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Second-Order Optimality Conditions in Generalized Semi-Infinite Programming ⋆

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Abstract. This paper deals with generalized semi-infinite optimization problems where the (infinite) index set of inequality constraints depends on the state variables and all involved functions are twice continuously differentiable. Necessary and sufficient second-order optimality conditions for such problems are derived under assumptions which imply that the corresponding optimal value function is second-order (parabolically) directionally differentiable and second-order epiregular at the considered point. These sufficient conditions are, in particular, equivalent to the second-order growth condition.

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1. Introduction

In the present paper we consider a *generalized semi-infinite optimization problem*, given in the form

$$\text{Minimize } f(x) \text{ subject to } x \in S, \tag{1.1}$$

where the feasible set S is defined as

$$S := \{x \in \mathbb{R}^n \mid g(x, y) \leqslant 0,\ y \in Y(x)\}$$

and the index set of inequality constraints of (1.1) is given as

$$Y(x) := \{y \in \mathbb{R}^k \mid h_i(x, y) = 0,\ i = 1, \dots, p,\\ h_j(x, y) \leqslant 0,\ j = p + 1, \dots, q\}.$$

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The principal difference to standard semi-infinite optimization problems is that in the standard case the set $Y(x)$ does *not* depend on x, i.e., $Y(x) = Y$ for all $x \in \mathbb{R}^n$.

The goal of this paper is to derive necessary and sufficient second-order optimality conditions for the problem (1.1). These conditions can be extended straightforwardly to the case with additional equality constraints in the definition of S. Therefore, for the sake of simplicity we focus ourselves on the given case without equality constraints.

Clearly the feasible set can be written in the following equivalent form

$$S = \{x \in \mathbb{R}^n \mid v(x) \leqslant 0\},$$

where $v(x)$ is the *optimal value function*

$$v(x) := \sup_{y \in Y(x)} g(x, y). \tag{1.2}$$

Optimality conditions derived in this paper are based on differentiability properties of the optimal value function $v(x)$.

Recently generalized semi-infinite optimization became an active research topic in applied mathematics. This is due to the fact that several engineering tasks lead to problems of type (1.1), e.g., design problems, problems of maneuverability of robots, reverse Chebyshev approximation problems (cf., e.g., [7, 9, 15]).

This paper is organized as follows. In Section 2 we give a general discussion of second order optimality conditions for an unconstrained problem with a locally Lipschitz continuous objective function. These optimality conditions are based on concepts of second-order (parabolic) directional differentiability and second order epiregularity. In Sections 3 and 4 we apply these general results to our generalized semi-infinite problem (1.1). Section 3 contains a discussion of the employed assumptions and in Section 4 we present second order necessary and sufficient optimality conditions for the problem (1.1). Note that the presented results are valid without assuming any kind of reduction approach. Finally, two illustrating examples in Section 5 conclude the paper.

We refer to the related papers [12, 21, 26] on first order optimality conditions for the problem (1.1) as well as to [13, 14, 27]. In [10] second-order necessary and sufficient optimality conditions for (1.1) are discussed under the assumption that a so-called reduction approach is applicable at the considered point. Under this assumption, the original problem can be locally reformulated as a problem with *finitely many* constraints, and the corresponding optimality conditions are obtained from the corresponding optimality conditions for finitely constrained problems. In our investigation, and in particular in the considered examples, we do not assume applicability of such reduction approach.

For the sake of completeness, we also refer to the survey papers [8, 18] and the recent book [19], which contains several survey and tutorial papers reflecting the state-of-the-art in semi-infinite programming and its relations to some other topics.

