SIAM J. OPTIMIZATION
Vol. 4, No. 1, pp. 130–141, February 1994

006

EXISTENCE AND DIFFERENTIABILITY OF METRIC PROJECTIONS IN HILBERT SPACES*

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Abstract. This paper considers metric projections onto a closed subset S of a Hilbert space. If the set S is convex, then it is well known that the corresponding metric projections always exist, unique and directionally differentiable at boundary points of S. These properties of metric projections are considered for possibly nonconvex sets S. In particular, existence and directional differentiability of metric projections for certain classes of sets are established and will be referred to as "nearly convex" sets.

Key words. metric projection, Hilbert space, directional differentiability, tangent cones, distance function

AMS subject classifications. 41A50, 41A65, 46C99, 52A30, 52A40

1. Introduction.

Consider a Hilbert space X and let S be a closed subset of X. With the set S are associated the distance function $d_S(x) = \text{dist}(x, S)$,

$$\text{dist}(x, S) = \inf\{\|x - y\| : y \in S\},$$

and the set-valued metric projection

$$\Pi_S(x) = \{y \in S : \|x - y\| = d_S(x)\}.$$

We also consider a selection mapping $P_S(x) \in \Pi_S(x)$, referred to as a metric projection of X onto S. Of course, $P_S(x)$ is defined only at such points x where the set $\Pi_S(x)$ is nonempty.

In cases when the set S is convex there are some well-known properties of P_S. That is, for all $x \in X$ the metric projection $P_S(x)$ exists and is unique, P_S is Lipschitz continous and for every $x \in S$, P_S is directionally differentiable at x (see, e.g., Zarantonello [18]). In this paper we discuss extensions of these results to situations where the set S is not necessarily convex. We also study directional differentiability of P_S at a point $x \notin S$.

Recall that the contingent (Bouligand) cone to S at $x \in S$, denoted by $T_S(x)$ or $T(x, S)$, is formed by vectors y such that there exist $x_n \in S$, $x_n \to x$, and $t_n \to 0^+$ with $t_n^{-1}(x_n - x) \to y$. Its polar (negative dual) cone,

$$N_S(x) = \{y \in X : (y, z) \leq 0 \text{ for all } z \in T_S(x)\}, \tag{1.1}$$

will be referred to as the normal cone to S at x. By $B(x, r)$ we denote the ball

$$B(x, r) = \{y : \|y - x\| \leq r\}.$$

It is said that a mapping $F : X \to Y$, from X into a Banach space Y, is directionally differentiable at x_0 (in the sense of Gâteaux) if the directional derivative

$$F'(x_0, y) = \lim_{t \to 0^+} \frac{F(x_0 + ty) - F(x_0)}{t} \tag{1.2}$$

* Received by the editors February 10, 1992; accepted for publication (in revised form) December 4, 1992.

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exists for all $y \in X$. We also say that F is directionally differentiable at x_0 in the Hadamard sense if

$$F'(x_0, y) = \lim_{\substack{y' \to y \\ t \to 0+}} \frac{F(x_0 + ty') - F(x_0)}{t}, \tag{1.3}$$

and that F is directionally differentiable in the Fréchet sense if

$$\lim_{y \to 0} \frac{F(x_0 + y) - F(x_0) - F'(x_0, y)}{\|y\|} = 0. \tag{1.4}$$

For a discussion of various relations between these concepts of directional differentiability see, e.g., [4], [9], and [14].

We shall use the following variational principle (see [15] for details). Consider the optimization problems

$$\min_{x \in S} f(x) \tag{1.5}$$

and

$$\min_{x \in T} g(x), \tag{1.6}$$

where $f, g : X \to \Re$ and $S, T \subset X$. Let x_0 be an optimal solution of the problem (1.5), and let $\bar{x}$ be an ϵ-optimal solution of (1.6), i.e., $\bar{x} \in T$ and

$$g(\bar{x}) \leq \inf_{x \in T} g(x) + \epsilon.$$

Suppose that there exist a positive constant α and a neighborhood W of x_0 such that for all $x \in S \cap W$,

$$f(x) \geq f(x_0) + \alpha \|x - x_0\|^2. \tag{1.7}$$

Suppose further that $f(x)$ and $g(x)$ are Lipschitz continuous on W modulus k_1 and k_2, respectively, and that $\bar{x} \in W$. Then

$$\|\bar{x} - x_0\| \leq \alpha^{-1}\kappa + \alpha^{-1/2}\epsilon^{1/2} + 2\delta_1 + \alpha^{-1/2}(k_1\delta_1 + k_2\delta_2)^{1/2}, \tag{1.8}$$

where

$$\delta_1 = \sup_{x \in T \cap W} \text{dist}\,(x, S \cap W), \tag{1.9}$$

$$\delta_2 = \text{dist}(x_0, T \cap W), \tag{1.10}$$

and κ is a Lipschitz constant of the function $h(x) = g(x) - f(x)$ on W. Note that if $S = T$, then $\delta_1 = \delta_2 = 0$, and if $f \equiv g$, then $\kappa = 0$.

