(a) This is a cost problem with penalty cost $p = 90$, procurement cost $c_v = 15$, and holding cost $h = -8$ cents. The critical ratio for part 947A is therefore given by

$$c^*_A = \frac{p - c_v}{p + h} = \frac{90 - 15}{90 - 8} = .914.$$ 

We would now like to find $q^*_A$ such that $F_A(q^*_A) = 0.765$, where $F_A$ is the CDF for the demand for part 947A. Letting $D_B$ denote the demand for part 947B, we have that

$$Z = \frac{D_A - 1,500,000}{500,000}$$

is a $N(0, 1)$ random variable. From a lookup table, we find that $P(Z \leq 1.365) \approx .914$ and so

$$.914 = P\left(\frac{D_A - 1,500,000}{500,000} \leq 1.365\right) = P(D_A \leq 2,182,500).$$

Thus, $q^*_A = 2,182,500$ part 947A’s should be produced.

(b) This is again a cost problem with penalty cost $p = 90$, procurement cost $c_v = 15$, and holding cost $h = -8$ cents and so the critical ratio for part 947B is given by $c^*_B = .914$. As in part (a), we would now like to find $q^*_B$ such that $F_B(q^*_B) = 0.914$, where $F_B$ is the CDF for the demand for part 947B. Letting $D_B$ denote the demand for part 947B, we have that

$$Z = \frac{D_B - 500,000}{100,000}$$

is a $N(0, 1)$ random variable. From a lookup table, we have that $P(Z \leq 1.365) \approx .914$ and so

$$.914 = P\left(\frac{D_A - 500,000}{100,000} \leq 1.365\right) = P(D_A \leq 636,500).$$

Thus, $q^*_B = 636,500$ part 947B’s should be produced.

(c) Note first that the demand for part 947C is $D_C = D_A + D_B$. That is, the demand for part 947C is equal to the sum of the demand for parts 947A and 947B. Thus, since the demands for part 947A and 947B are $N(1,500,000, (500,000)^2)$
and $N(500,000,(100,000)^2)$ random variables, respectively, we have that $D_C$ is 
a $N(2,000,000,(509,902)^2)$ random variable. Thus, letting 
\[
Z = \frac{D_C - 2,000,000}{509,902},
\]
we have that is a $N(0,1)$ random variable. Again, from a lookup table, we have 
that $P(Z \leq 1.365) \approx .914$ and so 
\[
.914 = P\left(\frac{D_C - 2,000,000}{509,902} \leq 1.365\right) = P(D_A \leq 2,696,016).
\]
Thus, $q_c = 2,696,016$ part 947C’s should be produced in order to meet the same 
fraction of demand.

(d) We shall now derive a general cost formula for each of the three demands 
and afterwards we will plug in the proper values for each of our three cases. Let 
\[
 l(q) = c_v q + pE((D - q)^+) + hE((q - D)^+)
\]
be the cost if we order $q$ parts. We then have that 
\[
l(q) = c_v q + p \int_q^{\infty} (x - q) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx - h \int_{-\infty}^q (x - q) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx
\]
\[
= c_v q + (\mu - q)(p(1 - F(q)) - h F(q)) + p \int_{-\infty}^q (x - \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx
\]
\[
-h \int_{-\infty}^q (x - \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx.
\]
Now consider the integral 
\[
\int_q^{\infty} (x - \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx
\]
appearing above. After making the change of variable $y = (x - \mu)^2$ and performing 
a standard u-substitution argument, we see that the above integral is equal to 
\[
\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(\mu-q)^2}{2\sigma^2}}.
\]
Furthermore, since $E[X - E[X]] = 0$, 
\[
\int_{-\infty}^q (x - \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = -\int_{-q}^{\infty} (x - \mu) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx.
\]
Therefore, we have the formula
\[ l(q) = c_v q + (\mu - q)(p(1 - F(q)) - hF(q)) + (p + h) \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(\mu - q)^2}{2\sigma^2}}. \]

Now, after plugging in the appropriate values for each demand, we see that
\[ l(q_A^*) = 28,907,682 \]
\[ l(q_B^*) = 8,781,536 \]
\[ l(q_C) = 50,014,664 \]

Our calculations now tell us that if we produce C, we will lose \[ l(q_C) - l(q_A^*) - l(q_B^*) = 12,325,446 \] cents.

