Solutions to Homework 1, ISyE 3232

Problem 1

(a) Poisson (b) exponential (c) geometric (d) Bernoulli (e) binomial (f) normal

Problem 2

We are given that $E[X] = 6$, $Var(X) = 80$. Thus

$E(6 - 3X) = 6 - 3E(X) = 6 - 3(6) = -12$

$Var(6 - 3X) = 9Var(X) = 9(80) = 720$

$E((2X - 3)/5) = (2/5)6 - 3/5 = 9/5$

$Var((2X - 3)/5) = (4/25)Var(X) = 320/25$

Problem 3

Recall the following formulas from algebra (they will prove to be useful below):

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

\[ \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \]

(a) We know that

\[ 1 = \sum_{k=1}^{6} P(X = 2k) = \sum_{k=1}^{6} 2ck = 42c \]

so it follows that $c = 1/42$.

(b)

\[ E(X) = \sum_{i=1}^{6} 2iP(X = 2i) = \left(\frac{4}{42}\right) \sum_{i=1}^{6} i^2 \]

\[ = \frac{(2/21)(6)(7)(13)}{6} = 26/3. \]

(c)
\[ E(X^2) = \sum_{i=1}^{6} 4i^2 P(X = 2i) = \frac{8}{42} \sum_{i=1}^{6} i^3 \]

\[ = (\frac{8}{42})(36)(49)/4 = 84. \]

(d)

\[ \text{Var}(X) = E(X^2) - (E(X))^2 = 84 - (26/3)^2. \]

(e)

\[ E((X-5)^+) = E(\text{max}(X-5,0)) = 1P(X = 6)+3P(X = 8)+5P(X = 10)+7P(X = 12) = 82/21. \]

Problem 4

(a)

\[ P(X = k) = \frac{5^k e^{-5}}{k!}, \text{for } k = 0, 1, 2, \ldots \]

(b)

\[ E(X) = 5 \]

(c)

\[ \text{Var}(X) = 5 \]

(d) Suppose \(0 \leq k \leq 4\), \(k\) integer. Then

\[ P(Y = k) = P(\text{min}(X, 5) = k) = P(X = k) = \frac{5^k e^{-5}}{k!}, \]

\[ P(Y = 5) = P(\text{min}(X, 5) = 5) = \sum_{k=5}^{\infty} P(X = k) = 1 - \sum_{k=0}^{4} \frac{5^k e^{-5}}{k!} \]

(e)

\[ E(Y) = \sum_{k=1}^{4} \frac{5^k e^{-5}}{(k-1)!} + 5 - 5 \sum_{k=0}^{4} \frac{5^k e^{-5}}{k!} \]

\[ \approx 4.123 \]
Problem 5

(a) We know that

\[ 1 = \int_0^{\infty} e^{-6s} ds = c/6 \]

so it follows that \( c = 6 \) (thus, \( Y \) is an exponential random variable with rate 6).

(b) Let \( \gamma \) denote the squared coefficient of variation of \( Y \). Then
\[ E(Y) = 1/6, \ Var(Y) = 1/36, \ \gamma = (1/36)/(1/6)^2 = 1. \]

(c) \[ P(Y > 6) = \int_6^{\infty} 6e^{-6t} dt = e^{-36}. \]

(d) \[ P(Y > 8|Y > 2) = P(Y > 8)/P(Y > 2) = e^{-48+12} = e^{-36}. \]

(e) We know that \( x^* \) satisfies the following:

\[ 1/3 = P(Y > x^*) = e^{-6x^*}. \]

After simplifying, we see that \( x^* = -\ln(1/3)/6. \)

Problem 6

(a) \( P(X = Y) = 0 \) (to see this, try setting up the limits of integration)

(b) \[ P(\text{min}(X,Y) > 1/5) = P(X > 1/5, Y > 1/5) = \int_{1/5}^{\infty} \int_{1/5}^{\infty} 4e^{-4x}5e^{-5y} dy dx \]

\[ = e^{-9/5}. \]

(c) \[ P(X \leq Y) = \int_0^{\infty} \int_x^{\infty} 4e^{-4x}5e^{-5y} dy dx \]

\[ = 4/9. \]

(d) Let \( x \geq 0 \). Then

\[ f_X(x) = \int_0^{\infty} 4e^{-4x}5e^{-5y} dy = 4e^{-4x} \]

(e) \[ E[XY] = \int_0^{\infty} \int_0^{\infty} xy4e^{-4x}5e^{-5y} dy dx = 1/20. \]
Problem 7

Let $X_k$ denote the processing time (measured in minutes) of the $k$th item, where $1 \leq k \leq 400$. Then the total processing time is

$$
\sum_{k=1}^{400} X_k.
$$

Then

$$
P(\sum_{k=1}^{400} X_k \leq 420) = P \left( \frac{\sum_{k=1}^{400} X_k - 800}{2\sqrt{400}} \leq \frac{420 - 800}{2\sqrt{400}} \right)
$$

$$
\approx P(Z \leq -9.5) \approx 0
$$

where $Z$ denotes a standard normal random variable (mean 0 and variance 1) (notice that the first approximation follows from the Central Limit Theorem; the second can be found in a standard normal table or by using Excel).