2. Existence of metric projections. In this section we give some sufficient conditions for existence and uniqueness of $P_S(x)$. We consider separately two cases: (i) when points x are sufficiently close to a point $x_0 \in S$, and (ii) when x are near a point $\bar{x}$ such that $P_S(\bar{x})$ exists and is unique. We shall use the following concepts of "nearly convex" sets.

DEFINITION 2.1. We say that the set S is $O(m)$-convex at a point $x_0 \in S$ if there are a neighborhood N of x_0 and a positive constant K such that for all $x, y \in S \cap N$,

$$\text{dist}(y - x, T_S(x)) \leq K\|y - x\|^m. \tag{2.1}$$

DEFINITION 2.2. We say that S is $o(m)$-convex at $x_0 \in S$ if there exist a function $k(x,y)$ and a neighborhood N of x_0 such that for all $x, y \in S \cap N$,

$$\operatorname{dist}(y-x, T_S(x)) \leq k(x,y)\|y-x\|^m \tag{2.2}$$

and

$$\lim_{x,y\to x_0} k(x,y) = 0. \tag{2.3}$$

Remarks. It is not difficult to see that $o(m)$-convexity at a point $x_0 \in S$ implies $O(m)$-convexity at the same point. Also if S is $O(m')$-convex at x_0 and $m' > m$, then $o(m)$-convexity at x_0 follows. (Indeed, take $k(x,y) = K\|y-x\|^{m'-m}$.) When the set S is convex, we have that for all $x, y \in S$, $y-x \in T_S(x)$ and hence $\operatorname{dist}(y-x, T_S(x)) = 0$.

The concept of $o(1)$-convexity was introduced in Shapiro and Al-Khayyal [16] under the name "*near convexity.*" In this paper we shall be dealing mainly with $O(2)$- and $o(2)$-convexity. We shall need the following result. Its proof is similar to the proof of [16, Lem. 1].

LEMMA 2.1. *Suppose that S is $o(m)$-convex at x_0. Then there is a neighborhood N of x_0 such that for all $x_1, x_2 \in S \cap N$ and all $y_1 \in N_S(x_1)$, $y_2 \in N_S(x_2)$,*

$$(y_1 - y_2, x_1 - x_2) \geq -\{k(x_1,x_2)\|y_1\| + k(x_2,x_1)\|y_2\|\}\|x_1 - x_2\|^m. \tag{2.4}$$

Proof. It follows from (2.2) that for any $\epsilon > 0$ there is a point $\bar{x}_2 \in x_1 + T_S(x_1)$ such that

$$\|x_2 - \bar{x}_2\| \leq k(x_1,x_2)\|x_2 - x_1\|^m + \epsilon. \tag{2.5}$$

Since $y_1 \in N_S(x_1)$ and $\epsilon > 0$ is arbitrary we obtain then

$$\begin{aligned}(y_1, x_1 - x_2) &= (y_1, x_1 - \bar{x}_2) + (y_1, \bar{x}_2 - x_2) \geq (y_1, \bar{x}_2 - x_2)\\ &\geq -\|y_1\|\|\bar{x}_2 - x_2\| \geq -k(x_1,x_2)\|y_1\|\|x_1 - x_2\|^m.\end{aligned}$$

Similarly, for $y_2 \in N_S(x_2)$,

$$(y_2, x_2 - x_1) \geq -k(x_2,x_1)\|y_2\|\|x_1 - x_2\|^m.$$

Adding these two inequalities together we obtain (2.4). □

It can be shown in the same way that if S is $O(m)$-convex at x_0, then for all $x_1, x_2 \in S \cap N$ and all $y_1 \in N_S(x_1), y_2 \in N_S(x_2)$,

$$(y_1 - y_2, x_1 - x_2) \geq -2K(\|y_1\| + \|y_2\|)\|x_1 - x_2\|^m. \tag{2.6}$$

We can now formulate the first existence theorem.