(e) We have a cost problem with penalty cost \( p = 90 \), procurement cost \( c_v = 20 \), and holding cost \( h = -8 \) cents and so the critical ratio for part 947C is given by
\[ c_C^* = \frac{p - c_v}{p + h} = \frac{90 - 20}{90 - 8} = .854. \]

We would thus like to find \( q_C^* \) such that \( F_C(q_C^*) = 0.714 \), where \( F_C \) is the CDF for the demand for part 947C. Recall from part (c) that \( D_C \) is a \( N(2,000,000, (509,902)^2) \) random variable. Thus, letting \( D_C \) denote the demand for part 947C, we have that
\[ Z = \frac{D_C - 2,000,000}{509,902} \]
is a \( N(0,1) \) random variable. From a lookup table, we have that \( P(Z \leq 1.055) \approx .854 \) and so
\[ .854 = P \left( \frac{D_C - 2,000,000}{509,902} \leq 1.055 \right) = P(D_C \leq 2,537,947). \]

Thus, \( q_C^* = 2,537,947 \) is the optimal number of part 947C’s to be produced and so our answer from part (c) was incorrect.

(f) Using \( l(q) \) from part (d), we see that \( l(q_C^*) = 49,576,439 \). Therefore, we should not produce C at all since it would be cheaper to produce parts 947A and 947B separately.

**Problem 2**
This is a cost problem with fixed ordering cost \( c_f = 2,000 \), variable ordering cost \( c_v = 100 \), holding cost \( h = 20 \), and penalty \( p = 200 \). The critical ratio is therefore given by

\[
c^* = \frac{p - c_v}{p + h} = \frac{200 - 100}{200 + 20} = .45
\]

and so, since the demand is \( U[400,800] \), the optimal ordering quantity for the same problem but without a fixed cost is given by

\[
q^* = 400 + .45 * (400) = 580.
\]

We will next calculate the expected total loss \( L(q) \) given that \( q \) units of solvent are ordered. Letting \( D \) be the demand for the solvent, we have that

\[
L(q) = c_v q + h E[(q - D)^+] + p E[(D - q)^+].
\]

Thus, since \( D \sim U[400,800] \), we have that for \( 400 \leq q \leq 800 \),

\[
L(q) = 100q + 20 E[(q - D)^+] + 200 E[(D - q)^+]
\]

\[
= 100q + 20 \int_{400}^{q} (q - x) \frac{1}{400} dx + 200 \int_{q}^{800} (x - q) \frac{1}{400} dx
\]

\[
= 100q + \frac{1}{20} \int_{400}^{q} (q - x) dx + \frac{1}{2} \int_{q}^{800} (x - q) dx
\]

\[
= 100q + \frac{1}{20} \left( qx - \frac{x^2}{2} \right)_{400}^{q} + \frac{1}{2} \left( x^2 - qx \right)_{q}^{800}
\]

\[
= 100q + \frac{1}{20} \left( q^2 - 400q + 80,000 \right) + \frac{1}{2} \left( 320,000 - 800q + \frac{q^2}{2} \right)
\]

\[
= .275q^2 - 320q + 164,000.
\]

Substituting \( q^* = 580 \) into the equation above, we find that \( L(q^*) = 70,910 \).

Setting

\[
L(s) = L(q^*) + c_F
\]

and solving for \( s \) we now find that

\[
.275s^2 - 320s + 164,000 = 70,910 + 2,000,
\]

or,

\[
.275s^2 - 320s + 91,090 = 0,
\]

and so by the quadratic formula

\[
s = \frac{320 \pm \sqrt{(320)^2 - 4(.275)(91,090)}}{2(.275)}
\]

\[
= 630.7 \text{ or } 496.4
\]
Since the optimal ordering quantity was 580, we take the smaller quantity of 496.4.

(a) Thus, if start out with zero units of solvent, we order 580 units. If we start out with 100 units, we order 480 units. If we start out with 300 units, we order 280 units. If we start out with either 500 or 800 units, we do not order since we are above the level \( s = 496.4 \).

(b) In general, the optimal ordering policy to order \( q^* - x = 580 - x \) units if we start out with \( x \leq s = 496.4 \) units and order nothing if we start out with \( x > 496.4 \) units.