2. General Discussion of Second-Order Optimality Conditions

In this section we give a general discussion of first and especially second-order optimality conditions. Consider the following (unconstrained) optimization problem

(P) Minimize $\psi(x)$ subject to $x \in \mathbb{R}^n$,

where ψ is a real valued function. Throughout this section we assume that ψ is locally Lipschitz continuous and *second-order* (*parabolically*) *directionally differentiable* at a considered point $x_0 \in \mathbb{R}^n$. That is, ψ is directionally differentiable at x_0 and, moreover, the limit

$$\psi''(x_0; d, w) := \lim_{t \downarrow 0} \frac{\psi(x_0 + td + \frac{1}{2}t^2 w) - \psi(x_0) - t\psi'(x_0, d)}{\frac{1}{2}t^2}$$

exists for any $d, w \in \mathbb{R}^n$. Note that directional differentiability together with local Lipschitz continuity of ψ imply that

$$\psi'(x_0, d) = \lim_{\substack{t \downarrow 0 \\ d' \to d}} \frac{\psi(x_0 + td') - \psi(x_0)}{t},$$

i.e., ψ is directionally differentiable at x_0 in the sense of Hadamard, e.g., [24].

It can be easily shown, and is well known, that the following are (first-order) necessary conditions for x_0 to be a locally optimal solution of the problem (P):

$$\psi'(x_0, d) \geqslant 0, \quad \forall d \in \mathbb{R}^n. \tag{2.1}$$

Consider the set

$$C(x_0) := \{d \in \mathbb{R}^n : \psi'(x_0, d) = 0\}. \tag{2.2}$$

Since the function $\psi'(x_0, \cdot)$ is positively homogeneous, the set $C(x_0)$ is a cone and, because of (2.1), $C(x_0)$ is convex if $\psi'(x_0, \cdot)$ is convex. The cone $C(x_0)$ is called the *cone of critical directions* or, simply, the *critical cone*. It represents those directions for which first-order analysis does not provide an information about optimality of x_0. Note that because of (2.1), the critical cone can be written in the following equivalent form $C(x_0) = \{d : \psi'(x_0, d) \leqslant 0\}$.

It is said that the *second-order growth condition* holds at x_0 if there exist a constant $c > 0$ and a neighborhood N of x_0 such that

$$\psi(x) \geqslant \psi(x_0) + c\|x - x_0\|^2, \quad \forall x \in N. \tag{2.3}$$

Clearly the above second-order growth condition implies local optimality of x_0. (Here $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$, e.g., the Euclidean norm.)

Since for any $d \in C(x_0)$ and $t \geqslant 0$ we have

$$\psi(x_0 + td + \tfrac{1}{2}t^2 w) = \psi(x_0) + \tfrac{1}{2}t^2 \psi''(x_0; d, w) + \mathrm{o}(t^2),$$

it is straightforward to verify the following second-order necessary conditions. If x_0 is a locally optimal solution of (P), then

$$\psi''(x_0; d, w) \geqslant 0, \quad \forall d \in C(x_0),\ \forall w \in \mathbb{R}^n. \tag{2.4}$$

Moreover, if the second-order growth condition (2.3) holds at x_0, then

$$\inf_{w\in\mathbb{R}^n} \psi''(x_0; d, w) > 0, \quad \forall d \in C(x_0) \setminus \{0\}. \tag{2.5}$$

Such conditions were already used in [1], and are discussed, in a more general framework, in [20] and [4], for example.

Second-order necessary conditions (2.4) are derived by verifying local optimality of x_0 along parabolic paths of the form $x(t) := x_0 + td + \frac{1}{2}t^2 w$, i.e., by verifying that $\psi(x(t)) \geqslant \psi(x_0)$ for sufficiently small $t > 0$. There is no reason, a priori, that (local) optimality of x_0 should be verified along such parabolic paths only, and in general the corresponding conditions (2.5) do not guarantee local optimality of x_0. Nevertheless, for a large class of problems conditions (2.5) imply the second-order growth condition, and hence are sufficient second order optimality conditions. The following type of regularity conditions was introduced in [2] and is extensively discussed in [4, Section 3.3.4].

DEFINITION 2.1. It is said that ψ is second order epiregular at x_0, if for any d and for $t \geqslant 0$ and any path $w(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}^n$ such that $tw(t) \to 0$ as $t \downarrow 0$, the following inequality holds

$$\begin{aligned}\psi\big(x_0 + td + \tfrac{1}{2}t^2 w(t)\big) \;\geqslant\; & \psi(x_0) + t\psi'(x_0, d) + \\ & + \tfrac{1}{2}t^2\psi''(x_0; d, w(t)) + \mathrm{o}(t^2)\end{aligned} \tag{2.6}$$

(where $\mathbb{R}_+$ denotes the set of nonnegative reals).