THEOREM 2.2. *Suppose that S is $O(2)$-convex at $x_0 \in S$. Then $P_S(x)$ exists and is unique and locally Lipschitzian for all x in a neighborhood of x_0.*

Proof. Consider the neighborhood N specified in Definition 2.1, and let r be a positive number such that $B(x_0, r) \subset N$. For any given point x and $\epsilon > 0$ we can find a point $z \in S$ such that

$$\|x - z\|^2 \leq d_S(x)^2 + \epsilon^2.$$

We then have

$$\|z - x_0\| \leq \|z - x\| + \|x - x_0\| \leq d_S(x) + \epsilon + \|x - x_0\| \leq 2\|x - x_0\| + \epsilon.$$

Moreover, by Ekeland's variational principle [2, p. 255] we can find $\bar{z} \in S$ such that

$$\|x - \bar{z}\|^2 \leq d_S(x)^2 + \epsilon^2, \tag{2.7}$$

$\|z - \bar{z}\| \leq \epsilon$, and $\bar{z}$ is the minimizer of the function

$$g_\epsilon(z) = \|x - z\|^2 + \epsilon\|z - \bar{z}\| \tag{2.8}$$

over S. It follows that

$$\|\bar{z} - x_0\| \leq 2\|x - x_0\| + 2\epsilon. \tag{2.9}$$

We denote by $\Pi(x, \epsilon)$ the set of points $\bar{z} \in S$ satisfying conditions (2.7) and (2.9) and being minimizers over S of the corresponding functions given in (2.8). Clearly, for $\epsilon_1 > \epsilon_2$ we have $\Pi(x, \epsilon_2) \subset \Pi(x, \epsilon_1)$.

Now consider a point x and $\epsilon > 0$ such that $\|x - x_0\| < r/4$ and $\epsilon < r/4$, and let $\bar{z}_1, \bar{z}_2 \in \Pi(x, \epsilon)$. It then follows from (2.9) that $\bar{z}_1, \bar{z}_2 \in S \cap N$. Moreover, by first-order optimality conditions we have that there exist v_1 and v_2 such that $\|v_1\| \leq \epsilon$, $\|v_2\| \leq \epsilon$, and

$$y_i = x - \bar{z}_i + v_i \in N_S(\bar{z}_i), \qquad i = 1, 2. \tag{2.10}$$

We then obtain from (2.6) that

$$-(y_1 - y_2, \bar{z}_1 - \bar{z}_2) \leq 2K\|\bar{z}_1 - \bar{z}_2\|^2(\|y_1\| + \|y_2\|)$$

and hence

$$\|\bar{z}_1 - \bar{z}_2\|^2 - \|v_1 + v_2\|\|\bar{z}_1 - \bar{z}_2\| \leq 2K\|\bar{z}_1 - \bar{z}_2\|^2(r + 2\epsilon).$$

It follows that

$$\|\bar{z}_1 - \bar{z}_2\|^2 - 2\epsilon\|\bar{z}_1 - \bar{z}_2\| \leq 2K\|\bar{z}_1 - \bar{z}_2\|^2(r + 2\epsilon)$$

and, since $\epsilon < r/4$,

$$(1 - 3rK)\|\bar{z}_1 - \bar{z}_2\| \leq 2\epsilon. \tag{2.11}$$

We further take such $r > 0$ that $r < (3K)^{-1}$ and hence $\alpha = 1 - 3rK > 0$.

Let $\epsilon_n \downarrow 0$ and $\bar{z}_n \in \Pi(x, \epsilon_n)$. It then follows from (2.11) that for $m \geq n$,

$$\|\bar{z}_n - \bar{z}_m\| \leq 2\alpha^{-1}\epsilon_n;$$

hence $\{\bar{z}_n\}$ is a Cauchy sequence. Consequently, $\{\bar{z}_n\}$ converges to a point $z^* \in S$. It also follows from (2.7) that $\|x - z^*\| = d_S(x)$ and hence $z^* = P_S(x)$. Uniqueness of $P_S(x)$ follows now from (2.11) by taking $\epsilon = 0$.

Finally, consider two points x_1, x_2 sufficiently close to x_0 such that $z_1 = P_S(x_1)$ and $z_2 = P_S(x_2)$ do exist. We have that $x_i - z_i \in N_S(z_i)$, $i = 1, 2$, and hence by (2.6),

$$-(x_1 - z_1 - x_2 + z_2, z_1 - z_2) \leq 2K\|z_1 - z_2\|^2(\|x_1 - z_1\| + \|x_2 - z_2\|).$$

Consequently, by taking x_1 and x_2 sufficiently close to x_0 such that

$$\beta = 2K(\|x_1 - z_1\| + \|x_2 - z_2\|) < 1,$$

we obtain

$$\|z_1 - z_2\|^2 - \|x_1 - x_2\|\|z_1 - z_2\| \leq \beta\|z_1 - z_2\|^2$$

and hence

$$\|z_1 - z_2\| \leq (1-\beta)^{-1}\|x_1 - x_2\|.$$

This completes the proof. □

In situations where points x are not necessarily close to S we shall have to impose stronger conditions on "near convexity" of S.

