Name

Please be neat and show all your work so that I can give you partial credit. GOOD LUCK.

Question 1
Question 2
Question 3
Question 4

Total
1. (25) Suppose \( g \) is a subadditive function on \( X \times Y \) and for each \( x \in X \), \( \max_{y \in Y} g(x, y) \) exists. Then show that

\[
f(x) = \max\{y' \in \arg\max_{y \in Y} g(x, y)\}
\]

is nonincreasing in \( x \).
2.a. (15) Suppose you have solved a discounted Markov decision process (with finite state space and finite action set) under maximization and have computed $v^*_\lambda(s)$ and an optimal policy $d^\infty$ for which $v^d_{\lambda} = v^*_\lambda$. Suppose a new action $a'$ becomes available in state $s'$. How can you determine whether $d^\infty$ is still optimal without resolving the problem?

2.b. (15) Suppose action $a^*$ is optimal in state $s^*$, that is $d^*(s) = a^*$, and you find that the reward in state $s$ under action $a^*$ decreases by $\Delta$. Provide an efficient way for determining whether $d^\infty$ is still optimal.
3. (25) Consider a finite horizon Markov decision process with \( T = \{1, \ldots, N\} \),
state space \( S = \{0, 1, \ldots\} \), and a finite set of actions \( A_s = A \) for all \( s \in s \).
Suppose

1. \( r_t(s, a) \) is non decreasing in \( s \) for all \( a \in A \)
2. \( \sum_{j=k}^{\infty} p_t(j|s, a) \) is nondecreasing in \( s \) for all \( k \in S \) and \( a \in A \)
3. \( r_1(s, a) \) is a subadditive function on \( S \times A \)
4. \( \sum_{j=k}^{\infty} p_t(j|s, a) \) is a subadditive function on \( S \times A \) for all \( k \in S \) and
5. \( r_N(s) \) is nondecreasing on \( s \).

Does there exist an optimal decision rule \( d^*_t(s) \) which is monotone is \( s \)? If it does, is it monotone nondecreasing or nonincreasing?
4. (20) Let $S = \{0,1,\ldots\}$ denote the state of an equipment at each decision epoch. State 0 refers to a new equipment and the greater the state index is the poorer the condition of the equipment is. For any $s \in S$, there is a probability $p_s$ that the equipment fails and a probability $1 - p_s$ that it ages by one unit between decision epochs. Let $R$ denote the fixed income per period, $h(s)$ cost of operating the equipment when it is in state $s$, $K$ the cost of replacement and $F$ cost of failure. Clearly, when the equipment fails, it has to be replaced. Otherwise, at each decision epoch the available actions are replace the equipment or continue to operate as it is. Formulate this problem as a discounted Markov decision problem with discount factor $\lambda$ and provide the optimality equations.