The following result is taken from [4, Section 3.3.5], since the proof is easy we only give it for the sake of completeness.

PROPOSITION 2.1. *Suppose that the first-order necessary conditions* (2.1) *hold and that ψ is second-order epiregular at x_0. Then the second-order growth condition holds at x_0 if and only if conditions* (2.5) *are satisfied.*

Proof. As we discussed above verification of necessity of conditions (2.5) is straightforward. In order to show sufficiency we argue by a contradiction. Suppose that conditions (2.5) are satisfied and that (2.3) is false. This means that there exists a sequence $x_\nu \to x_0$, $x_\nu \neq x_0$, such that

$$\psi(x_\nu) \leqslant \psi(x_0) + \mathrm{o}(\|x_\nu - x_0\|^2). \tag{2.7}$$

Consider $t_\nu := \|x_\nu - x_0\|$ and $d_\nu := (x_\nu - x_0)/t_\nu$. Using compactness arguments we can assume, by passing to a subsequence if necessary, that d_ν converge to a vector d. Since $\psi(x_\nu) = \psi(x_0) + t_\nu\psi'(x_0, d) + \mathrm{o}(t_\nu)$, it follows from (2.7) that

$\psi'(x_0, d) \leqslant 0$ and hence $d \in C(x_0)$. Define $w_\nu := (d_\nu - d)/(\frac{1}{2}t_\nu)$. We have then that $t_\nu w_\nu \to 0$, that $x_0 + t_\nu d_\nu = x_0 + t_\nu d + \frac{1}{2}t_\nu^2 w_\nu$ and, hence, by (2.6)

$$\psi(x_\nu) \geqslant \psi(x_0) + t_\nu \psi'(x_0, d) + \tfrac{1}{2}t_\nu^2 \psi''(x_0; d, w_\nu) + o(t_\nu^2).$$

Together with (2.5) this implies

$$\psi(x_\nu) \geqslant \psi(x_0) + t_\nu^2 \alpha + o(t_\nu^2),$$

for some constant $\alpha > 0$. This, however, contradicts (2.7). □

3. Assumptions

We return now to the generalized semi-infinite problem (1.1). In this section we present assumptions which are used in Section 4 in the discussion of second-order optimality conditions for (1.1). In the remainder of this paper we assume that $x_0 \in S$ is a considered feasible point of the problem (1.1) and that the following assumptions hold.

(A1) The functions $f(x)$, $g(x, y)$, $h_i(x, y)$, $i = 1, \ldots, q$ are real-valued and twice continuously differentiable.

By $\nabla g(x_0, y_0)$ and $\nabla_x g(x_0, y_0)$ we denote the gradient and the partial gradient with respect to x, respectively, of g at (x_0, y_0), and by $\nabla^2 g(x_0, y_0)$, $\nabla^2_{xx} g(x_0, y_0)$, $\nabla^2_{xy} g(x_0, y_0)$, etc., we denote the corresponding Hessian matrices of second-order partial derivatives.

(A2) The sets $Y(x)$ are nonempty and *uniformly bounded* in a neighborhood of x_0, i.e., there exist a neighborhood M of x_0 and a bounded set $T \subset \mathbb{R}^k$ such that $Y(x) \neq \emptyset$ and $Y(x) \subset T$ for all $x \in M$.

Assumptions (A1) and (A2) imply that the sets $Y(x)$ are compact and, hence, the supremum in (1.2) is attained for all x near x_0. Furthermore, the set-valued mapping $x \mapsto Y(x)$ is closed at x_0 and the optimal value function $v(x)$ is upper semicontinuous at x_0. Hence, if $v(x_0) < 0$, then x_0 is an interior point of S. In that case the problem is locally unconstrained and, hence, $\nabla f(x_0) = 0$, and the corresponding second-order necessary (sufficient) optimality conditions are positive semi-definiteness (positive definiteness) of the Hessian matrix $\nabla^2 f(x_0)$. Therefore, we assume subsequently that $v(x_0) = 0$ or, in other words, that the set of active inequality constraints

$$Y_0(x_0) := \{y \in Y(x_0) \mid g(x_0, y) = 0\} \tag{3.1}$$

is nonempty.