THEOREM 2.3. *For a given point $\bar{x}$, let $x_0 \in \Pi_S(\bar{x})$ and suppose that*

(i) *S is $o(2)$-convex at x_0, and*

(ii) *There is a positive constant α such that*

$$\|\bar{x} - x\|^2 \geq \|\bar{x} - x_0\|^2 + \alpha\|x - x_0\|^2 \tag{2.12}$$

for all $x \in S$.

Then $P_S(x)$ exists, is unique, and is locally Lipschitz continuous for all x in a neighborhood of $\bar{x}$.

Proof. Consider a point x, a positive number ϵ, and let $z^* \in S$ be such that

$$\|x - z^*\|^2 \leq d_S(x)^2 + \epsilon^2.$$

That is, z^* is an ϵ^2-optimal solution of the problem of minimization of $\|x-z\|^2$ subject to $z \in S$. Note that condition (2.12) corresponds to the second-order growth condition (1.7) for $f(z) = \|\bar{x} - z\|^2$. It then follows from (1.8) that

$$\|z^* - x_0\| \leq \alpha^{-1}\kappa + \alpha^{-1/2}\epsilon,$$

where $\kappa = 2\|x - \bar{x}\|$. Furthermore, by Ekeland's variational principle we can then find a point $\bar{z} \in S$ such that $\|z^* - \bar{z}\| \leq \epsilon$, that inequality (2.7) holds, and that $\bar{z}$ is the minimizer of the corresponding function given in (2.8). Proceeding now as in the proof of Theorem 2.2 and using (2.4) (instead of (2.6)), we obtain vectors v_1, v_2, such that $\|v_1\| \leq \epsilon$, $\|v_2\| \leq \epsilon$, and

$$\|\bar{z}_1 - \bar{z}_2\|^2 - \|v_1 + v_2\|\|\bar{z}_1 - \bar{z}_2\| \leq \{k(\bar{z}_1, \bar{z}_2)\|y_1\| + k(\bar{z}_2, \bar{z}_1)\|y_2\|\}\|\bar{z}_1 - \bar{z}_2\|^2,$$

where $y_i = x - \bar{z}_i + v_i$. Now for x sufficiently close to $\bar{x}$ and ϵ sufficiently small we can choose $\bar{z}_1, \bar{z}_2$ sufficiently close to x_0 such that

$$\beta = k(\bar{z}_1, \bar{z}_2)\|y_1\| + k(\bar{z}_2, \bar{z}_1)\|y_2\| < 1.$$

Then

$$\|\bar{z}_1 - \bar{z}_2\| \leq 2(1-\beta)^{-1}\epsilon,$$

and we can complete the proof proceeding as in the proof of Theorem 2.2. □

Consider now a situation where the set S is given in the form

$$S = \{x \in X : g(x) \in K\}, \tag{2.13}$$

with $g(x)$ being a continuously differentiable mapping from X into a Banach space Y and K being a closed convex cone in Y. Consider a point $x_0 \in S$ regular in the sense of Robinson [10], that is,

$$0 \in \text{ int } \{g(x_0) + Dg(x_0)X - K\}. \tag{2.14}$$

It then follows from the Robinson–Ursescu [11], [17] stability theorem that there is a neighborhood N of x_0 and a constant $c > 0$ such that for all $x, y \in S \cap N$,

$$\operatorname{dist}(y - x, T_S(x)) \leq c\|g(y) - g(x) - Dg(x)(y - x)\| \tag{2.15}$$

(see the proof of Theorem 2 in [16]). By the Mean Value Theorem we obtain from (2.15) that if, in addition, $Dg(x)$ is Lipschitz continuous near x_0 (in the operator norm), then S is $O(2)$-convex at x_0. Moreover, if $g(x)$ is twice continuously differentiable and $D^2_{xx}g(x_0) = 0$, then $o(2)$-convexity follows.

Note that here the contingent cone $T_S(x)$ is given (for all points x sufficiently close to x_0) by

$$T_S(x) = \{y : Dg(x)y \in T_K(g(x))\} \tag{2.16}$$

and hence is *convex*.

3. Directional differentiability of metric projections at boundary points. It is known that if the set S is convex and $x_0 \in S$, then for any $y \in X$ and $t \geq 0$,

$$P_S(x_0 + ty) = x_0 + tP_{T(x_0,S)}(y) + o(t) \tag{3.1}$$

(see [18]). That is, P_S is directionally differentiable at x_0 (in the sense of Gâteaux) and

$$P'_S(x_0; y) = P_{T(x_0,S)}(y). \tag{3.2}$$

Moreover, since for any $x_1, x_2 \in X$,

$$\|P_S(x_1) - P_S(x_2)\| \leq \|x_1 - x_2\|,$$

i.e., P_S is Lipschitz continuous modulus one, this implies that P_S is directionally differentiable at x_0 in the Hadamard sense. We extend this result to nonconvex sets as follows.

