A > 5] = E[5 + (T_A - 5) \mid T_A > 5] \\
= E[5 \mid T_A > 5] + E[T_A - 5 \mid T_A > 5] \\
= 5 + E[T_A] \\
= 5 + 2 = 7 \text{ minutes.}
\]

(d) $\Pr\{T_A < T_B\} = \mu_1/(\mu_1 + \mu_2) = 1/2(1/2 + 1/4) = 2/3$.

(e) $E[\min(T_A, T_B)] = 1/(\mu_1 + \mu_2) = 4/3 \text{ minutes.}$

(f) $E[\min(T_A, T_B)] = 1/(\mu_1 + \mu_2) = 4/3 \text{ minutes.}$ (Yes, this is the same question just expressed a different way.)

(g) 

\[
E[T_C] = E[\min\{T_A, T_B\} + (T_C - \min\{T_A, T_B\})] \\
= 4/3 + E[(T_C - \min\{T_A, T_B\})] \\
= 4/3 + \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2} \\
= 4/3 + 2/(\mu_1 + \mu_2) = 12/3 \text{ minutes.}
\]

(h) We need to calculate $E[\max\{T_A, T_B, T_C\}]$. Note that this is not the same as the last question since $C$ may not be the last customer to leave. It’s easiest to break $\max\{T_A, T_B, T_C\}$ into the sum of three quantities: the time until the first departure, the time between the first and second departure, and the time between the second and third departure. Define middle to give the middle of 3 numbers just like max gives the largest and min gives the smallest. So we can write $\max\{T_A, T_B, T_C\} = \min\{T_A, T_B, T_C\} + (\text{middle}\{T_A, T_B, T_C\} - \min\{T_A, T_B, T_C\}) + (\max\{T_A, T_B, T_C\} - \text{middle}\{T_A, T_B, T_C\})$. Hence, $E[\max\{T_A, T_B, T_C\}] = E[\min\{T_A, T_B, T_C\}] + E[\text{middle}\{T_A, T_B, T_C\} - \min\{T_A, T_B, T_C\}] + E[\max\{T_A, T_B, T_C\} - \text{middle}\{T_A, T_B, T_C\}]$. The first term $E[\min\{T_A, T_B, T_C\}] = E[\min\{T_A, T_B\}] = 4/3$ since either $A$ or $B$ leaves first. The second term giving the time between the first departure and the second departure is again the minimum of two independent exponential random variables since both servers are busy and using the memoryless property which is again $4/3$. Note that it does not matter whether $A$ or $B$ left first. However, the third term, the difference between the second and third departure is simply the remaining service time of the last customer in the system which could be at either server. Hence, the last term is simply

\[
E[\max\{T_A, T_B, T_C\} - \text{middle}\{T_A, T_B, T_C\}] = \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2} = 8/3 \text{ minutes.}
\]

So $E[\max\{T_A, T_B, T_C\}] = 2(4/3) + 8/3 = 16/3$.

(i) $\Pr\{T_C < T_A \mid T_B < T_A\} = \mu_2/(\mu_1 + \mu_2) = 1/3$.

(j) $\Pr\{T_A > \max\{T_B, T_C\}\} = [\mu_2/(\mu_1 + \mu_2)]^2 = 1/9$, $\Pr\{T_B > \max\{T_A, T_C\}\} = [\mu_1/(\mu_1 + \mu_2)]^2 = 4/9$, $\Pr\{T_C > \max\{T_A, T_B\}\} = 2[\mu_1/(\mu_1 + \mu_2)][\mu_2/(\mu_1 + \mu_2)] = 4/9$. 

Solutions to Homework 7
November 12, 2006
(k) First we need two departures for D to start service. Since both servers are busy for these departures, the expected time until the second departure is double the answer to (e), i.e., $8/3$. Then we need the expected service time of D which depends on which server D receives service from. As in (g), the expected service time of customer D is $8/3$, so $E[W_D] = 16/3$.

2. (a) $\Pr\{T < 1\} = 1 - e^{-10/8} \approx .71$ and $\Pr\{T < 2\} = 1 - e^{-20/8} \approx .92$.

(b) The expected tardiness is $E[(T - 1)^+]$ where the $+$ denotes positive part; i.e., $x^+ = x$ if $x > 0$ and 0 otherwise. Note that the tardiness cannot be negative!

$$
E[(T - 1)^+] = E[(T - 1)^+ | T \leq 1] \Pr\{T \leq 1\} + E[(T - 1)^+ | T > 1] \Pr\{T > 1\}
$$

$$
= 0 \times (1 - e^{-10/8}) + E[T] e^{-10/8}
$$

$$
= \frac{8}{10} e^{-10/8} \approx .23
$$

(c) The expected revenue for this contract and the other two Littlefield contracts is $E[g(T)]$ where

$$
g(s) = \begin{cases} 
1000 & s \leq 1, \\
1000(2 - s) & 1 < s \leq 2, \\
0 & 2 < s. 
\end{cases}
$$

Let $\lambda = 10/8$.

$$
E[g(T)] = \int_{0}^{\infty} g(s) \lambda e^{-\lambda s} \, ds
$$

$$
= \int_{0}^{1} 1000 \lambda e^{-\lambda s} \, ds + \int_{1}^{2} 1000(2 - s) \lambda e^{-\lambda s} \, ds
$$

$$
\approx \$836.46
$$

For the $1250 contract, changing things appropriately gives $752.49. For the $750 contract, we get $799.99. Thus, the $1000 contract would be optimal.

(d) The $1250 is best when $\delta \leq .536$ days. If $.536 < \delta \leq 1.054$ days, the $1000 contract is optimal. If the expected time until delivery is longer than 1.054 days, we should use the $750 contract. (Even though we do not know whether the length of time to meet an order in Littlefield is exponentially distributed, the above should provide a useful rule of thumb for deciding which contract should be used when playing the simulation.)

3. (a) The rate of defects per meter of wire is $\lambda = .01$

(b) The expected number of defects in the first spool is $E[N(100)] = 1$

(c) Recall that the Poisson process has stationary increments, thus $E[N(1000) - N(900)] = E[N(100)] = 1$.

(d) $\Pr\{N(100) = 0\} = \frac{(10e^{-10})^0}{0!} = e^{-1}$.

(e) $\Pr\{N(1000) - N(900) = 1\} = \Pr\{N(100) = 1\} = e^{-1}$.

(f) Here the Poisson process has independent increments, our desired solution is $e^{-1}$.

(g) $\Pr\{N(1000) = 1\} = 10e^{-10}$.

(h) Let $T_1$ be the time when the first defect occurs. Here $E[T_1] = \frac{1}{0.01} = 100$.

(i) $E[S_{10}] = \sum_{i=1}^{10} E[T_i] = 1000$.

(j) $E[T_1 | T_1 > 100] = E[T_1 - 100 + 100 | T_1 > 100] = 100 + E[T_1] = 200$.

(k) Since the Poisson process regenerates at all time points, we have that $E[T_1] = 100$. And thus, the answer is $1000 + 100 = 1100$.

(l) The expected number of major defects in the first 5 spools is $(.2)(.01)(500) = 1.$