

October 12, 1987

A. Shapiro
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Dear Prof Shapiro,

The statement of your conjecture, appeared in [1], is a known result of integral geometry. It is a simple consequence of the theorem of Gauss-Bonnet [2] (see [3] for an elementary proof). Moreover the quantities w_i satisfy further relations (see [2] pp. 310 or [4] pp. 184):

$$(1) \quad \int \chi(C \cap L_i) dL_i = 2 \sum_{m \geq 0} w_{n+1-i+2m}(C)$$

for $i = 1, \dots, n$ and convex cones C ; χ is the Euler characteristic, and the integral is over the Grassmanian manifolds of all i -flats L_i through 0, with the invariant measure normalised to total measure 1. In particular, your conjecture can be derived from these relations. If we apply (1) for $i = n$ then we have

$$1 = \int \chi(C \cap L_n) = 2 \sum_{i \text{ odd}} w_i$$

and subtract this from the obviously true equation

$$1 = \sum_{i=0}^n w_i$$

then we get the relation in question.

- [1] A. Shapiro, A conjecture related to chi-bar-squared distribution, *Amer. Math. Monthly*, (94) 1(1987), p. 46.
- [2] L. A. Santaló, *Integral geometry and geometric probability*, Addison-Wesley Publ. Comp., Reading, Mass., 1976.
- [3] P. McMullen, Non-linear angle-sum relations for polyhedral cones and polytopes, *Math. Proc. Camb. Phil. Soc.*, 78(1975), p. 247.
- [4] *Convexity and its Applications*, ed. by M. Gruber and M. Wills, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983, pp.184.

Sincerely,

J. Kincses

J. Kincses

George Mason University

February 1, 1987

Professor Alexander Shapiro
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Dear Professor Shapiro:

I am writing concerning your recent article in the Unsolved Problems section of the American Mathematical Monthly.

Any rotationally invariant, continuous probability distribution on \mathbb{R}^d induces a rotationally invariant measure μ on the unit sphere S^{d-1} by the rule $\mu(A) = (\text{probability that } x/\|x\| \in A)$. The only such measure μ , having $\mu(S^{d-1}) = 1$, is the standard surface measure, suitably normalized. We see from this that if D is any (convex) cone then the probability that x is in D is the solid angle of D , $\mu(S^{d-1} \cap D)$.

It is easy to see that if, now, $C \subseteq \mathbb{R}^d$ is a closed, polyhedral cone, F is a face of C , and $D_F = \{x : p(x) \in F\}$ (where $p : \mathbb{R}^d \rightarrow C$ is the nearest point mapping), then the solid angle of D_F is $\beta(0, F)\gamma(F, C)$, where $\beta(0, F)$ represents the internal angle of F at the origin 0 , and $\gamma(F, C)$ represents the external angle of C at F . Comparing (3.31) of

(*) P. McMullen and R. Schneider, "Valuations on Convex Bodies", in Convexity and its Applications, ed. M. Gruber and J. Wills, Birkhauser, 1983, 170-247,

we see that your w_k is $\varphi_k(C)$ of (*). This gives the geometrical interpretation for the w_k 's that you desired. Your conjecture is Theorem 8.9(b) of (*).

I would like to describe a direct proof of your conjecture which was noticed by Jon Spingarn and myself in another connection. Let $q(x) = 2p(x) - x$, for $x \in \mathbb{R}^d$. It is easily seen that $q|_{D_F}$ is a linear isometry for each face F of C . This isometry is orientation-preserving precisely for faces F of even codimension in \mathbb{R}^d . Also, it is easy to see that the restriction to the sphere $q_0 = q|_{S^{d-1}}$ has as its image, $q_0(S^{d-1})$, a proper subset of S^{d-1} (assuming, as we do, that C is not a linear subspace). Therefore its topological degree is 0. We can compute the degree of q_0 by choosing any point $y \in S^{d-1}$ not on the boundary of any of the images $q_0(D_F \cap S^{d-1})$, listing the inverse images x_1, \dots, x_n of y under q_0 , and computing the number of those at which q_0 (or q) preserves orientation minus the number at which it reverses orientation. That is, we evaluate the function

$$F = \sum (-1)^{d-\dim F} \chi_{q(D_F)}$$

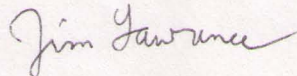
at y . (Here, $\chi_{q(D_F)}$ is the characteristic function of the set $q(D_F)$.) Since the degree is 0, we find that $F(y) = 0$ (except, possibly, for y in the boundary of one of the sets $q(D_F)$ -- a set of measure 0.) We now see that

$$0 = \int_{S^{d-1}} F \, d\mu = (w_d - w_{d-1} + \dots + (-1)^d w_0),$$
 since clearly

$$w_k = \int_{S^{d-1}} \left(\sum_{\dim F = k} \chi_{q(D_F)} \right) d\mu.$$

It is nice to see from your paper that this result has a use!

Sincerely,



Jim Lawrence

cc: R. K. Guy;
J. Spingarn