THEOREM 3.1. *Suppose that $x_0 \in S$ and that*

(i) *The contingent cone $T_S(x_0)$ is convex*

(ii)

$$\lim_{\substack{x \to x_0 \\ x \in S}} \frac{\operatorname{dist}(x - x_0, T_S(x_0))}{\|x - x_0\|} = 0, \tag{3.3}$$

(iii) *For every $v \in T_S(x_0)$,*

$$\lim_{t \to 0^+} \frac{\operatorname{dist}(x_0 + vt, S)}{t} = 0. \tag{3.4}$$

Then for any fixed $y \in X$ and for $P_S(x_0 + ty) \in \Pi_S(x_0 + ty)$ with $t \geq 0$,

$$P_S(x_0 + ty) = x_0 + tP_{T(x_0,S)}(y) + o(t). \tag{3.5}$$

Proof. Without loss of generality we can assume that $x_0 = 0$. For a given vector y and $t > 0$ consider the optimization problems

$$\min_{x \in T(0,S)} f(x, ty) \tag{3.6}$$

and

$$\min_{x \in S} f(x, ty), \tag{3.7}$$

where $f(x_1, x_2) = \|x_1 - x_2\|^2$. Note that the optimal solution of (3.6) is $x^* = P_{T(0,S)}(ty)$, which always exists since $T_S(0)$ is convex (and closed). An optimal solution of (3.7) (if it exists) is $\bar{x} = P_S(ty)$. We have here that for any $x \in T_S(0)$,

$$f(x, ty) \geq f(x^*, ty) + \|x - x^*\|^2, \tag{3.8}$$

i.e., the second-order growth condition for (3.6) holds. Indeed,

$$\|x - ty\|^2 - \|x^* - ty\|^2 = 2(ty - x^*, x^* - x) + \|x^* - x\|^2$$

and, since $ty - x^* \in N_{T(0,S)}(x^*)$ and $T_S(0)$ is convex,

$$(ty - x^*, x^* - x) \geq 0.$$

It follows then from (1.8) that

$$\|\bar{x} - x^*\| \leq 2\delta_1 + (k\delta_1 + k\delta_2)^{1/2}, \tag{3.9}$$

where

$$\delta_1 = \delta_1(t) = \sup\{\operatorname{dist}(x, T_S(0)) : x \in S \cap B(0, r)\},$$
$$\delta_2 = \delta_2(t) = \operatorname{dist}(x^*, S),$$

$r = 2t\|y\|$, and k is the Lipschitz constant of $f(\cdot, ty)$ on the ball $B(0, r)$. Note that $x^*, \bar{x} \in B(0, r)$ and that

$$k \leq 2(t\|y\| + r) = O(t).$$

Now it follows from condition (ii) that $\delta_1(t) = o(r) = o(t)$ and from (iii), for $v = P_{T(0,S)}(y) \in T_S(0)$, that

$$\delta_2(t) = \operatorname{dist}(tv, S) = o(t). \tag{3.10}$$

It follows then from (3.9) that

$$\|P_S(ty) - P_{T(0,S)}(ty)\| = \|\bar{x} - x^*\| = o(t),$$

and hence the proof is complete. □

Remarks. If $P_S(x)$ exists for all x in a neighborhood of x_0 (for example, if S is $O(2)$-convex at x_0), then formula (3.5) implies that P_S is directionally differentiable at x_0 and the directional derivative $P'_S(x_0, y)$ is given by $P_{T(x_0,S)}(y)$. Moreover, since $T(x_0, S)$ was supposed to be convex, we have that for any y and y',

$$\|P_{T(x_0,S)}(y') - P_{T(x_0,S)}(y)\| \leq \|y' - y\|.$$

Then, by replacing $v = P_{T(x_0,S)}(y)$ with $v' = P_{T(x_0,S)}(y')$ in (3.10) and taking $y' \to y$, we obtain that assumptions (i)–(iii) of Theorem 3.1 actually imply Hadamard directional differentiability of P_S at x_0.

If the set S is defined in the form (2.13) and regularity condition (2.14) holds, then conditions (i)–(iii) of Theorem 3.1 follow by the Robinson–Ursescu stability theorem. In this case we obtain that $P_S(x)$ exists for all x near $x_0 \in S$ and is Hadamard directionally differentiable at x_0.

In order to ensure directional differentiability of P_S, at $x_0 \in S$, in the sense of Fréchet we will have to replace condition (iii) by

$$\lim_{\substack{y \to 0 \\ y \in T(x_0, S)}} \frac{\operatorname{dist}(x_0 + y, S)}{\|y\|} = 0. \tag{3.11}$$

This condition is considerably stronger than the corresponding condition (3.4) and, in infinite-dimensional spaces, does not necessarily hold even for convex sets S. (In finite-dimensional spaces conditions (3.4) and (3.11) are equivalent.) Cones $T_S(x_0)$ satisfying (3.3) and (3.11) were called the *approximating cones* in [12].

