Problem 1

Consider a constant stepsize algorithm, so that $x_{k+1} = x_k + sd_k$ for some constant stepsize $s > 0$. There are a variety of conditions under which the sequence of iterates $\{x_k\}$ of such an algorithm converges to a stationary point of $f$.

(1) Consider the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by $f(x) := \|x\|_2^{2+a}$, with $a \geq 0$. Consider the application of the steepest descent algorithm with constant stepsize to $f$, that is, $x_{k+1} = x_k - s\nabla f(x_k)$ for some constant stepsize $s > 0$. Determine for which values of $s$ and $x_0$ the sequence of iterates $\{x_k\}$ converges to $x^* = 0$.

(2) Consider the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by $f(x) := \|x\|_3^{3/2}$.

(a) Show that $f$ is not Lipschitz continuously differentiable, that is, there is no constant $L$ such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2$$

for all $x,y \in \mathbb{R}^n$. (In fact, $f$ is not even locally Lipschitz continuously differentiable at the optimal solution $x^* = 0$, that is, there is no neighborhood of the optimal solution $x^* = 0$ and constant $L$ such that the Lipschitz inequality above holds for all $x,y$ in the neighborhood.)

(b) Consider the application of the steepest descent algorithm with constant stepsize to $f$, that is, $x_{k+1} = x_k - s\nabla f(x_k)$ for some constant stepsize $s > 0$. Show that, for any value of $s > 0$, the sequence of iterates $\{x_k\}$ either converges to $x^* = 0$ in a finite number of iterations (and only in a very special case), or else the iterates do not converge to $x^*$.

(3) Consider a quadratic function $f : \mathbb{R}^n \mapsto \mathbb{R}$ given by $f(x) := \frac{1}{2}x^T G x + d^T x$, where $G \in \mathbb{R}^{n \times n}$ is symmetric positive definite and $d \in \mathbb{R}^n$. In the previous homework you were asked to show that $f$ is Lipschitz continuously differentiable, that is, there is a constant $L$ such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2$$

for all $x,y \in \mathbb{R}^n$, with the smallest such constant $L$ given by the largest eigenvalue of $G$.

(a) Consider a steepest descent algorithm with a constant stepsize $s$ applied to $f$. Show that $\{x_k\}$ converges to $x^* = -G^{-1}d$ for any starting point $x_0$ if and only if $0 < s < 2/L$.

(b) Consider a gradient search algorithm with a constant stepsize $s$ and constant symmetric positive definite deflection matrix $B$ applied to $f$, that is, $x_{k+1} = x_k - sB\nabla f(x_k)$. Let $L$ be the largest eigenvalue of $B^{1/2}GB^{1/2}$. Show that $\{x_k\}$ converges to $x^* = -G^{-1}d$ for any starting point $x_0$ if and only if $0 < s < 2/L$. 
(4) Consider a quadratic function \( f : \mathbb{R}^n \to \mathbb{R} \) given by \( f(x) := \frac{1}{2} x^T G x \), where \( G \in \mathbb{R}^{n \times n} \) is non-singular symmetric indefinite. Consider a steepest descent algorithm with a constant stepsize \( s \) applied to \( f \). Show that if the starting point \( x_0 \) does not belong to the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues of \( G \), the sequence \( \{x_k\} \) diverges.

(5) Suggest some conditions that you think would be required for a constant stepsize algorithm to converge to a stationary point of \( f \).

**Problem 2**
Nocedal and Wright, Problem 3.1

**Problem 3**
Nocedal and Wright, Problem 3.5

**Problem 4**
Nocedal and Wright, Problem 3.8