A. Shapiro  
Department of Mathematics and Applied Mathematics  
University of South Africa  
Pretoria  
0001 South Africa

Dear Prof Shapiro,

The statement of your conjecture, appeared in [1], is a known result of integral geometry. It is a simple consequence of the theorem of Gauss-Bonnet [2] (see [3] for an elementary proof). Moreover the quantities \( w_i \) satisfy further relations (see [2] pp. 310 or [4] pp. 184):

\[
\int x(C \cap L_i) dL_i = 2 \sum_{m \geq 0} w_{n+1-i+2m}(C)
\]

for \( i = 1, \ldots, n \) and convex cones \( C \). \( x \) is the Euler characteristic, and the integral is over the Grassmanian manifolds of all \( i \)-flats \( L_i \) through \( 0 \), with the invariant measure normalised to total measure 1. In particular, your conjecture can be derived from these relations. If we apply (1) for \( i = n \) then we have

\[
1 = \int x(C \cap L_n) = 2 \sum_{i \text{ odd}} w_i
\]

and subtract this from the obviously true equation

\[
1 = \sum_{i=0}^{n} w_i
\]

then we get the relation in question.


Sincerely,

J. Kincses
February 1, 1987

Professor Alexander Shapiro  
Department of Mathematics and Applied Mathematics  
University of South Africa  
Pretoria, 0001  
South Africa

Dear Professor Shapiro:

I am writing concerning your recent article in the Unsolved Problems section of the American Mathematical Monthly.

Any rotationally invariant, continuous probability distribution on $\mathbb{R}^d$ induces a rotationally invariant measure $\mu$ on the unit sphere $S^{d-1}$ by the rule $\mu(A) = \text{(probability that } x/\|x\| \in A)$. The only such measure $\mu$, having $\mu(S^{d-1}) = 1$, is the standard surface measure, suitably normalized. We see from this that if $D$ is any (convex) cone then the probability that $x$ is in $D$ is the solid angle of $D$, $\mu(S^{d-1} \cap D)$. 
It is easy to see that if, now, $C \subseteq \mathbb{R}^d$ is a closed, polyhedral cone, $F$ is a face of $C$, and $D_F = \{x : p(x) \in F\}$ (where $p : \mathbb{R}^d \to C$ is the nearest point mapping), then the solid angle of $D_F$ is $\beta(0, F) \gamma(F, C)$, where $\beta(0, F)$ represents the internal angle of $F$ at the origin $0$, and $\gamma(F, C)$ represents the external angle of $C$ at $F$. Comparing (3.31) of

(*) P. McMullen and R. Schneider, "Valuations on Convex Bodies", in Convexity and its Applications, ed. M. Gruber and J. Wills, Birkhauser, 1983, 170-247,

we see that your $w_k$ is $v_k(C)$ of (*). This gives the geometrical interpretation for the $w_k$'s that you desired. Your conjecture is Theorem 8.9(b) of (*).

I would like to describe a direct proof of your conjecture which was noticed by Jon Spingarn and myself in another connection. Let $q(x) = 2p(x) - x$, for $x \in \mathbb{R}^d$. It is easily seen that $q|D_F$ is a linear isometry for each face $F$ of $C$. This isometry is orientation-preserving precisely for faces $F$ of even codimension in $\mathbb{R}^d$. Also, it is easy to see that the restriction to the sphere $q_0 = q|S^{d-1}$ has as its image, $q_0(S^{d-1})$, a proper subset of $S^{d-1}$ (assuming, as we do, that $C$ is not a linear subspace). Therefore its topological degree is 0. We can compute the degree of $q_0$ by choosing any point $y \in S^{d-1}$ not on the boundary of any of the images $q_0(D_F \cap S^{d-1})$, listing the inverse images $x_1, \ldots, x_n$ of $y$ under $q_0$, and computing the number of those at which $q_0$ (or $q$) preserves orientation minus the number at which it reverses orientation. That is, we evaluate the function

$$F = \sum (-1)^{d-\dim F} \chi_{q(D_F)}$$
at y. (Here, \( x_{q(D_F)} \) is the characteristic function of
the set \( q(D_F) \).) Since the degree is 0, we find that
\( F(y) = 0 \) (except, possibly, for \( y \) in the boundary of
one of the sets \( q(D_F) \) -- a set of measure 0.) We now
see that
\[
0 = \int_{S^{d-1}} F \, d\mu = (w_d - w_{d-1} + \ldots + (-1)^d w_0),
\]
since clearly
\[
w_k = \int_{S^{d-1}} \left( \sum_{\dim F = k} x_{q(D_F)} \right) \, d\mu.
\]
It is nice to see from your paper that this
result has a use!

Sincerely,

Jim Lawrence

cc: R. K. Guy;
    J. Spingarn