4. Directional differentiability of metric projections at nonboundary points. In cases when $\bar{x}$ is not contained in S, the metric projection P_S can be directionally nondifferentiable at $\bar{x}$ even if the set S is convex. An example of such a convex set S in $\Re^3$ is given in Kruskal [8]. We show that directional differentiability of P_S follows if the set S is sufficiently "flat" at $x_0 = P_S(\bar{x})$. With a point $\bar{x}$ and $x_0 = P_S(\bar{x})$ we associate the cone

$$C(\bar{x}) = \{v \in T_S(x_0) : (v, \bar{x} - x_0) = 0\}. \tag{4.1}$$

THEOREM 4.1. *For a given point $\bar{x}$, let $x_0 \in \Pi_S(\bar{x})$ and suppose that the following conditions hold:*

(i) *The contingent cone $T_S(x_0)$ is convex.*

(ii) *There is a positive constant α such that*

$$\|\bar{x} - x\|^2 \geq \|\bar{x} - x_0\|^2 + \alpha\|x - x_0\|^2 \tag{4.2}$$

for all $x \in S$.

(iii)

$$\lim_{\substack{x \to x_0 \\ x \in S}} \frac{\operatorname{dist}(x - x_0, T_S(x_0))}{\|x - x_0\|^2} = 0. \tag{4.3}$$

(iv) *For any curve $v(t) \in T_S(x_0)$, $t \geq 0$, such that $v(t) = t\bar{v} + o(t)$ and $\bar{v} \in C(\bar{x})$,*

$$\lim_{t \to 0+} \frac{\operatorname{dist}(x_0 + v(t), S)}{t^2} = 0. \tag{4.4}$$

Then for any fixed $y \in X$ and for $P_S(\bar{x} + ty) \in \Pi_S(\bar{x} + ty)$ with $t \geq 0$,

$$P_S(\bar{x} + ty) = x_0 + tP_{C(\bar{x})}(y) + o(t). \tag{4.5}$$

In order to prove Theorem 4.1 we shall use the results of the following lemmas, which may also be of independent interest.

LEMMA 4.2. *Let Σ be a closed convex set such that $P_\Sigma(\bar{x}) = x_0$ and*

$$\lim_{\substack{x \to x_0 \\ x \in S}} \frac{\operatorname{dist}(x, \Sigma)}{\|x - x_0\|^2} = 0, \tag{4.6}$$

(4.7) $$\lim_{t\to 0+} \frac{\operatorname{dist}(P_\Sigma(\bar{x}+ty), S)}{t^2} = 0,$$

and suppose that condition (ii) *of Theorem* 4.1 *holds.*
Then

(4.8) $$P_S(\bar{x}+ty) = P_\Sigma(\bar{x}+ty) + o(t), \qquad t \geq 0.$$

Proof. Let us first observe that, by (1.8), it follows from condition (4.2) that

(4.9) $$\|P_S(x) - P_S(\bar{x})\| \leq 2\alpha^{-1}\|x - \bar{x}\|.$$

Consider the following optimization problems:

(4.10) $$\min_{x\in\Sigma} f(x, \bar{x}+ty)$$

and

(4.11) $$\min_{x\in S} f(x, \bar{x}+ty),$$

where $f(x_1, x_2) = \|x_1 - x_2\|^2$. Let $x^* = P_\Sigma(\bar{x}+ty)$ and $x' = P_S(\bar{x}+ty)$, i.e., x^* and x' are optimal solutions of problems (4.10) and (4.11), respectively. Note that it follows from (4.9) that $\|x' - x_0\| \leq \alpha^{-1}t\|y\|$, and because of the convexity of Σ, $\|x^* - x_0\| \leq t\|y\|$. Therefore $x', x^* \in B(x_0, r)$, where $r = \max\{2\alpha^{-1}t\|y\|, t\|y\|\}$.

