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A. Shapiro
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Dear Prof Shapiro,

The statement of your conjecture, appeared in [1], is a known result of integral geometry. It is a simple consequence of the theorem of Gauss-Bonnet [2] (see [3] for an elementary proof). Moreover the quantities w_i satisfy further relations (see [2] pp. 310 or [4] pp. 184):

$$\int \chi(C \bigcap L_i) dL_i = 2 \sum_{m \geq 0} w_{n+1-i+2m}(C) \qquad (1)$$

for $i = 1, \ldots, n$ and convex cones C; χ is the Euler characteristic, and the integral is over the Grassmanian manifolds of all i-flats L_i through 0, with the invariant measure normalised to total measure 1. In particular, your conjecture can be derived from these relations. If we apply (1) for $i = n$ then we have

$$1 = \int \chi(C \bigcap L_n) = 2 \sum_{i \text{ odd}} w_i$$

and subtract this from the obviously true equation

$$1 = \sum_{i=0}^{n} w_i$$

then we get the relation in question.

[1] A. Shapiro, A conjecture related to chi-bar-squred distribution, *Amer. Math. Monthly*, (94) 1(1987), p. 46.
[2] L. A. Santaló, *Integral geometry and geometric probabilitiy*, Addison-Wesley Publ. Comp., Reading, Mass., 1976.
[3] P. Mcmullen, Non-linear angle-sum relations for polyhedral cones and polytopes, *Math. Proc. Camb. Phil. Soc.*, 78(1975), p. 247.
[4] *Convexity and its Applications*, ed. by M. Gruber and M. Wills, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983, pp.184.

Sincerely,

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Professor Alexander Shapiro
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Dear Professor Shapiro:

I am writing concerning your recent article in the Unsolved Problems section of the American Mathematical Monthly.

Any rotationally invariant, continuous probability distribution on $\mathbb{R}^d$ induces a rotationally invariant measure μ on the unit sphere S^{d-1} by the rule $\mu(A)$ = (probability that $x/\|x\| \in A$). The only such measure μ, having $\mu(S^{d-1}) = 1$, is the standard surface measure, suitably normalized. We see from this that if D is any (convex) cone then the probability that x is in D is the solid angle of D, $\mu(S^{d-1} \cap D)$.

It is easy to see that if, now, $C \subseteq \mathbb{R}^d$ is a closed, polyhedral cone, F is a face of C, and $D_F = \{x : p(x) \in F\}$ (where $p : \mathbb{R}^d \to C$ is the nearest point mapping), then the solid angle of D_F is $\beta(0, F)\gamma(F, C)$, where $\beta(0, F)$ represents the internal angle of F at the origin 0, and $\gamma(F, C)$ represents the external angle of C at F. Comparing (3.31) of

(*) P. McMullen and R. Schneider, "Valuations on Convex Bodies", in _Convexity and its Applications_, ed. M. Gruber and J. Wills, Birkhauser, 1983, 170-247,

we see that your w_k is $\varphi_k(C)$ of (*). This gives the geometrical interpretation for the w_k's that you desired. Your conjecture is Theorem 8.9(b) of (*).

I would like to describe a direct proof of your conjecture which was noticed by Jon Spingarn and myself in another connection. Let $q(x) = 2p(x) - x$, for $x \in \mathbb{R}^d$. It is easily seen that $q|D_F$ is a linear isometry for each face F of C. This isometry is orientation-preserving precisely for faces F of even codimension in $\mathbb{R}^d$. Also, it is easy to see that the restriction to the sphere $q_0 = q|S^{d-1}$ has as its image, $q_0(S^{d-1})$, a proper subset of S^{d-1} (assuming, as we do, that C is not a linear subspace). Therefore its topological degree is 0. We can compute the degree of q_0 by choosing any point $y \in S^{d-1}$ not on the boundary of any of the images $q_0(D_F \cap S^{d-1})$, listing the inverse images $x_1, \ldots, x_n$ of y under q_0, and computing the number of those at which q_0 (or q) preserves orientation minus the number at which it reverses orientation. That is, we evaluate the function

$$F = \sum (-1)^{d-\dim F} \chi_{q(D_F)}$$

at y. (Here, $\chi_{q(D_F)}$ is the characteristic function of the set $q(D_F)$.) Since the degree is 0, we find that $F(y) = 0$ (except, possibly, for y in the boundary of one of the sets $q(D_F)$ -- a set of measure 0.) We now see that

$$0 = \int_{S^{d-1}} F \, d\mu = (w_d - w_{d-1} + \ldots + (-1)^d w_0),$$

since clearly

$$w_k = \int_{S^{d-1}} \Big(\sum_{\dim F = k} \chi_{q(D_F)} \Big) \, d\mu.$$

It is nice to see from your paper that this result has a use!

Sincerely,

Jim Lawrence

Jim Lawrence

cc: R. K. Guy;
J. Spingarn