(c) If we start out with 0 units, then the optimal number to order is \( q^* = 580 - 0 = 580 \) units. The cost for doing this will be

\[
2,000 + (.275 \times (580)^2) - 320 \times 580 + 164,000 = 72,910 \text{ dollars.}
\]

If our initial inventory level is \( x = 700 \) liters, then, since \( 700 > 496.4 \), the optimal policy is to do nothing. The cost for doing this will be

\[
hE[(700 - D)^+] + pE[(D - 700)^+] = \frac{1}{20} \left( 700x - 400 \right)_{700}^{700} + \frac{1}{2} \left( \frac{x^2}{2} - 700x \right)_{700}^{800}
\]

\[
= 4,750 \text{ dollars.}
\]

(d) We perform the same procedure as at the start of the problem but instead substitute 2,000 with 10,000. This leads the quadratic equation

\[
.275s^2 - 320s + 164,000 = 70,910 + 10,000,
\]

or,

\[
.275s^2 - 320s + 83,090 = 0,
\]

and so by the quadratic formula

\[
s = \frac{320 \pm \sqrt{(320)^2 - 4(.275)(83,090)}}{2(.275)}
\]

\[
= 772.5 \text{ or } 391.1.
\]

However, since 772.5 > 580 and 391.1 < 400 which is out of the range of the valid range for the formula is valid for, we must now assume that we order less than 400 units. This then leads to the formula for \( 0 \leq q \leq 400 \),

\[
L(q) = 100q + 200E[(D - q)^+]
\]

\[
= 100q + 200E[D] - 200q
\]

\[
= -100q + 200 \times 600
\]

\[
= -100q + 120,000.
\]
Thus, setting

\[-100s + 120,000 = 70,910 + 10,000,\]

we find that that $s = 390.9$.

(d) If start out with zero units of solvent, we order 580 units. If we start out with 100 units, we order 480 units. If we start out with 300 units, we order 280 units. If we start out with either 500 or 800 units, we do not order since we are above the level $s = 390.9$.

(e) In general, the optimal ordering policy to order $q^* - x = 580 - x$ units if we start out with $x \leq s = 390.9$ units and order nothing if we start out with $x > 390.9$ units.

**Problem 3**

The time between the arrival of beams is 600 seconds and so the arrival rate of beams to the system is $\lambda = 1/600$ per second. Further, since the time between the arrivals of beams is deterministic, the coefficient of the variation of the interarrival times is zero, $c_A = 0$. The human operator takes an average of $\tau_H = 500$ seconds to paint a beam with a standard deviation of $\sigma_H = 300$ seconds. Thus, the service rate of the human operator is $\mu_H = 1/500$ per second and the coefficient of variation is $c_H = \sigma_H/\tau_H = 3/5$. The traffic intensity under the human operator is $\rho_H = \lambda/\mu_H = 5/6$. By Kingman’s approximation we now have that the average waiting for beams in the queue under the human operator is

$$W_H^q \approx \tau_H \frac{\rho_H c_H^2 + c_H^2}{2} = 500 \frac{5/6}{1 - 5/6} \frac{0 + 9/25}{2} = 450 \text{ seconds}.$$  

By Little’s Law expected number of steal beams waiting in queue under the human operator is given by

$$L_H^q = \lambda W_H^q = \frac{1}{600} \times 450 = .75 \text{ beams.}$$

For the automatic painter, it takes an average of $\tau_M = 560$ seconds to paint a beam but with a standard deviation of only $\sigma_M = 100$ seconds. Thus, the service rate of the machine is $\mu_M = 1/560$ per second and with a coefficient of variation of $c_M = \sigma_M/\tau_M = 5/28$. The traffic intensity under the human operator is $\rho_M = \lambda/\mu_M = 14/15$. By Kingman’s approximation, we now have that the average waiting for beams in the queue under the machine is

$$W_M^q \approx \tau_M \frac{\rho_M c_M^2 + c_M^2}{2} = 560 \frac{14/15}{1 - 14/15} \frac{0 + 25/784}{2} = 125 \text{ seconds}.$$
By Little’s Law expected number of steal beams waiting in queue is under the machine is given by

\[ L_q^M = \lambda W_q = \frac{1}{600} \times 125 = 0.25 \text{ beams.} \]

Thus, even though the machine takes on average 60 seconds more than the human operator to paint a beam, it has a lower standard deviation and there on average shorter waiting times in the queue and less beams waiting to be painted.