Since Σ is convex we have that

(4.12) $$f(x, \bar{x}+ty) \geq f(x^*, \bar{x}+ty) + \|x - x^*\|^2$$

for all $x \in \Sigma$. It follows then by (1.8) that

(4.13) $$\|x' - x^*\| \leq 2\delta_1 + (k\delta_1 + k\delta_2)^{1/2},$$

where

$$\delta_1 = \delta_1(t) = \sup\{\operatorname{dist}(x, \Sigma) : x \in S \cap B(x_0, r)\},$$
$$\delta_2 = \delta_2(t) = \operatorname{dist}(x^*, S),$$

and k is the Lipschitz constant of $f(\cdot\,, \bar{x}+ty)$ on the ball $B(x_0, r)$. Moreover, we have that

$$k \leq 2(r + \|x_0 - \bar{x}\| + t\|y\|)$$

and that $r = O(t)$. It follows from (4.6) that $\delta_1 = o(r^2)$ and hence $\delta_1 = o(t^2)$, and from (4.7) that $\delta_2 = o(t^2)$. We obtain then from (4.13) that

$$\|x' - x^*\| = o(t)$$

and hence the proof is complete. □

LEMMA 4.3. *Let K be a convex, closed cone, $K^- = N_K(0)$ be its negative dual, $v \in K^-$, and consider the cone*

$$K_0 = \{x \in K : (x, v) = 0\}.$$

Then for a given $y \in X$ and $t \geq 0$,

$$P_K(v + ty) = tP_{K_0}(y) + o(t). \tag{4.14}$$

Proof. We can write (see, e.g., [18])

$$P_K(v + ty) = v + ty - P_{K^-}(v + ty). \tag{4.15}$$

Also since K^- is convex and $v \in K^-$, we have that P_{K^-} is directionally differentiable at v and

$$P_{K^-}(v + ty) = v + tP_{T(v,K^-)}(y) + o(t). \tag{4.16}$$

Note that the contingent cone $T(v, K^-)$ is given by the topological closure of $K^- + [v]$, where $[v]$ denotes the linear space generated by v, and hence the polar (negative dual) of $T(v, K^-)$ is the cone K_0. Therefore,

$$P_{T(v,K^-)}(y) = y - P_{K_0}(y). \tag{4.17}$$

Equations (4.15)–(4.17) imply (4.14).

Proof of Theorem 4.1. Consider $\Sigma = x_0 + T_S(x_0)$. Since $\bar{x} - x_0 \in N_S(x_0)$ we have that $x_0 = P_\Sigma(\bar{x})$. Also, by Lemma 4.3,

$$P_\Sigma(\bar{x} + ty) = x_0 + tP_{C(\bar{x})}(y) + o(t). \tag{4.18}$$

In order to complete the proof we can now apply the result (4.8) of Lemma 4.2. Note that here condition (4.6) follows from (4.3) and condition (4.7) follows from (4.4) because of (4.18). □

If the set S is convex, conditions (i)–(iii) of Theorem 4.1 are automatically satisfied. In this case, condition (iv) implies that P_S is directionally differentiable at x_0, in the sense of Hadamard, and that

$$P'_S(\bar{x}, y) = P_{C(\bar{x})}(y). \tag{4.19}$$

It is known that in general differentiability properties of P_S are influenced by second-order behavior of the boundary of S (cf. [1], [5], [7], and [13]). In this respect condition (iv) ensures that S is sufficiently "directionally flat" at the point $x_0 = P_S(\bar{x})$ and consequently only first-order tangential properties of S at x_0 are reflected in formula (4.19).

Condition (iv) follows from and, in finite-dimensional spaces with $\bar{v} \in C(\bar{x})$ relaxed to $\bar{v} \in T_S(x_0)$, is equivalent to

$$\lim_{\substack{x \to x_0 \\ x \in \Sigma}} \frac{\operatorname{dist}(x, S)}{\|x - x_0\|^2} = 0, \tag{4.20}$$

where $\Sigma = x_0 + T_S(x_0)$. Sets S and Σ satisfying (4.6) and (4.20) were called 2-tangent at x_0, in Auslender and Cominetti [3].

Finally we compare Theorem 4.1 with corresponding results of Haraux [6]. It was shown in [6] that P_S is directionally differentiable and formula (4.19) holds for a class of convex sets S that were called *polyhedric*. In finite-dimensional spaces polyhedric convex sets in the sense of Haraux coincide with ordinary polyhedral convex sets. Clearly a set S does not need to be polyhedral in order to satisfy (4.20). We also give now an example of a convex, polyhedric set S that does not satisfy condition (iv). Therefore, the results of Theorem 4.1 and those of Haraux do not imply each other.

