1. Let $X_1, \ldots, X_n$ be an iid sample from $N(\theta, \sigma^2)$ and suppose that the prior distribution on $\theta$ is $N(\mu, \tau^2)$, where $\sigma^2$, $\mu$ and $\tau^2$ are assumed to be known. Find the posterior distribution of $\theta$, and calculate $E(\theta|x)$ and $Var(\theta|x)$.

We have that the posterior distribution of $\theta$ is proportional to

$$
\exp \left\{ -\sum_{i=1}^{n} \frac{(x_i - \theta)^2}{2\sigma^2} - \frac{(\theta - \mu)^2}{2\tau^2} \right\} = \exp \left\{ -\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{2\sigma^2} - \frac{n(\bar{x} - \theta)^2}{2\sigma^2} - \frac{(\theta - \mu)^2}{2\tau^2} \right\},
$$

and hence is proportional to

$$
\exp \left\{ -\frac{(\theta - \frac{\tau^2}{\tau^2 + \sigma^2/n} \bar{x} - \frac{\sigma^2/n}{\tau^2 + \sigma^2/n} \mu)^2}{2\frac{\sigma^2 \tau^2/n}{\tau^2 + \sigma^2/n}} \right\}.
$$

Consequently the posterior distribution of $\theta$ is normal with mean

$$
E(\theta|x) = \frac{\tau^2}{\tau^2 + \sigma^2/n} \bar{x} + \frac{\sigma^2/n}{\tau^2 + \sigma^2/n} \mu,
$$

and variance

$$
Var(\theta|x) = \frac{\sigma^2 \tau^2/n}{\tau^2 + \sigma^2/n}.
$$

2. Let $X_1, \ldots, X_n$ be iid Poisson ($\lambda$) and the prior distribution of $\lambda$ is Gamma ($\alpha, \beta$). Recall that the pdf of Gamma ($\alpha, \beta$) is

$$
f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0.
$$

Find the posterior distribution of $\lambda$, and the posterior mean and variance.

We have here

$$
f(x|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!} = \lambda^y e^{-n\lambda} \left( \prod_{i=1}^{n} x_i! \right)^{-1},
$$

where $y = \sum_{i=1}^{n} x_i$. Consequently $\pi(\lambda|x)$ is proportional to

$$
\lambda^y e^{-n\lambda} \lambda^{\alpha-1} e^{-\lambda/\beta} = \lambda^{y+\alpha-1} e^{-\lambda(n\beta+1)/\beta}.
$$

It follows that the posterior distribution of $\lambda$ is Gamma ($y + \alpha, \beta/(n\beta + 1)$), and hence the posterior mean is $(y + \alpha)/(n\beta + 1)$ and the posterior variance is $(y + \alpha)\beta^2/(n\beta + 1)^2$.
3. Let \( X_1, \ldots, X_n \) be independent random variables, where \( X_i \) has cdf \( F(x|\theta_i) \). Denote \( X = (X_1, \ldots, X_n) \) and \( \theta = (\theta_1, \ldots, \theta_n) \). Show that if \( \delta^{\pi_i}_i(X_i) \) is a Bayes rule for estimating \( \theta_i \) using loss \( L(\theta_i, a_i) \) and prior \( \pi_i(\theta_i) \), \( i = 1, \ldots, n \), then \( \delta^{\pi}(X) = (\delta^{\pi_1}_1(X_1), \ldots, \delta^{\pi_n}_n(X_n)) \) is a Bayes rule for estimating \( \theta \) using loss \( \sum_{i=1}^n L(\theta_i, a_i) \) and prior \( \pi(\theta) = \prod_{i=1}^n \pi_i(\theta_i) \).

We have that
\[
R(\theta, \delta) = \mathbb{E}_\theta[L(\theta, \delta(X))] = \mathbb{E}_\theta \left[ \sum_{i=1}^n L(\theta_i, \delta_i(X_i)) \right] = \sum_{i=1}^n R_i(\theta_i, \delta_i),
\]
where
\[
R_i(\theta_i, \delta_i) = \mathbb{E}_{\theta_i}[L(\theta_i, \delta_i(X_i))].
\]
Consequently
\[
\int R(\theta, \delta)\pi(\theta)d\theta = \sum_{i=1}^n \int R_i(\theta_i, \delta_i)\pi(\theta)d\theta_i.
\]
It follows that the minimization of the left hand side of the above equation can be accomplished by minimization of each term of the sum in the right hand side of the above equation.

4. Consider the following (linear-exponential) loss function
\[
L(\theta, a) = e^{c(a-\theta)} - c(a - \theta) - 1,
\]
where \( c \) is a positive constant. (i) Show that, indeed, \( L(\theta, a) \) is a loss function.

Since \( e^x \geq 1 + x \) for any \( x \in \mathbb{R} \), we have that \( L(\theta, a) \geq 0 \) for any \( \theta, a \in \mathbb{R} \). Also, clearly \( L(\theta, \theta) = 0 \).

(ii) Assuming \( X \sim F(x|\theta) \) and prior \( \pi(\theta) \), show that \( \delta^{\pi}(X) = k \log \mathbb{E}(e^{-c\theta}|X) \) and find the constant \( k \).

Consider
\[
r(x, a) = \mathbb{E}[L(\theta, a)|X = x] = e^{ca} \mathbb{E}(e^{-c\theta}|X = x) - ca + \mathbb{E}(\theta|X = x) - 1.
\]

For any \( x \in \mathbb{R} \) the function \( r(x, \cdot) \) is convex and
\[
\frac{\partial r(x, a)}{\partial a} = ce^{ca} \mathbb{E}(e^{-c\theta}|X = x) - c,
\]
and hence \( a_x \) which minimizes \( r(x, \cdot) \) is given by
\[
a_x = -c^{-1} \log \mathbb{E}(e^{-c\theta}|X = x).
\]
Consequently
\[
\delta^{\pi}(X) = -c^{-1} \log \mathbb{E}(e^{-c\theta}|X).
\]
(iii) Let \( X_1, \ldots, X_n \) be an iid sample from \( N(\theta, \sigma^2) \), where \( \sigma^2 \) is known, and suppose that \( \theta \) has the noninformative prior \( \pi(\theta) = 1 \). Find \( \delta^\pi(\bar{X}) \) with respect to the linear-exponential loss function.

By (ii) we have that

\[
\delta^\pi(\bar{X}) = -c^{-1} \log \mathbb{E}(e^{-\theta}\mid \bar{X}).
\]

Since \( \pi(\theta) = 1 \), it follows that the posterior distribution is proportional to \( f(x\mid \theta) \). Consequently, given \( \bar{X} = \bar{x} \), the posterior distribution \( \pi(\theta\mid \bar{x}) \) is normal \( N(\bar{x}, \sigma^2/n) \). Then

\[
\mathbb{E}(e^{-\theta}\mid \bar{X} = \bar{x}) = \int_{-\infty}^{+\infty} e^{-c\theta} \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n(\theta - \bar{x})^2/2\sigma^2} d\theta = \exp \left\{ -c\bar{x} + c^2\sigma^2/(2n) \right\}.
\]

Consequently

\[
\delta^\pi(\bar{X}) = \bar{X} - c\sigma^2/(2n).
\]