Consider the Hilbert space $X = L^2[-1,1]$. Then the set

$$S = \{x(t) \in X : x(t) \geq 0 \text{ for all } t \in [-1,1]\}$$

forms a convex, polyhedric set in X. Consider also the following points (functions) in X, $\bar{x}(t) = -1$ for $-1 \leq t \leq 0$, $\bar{x}(t) = 1$ for $0 < t \leq 1$, and $\bar{v}(t) = 0$ for $-1 \leq t \leq 0$, $\bar{v}(t) = -t^{-1/3}$ for $0 < t \leq 1$. It is then not difficult to see that the projection $x_0 = P_S(\bar{x})$ is given by $x_0(t) = 0$ for $-1 \leq t \leq 0$ and $x_0(t) = 1$ for $0 < t \leq 1$, and that

$$T_S(x_0) = \{x(t) \in X : x(t) \geq 0 \text{ for all } t \in [-1,0]\}.$$

Therefore, $\bar{v} \in T_S(x_0)$ and $(\bar{v}, \bar{x} - x_0) = 0$, i.e., $\bar{v} \in C(\bar{x})$. Moreover, for $0 < \tau \leq 1$,

$$P_S(x_0 + \tau\bar{v})(t) = \begin{cases} 0, & -1 \leq t \leq \tau^3, \\ 1 - \tau t^{-1/3}, & \tau^3 \leq t \leq 1, \end{cases}$$

and

$$\operatorname{dist}(x_0 + \tau\bar{v}, S), = \left[\int_0^{\tau^3} (1 - \tau t^{-1/3})^2 dt\right]^{1/2} = \tau^{3/2}.$$

Consequently, condition (4.4) does not hold here.

Haraux's result is an extension of the following construction. Let $X = X_1 \times X_2$ be the product of two Hilbert spaces X_1 and X_2, and let $S = S_1 \times S_2$, with S_1 being a closed convex subset of X_1 and S_2 being a closed convex cone in X_2. Then it follows from Lemma 4.3 that P_S is directionally differentiable at any point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ such that $\bar{x}_1 \in S_1$ and $\bar{x}_2 \in S_2^-$. Clearly, condition (4.4) does not need to hold here at the corresponding point $x_0 = (\bar{x}_1, 0)$.

REFERENCES

[1] T. J. ABATZOGLOU, *The minimum norm projection on C^2 manifolds in R^n*, Trans. Amer. Math. Soc., 243 (1978), pp. 115-122.

[2] J. P. AUBIN AND I. EKELAND, *Applied Nonlinear Analysis*, Wiley, New York, 1984.

[3] A. AUSLENDER AND R. COMINETTI, *A comparative study of multifunction differentiability with applications in mathematical programming*, Math. Oper. Res., 16 (1991), pp. 240–258.

[4] V. I. AVERBUKH AND O. G. SMOLYANOV, *The various definitions of the derivative in linear topological spaces*, Russian Mathematical Surveys, 23 (1968), pp. 67–113.

[5] S. FITZPATRICK AND R. R. PHELPS, *Differentiability of the metric projection in Hilbert space*, Trans. Amer. Math. Soc., 270 (1982), pp. 483–501.

[6] A. HARAUX, *How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities*, J. Math. Soc. Japan, 29 (1977), pp. 615–631.

[7] R. B. HOLMES, *Smoothness of certain metric projections on Hilbert space*, Trans. Amer. Math. Soc., 184 (1973), pp. 87–100.

[8] J. KRUSKAL, *Two convex counterexamples: A discontinuous envelope function and a non-differentiable nearest point mapping*, Proc. Amer. Math. Soc., 23 (1969), pp. 697–703.

[9] M. Z. NASHED, *Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials in nonlinear functional analysis*, Nonlinear Functional Analysis and Applications, L. B. Rall, ed., Academic Press, New York, 1971, pp. 103–309.

[10] S. M. ROBINSON, *Stability theorems for systems of inequalities, Part* II: *Differentiable nonlinear systems*, SIAM J. Numer. Anal., 13 (1976), pp. 497–513.

[11] ———, *Regularity and stability for convex multivalued functions*, Math. Oper. Res., 1 (1976), pp. 130–143.

[12] A. SHAPIRO, *On differentiability of metric projections in R^n, 1: Boundary case*, Proc. Amer. Math. Soc., 99 (1987), pp. 123–128.

[13] ———, *Sensitivity analysis of nonlinear programs and differentiability properties of metric projections*, SIAM J. Control and Optimization, 26 (1988), pp. 628–645.

[14] ———, *On concepts of directional differentiability*, J. Optim. Theory Appl., 66 (1990), pp. 477–487.

[15] ———, *Perturbation analysis of optimization problems in Banach spaces*, Numer. Funct. Anal. Optim., 13 (1992), pp. 97–116.

[16] A. SHAPIRO AND F. AL-KHAYYAL, *First order conditions for isolated locally optimal solutions*, J. Optim. Theory Appl., 77 (1993), pp. 189–196.

[17] C. URSESCU, *Multifunctions with convex closed graph*, Czechoslovak Math. J., 25 (1975), pp. 438–441.

[18] E. H. ZARANTONELLO, *Projections on convex sets in Hilbert space and spectral theory*, in contributions to *Nonlinear Functional Analysis*, Academic Press, New York, 1971, pp. 237–424.