We present now two sets of assumptions (regularity conditions) which will be used in the next section to derive second-order optimality conditions in two, somewhat different, situations.

(A3) The set $Y_0(x_0)$ is finite, say $Y_0(x_0) = \{y_1, \dots, y_m\}$.

Let us remark that the second-order sufficient conditions of the following assumption (A5) imply that every point of the set $Y_0(x_0)$ is an isolated point of $Y_0(x_0)$. Together with assumption (A2) this implies that the set $Y_0(x_0)$ is finite. That is, the above assumption (A3) follows from the assumptions (A1),(A2),(A4) and (A5). However, we write assumption (A3) explicitly in order to emphasize that in the present case we consider situations where the set of active constraints is finite.

Since x_0 is a feasible point of (1.1), we have that each $y_0 \in Y_0(x_0)$ is a global maximizer of the associated problem

$$\text{Maximize } g(x_0, y) \text{ subject to } y \in Y(x_0), \tag{3.2}$$

which is called the *lower level problem*. Then, for $y_0 \in Y_0(x_0)$ we define the corresponding Lagrangian

$$L(x_0, y_0, \alpha) := g(x_0, y_0) - \sum_{i=1}^{q} \alpha_i h_i(x_0, y_0), \quad \alpha \in \mathbb{R}^q,$$

the corresponding set of vectors of Lagrange multipliers

$$A(x_0, y_0) := \left\{ \alpha \in \mathbb{R}^q \;\middle|\; \begin{array}{l} \nabla_y L(x_0, y_0, \alpha) = 0,\ \alpha_j \geqslant 0,\ j = p+1, \dots, q \\ \alpha_j h_j(x_0, y_0, \alpha) = 0,\ j = p+1, \dots, q \end{array} \right\},$$

and the index set of active inequality constraints

$$I(x_0, y_0) := \{ j \in \{p+1, \dots, q\} \mid h_j(x_0, y_0) = 0 \}.$$

(A4) To each $y_\ell \in Y_0(x_0)$ corresponds a unique vector of Lagrange multipliers, i.e., $A(x_0, y_\ell) = \{\alpha_\ell\}$, $\ell = 1, \dots, m$.

Note that existence and uniqueness of α_ℓ in (A4) is, in itself, a constraint qualification. That is, if $y_\ell \in Y_0(x_0)$ and $\alpha_\ell \in A(x_0, y_\ell)$, then α_ℓ is *unique* if and only if the following two conditions are satisfied (cf. [16]):

(K1) The vectors $\nabla_y h_j(x_0, y_\ell)$, $j \in \{1, \dots, p\} \cup I_+(x_0, y_\ell)$ are linearly independent, where

$$I_+(x_0, y_\ell) := \{ j \in I(x_0, y_\ell) \mid \alpha_j^\ell > 0 \}.$$

(K2) There exists a vector $z \in \mathbb{R}^k$ satisfying

$$z^T \nabla_y h_i(x_0, y_\ell) = 0, \quad i \in \{1, \dots, p\} \cup I_+(x_0, y_\ell) \quad \text{and}$$
$$z^T \nabla_y h_j(x_0, y_\ell) < 0, \quad j \in I_0(x_0, y_\ell),$$

where

$$I_0(x_0, y_\ell) := I(x_0, y_\ell) \setminus I_+(x_0, y_\ell).$$

The assumption (A4) holds if (but not only if) for $\ell = 1, \ldots, m$ the linear independence constraint qualification (LICQ) is fulfilled at y_ℓ, i.e., if the vectors $\nabla_y h_i(x_0, y_\ell)$, $i \in \{1, \ldots, p\} \cup I(x_0, y_\ell)$ are linearly independent. It is clear that (A4) is stronger than the Mangasarian–Fromovitz constraint qualification (MFCQ), which is fulfilled at y_ℓ if:

- the vectors $\nabla_y h_i(x_0, y_\ell)$, $i = 1, \ldots, p$ are linearly independent, and
- there exists a vector $z \in \mathbb{R}^k$ satisfying

$$z^T \nabla_y h_i(x_0, y_\ell) = 0, \quad i = 1, \ldots, p,$$
$$z^T \nabla_y h_j(x_0, y_\ell) < 0, \quad j \in I(x_0, y_\ell).$$

(A5) The following second order sufficient conditions (for the lower level problem) hold at every $y_\ell \in Y_0(x_0)$, $\ell = 1, \ldots, m$:

$$\zeta^T \nabla^2_{yy} L(x_0, y_\ell, \alpha_\ell)\zeta < 0, \quad \forall \zeta \in Z(x_0, y_\ell) \setminus \{0\}, \tag{3.3}$$

where

$$Z(x_0, y_\ell) := \left\{ \zeta \in \mathbb{R}^k \;\middle|\; \begin{array}{l} \zeta^T \nabla_y h_i(x_0, y_\ell) = 0,\ i \in \{1, \ldots, p\} \cup I_+(x_0, y_\ell) \\ \zeta^T \nabla_y h_j(x_0, y_\ell) \leqslant 0,\ j \in I_0(x_0, y_\ell) \end{array} \right\} \tag{3.4}$$

is the critical cone, at y_ℓ, of the lower level problem.

Under assumption (A4), the second-order sufficient conditions (3.3), for the lower level problem, are weakest possible in the sense that the corresponding necessary conditions are obtained by replacing the strict inequality sign in (3.3) by the sign '$\leqslant$'.

Conditions (3.3) are weaker than the following so-called strong second-order sufficient conditions (SSOSC):

$$\zeta^T \nabla^2_{yy} L(x_0, y_\ell, \alpha_\ell)\zeta < 0 \quad \text{for all } \zeta \in \mathbb{R}^k \setminus \{0\} \text{ satisfying}$$
$$\zeta^T \nabla_y h_i(x_0, y_\ell) = 0, \quad i \in \{1, \ldots, p\} \cup I_+(x_0, y_\ell).$$

If the (LICQ) and (SSOSC) are satisfied at all points of $Y_0(x_0)$, then the reduction approach is applicable at x_0, i.e., the original problem (1.1) can be reformulated (locally) as a finitely constrained problem (cf. [10]). However, as it will become clear from our example in Section 5, under assumptions (A1)–(A5) the reduction approach cannot be applied in general.

We introduce now regularity conditions which do not require for the set $Y_0(x_0)$ to be finite.

(A6) For every $\bar{y} \in Y_0(x_0)$ the (LICQ) is fulfilled, i.e., vectors $\nabla_y h_i(x_0, \bar{y})$, $i \in \{1, \ldots, p\} \cup I(x_0, \bar{y})$, are linearly independent.

Under (A6), to each $\bar{y} \in Y_0(x_0)$ corresponds a unique vector of Lagrange multipliers, denoted $\alpha(\bar{y})$, and hence $A(x_0, \bar{y}) = \{\alpha(\bar{y})\}$.

For $\bar{y} \in Y_0(x_0)$ consider the set

$$\Theta(x_0, \bar{y}) := \{y \in \mathbb{R}^k \mid h_i(x_0, y) = 0,\ i \in \{1, \dots, p\} \cup I(x_0, \bar{y})\}.$$

We have that $\bar{y} \in \Theta(x_0, \bar{y})$ and that, restricted to a neighborhood of $\bar{y}$, this set is a subset of $Y(x_0)$. Moreover, it follows from the assumption (A6) that $\Theta(x_0, \bar{y})$ is a smooth manifold in a neighborhood of $\bar{y}$.

For a given $d \in \mathbb{R}^n$ consider the set

$$Y_1(x_0, d) := \arg\max_{y \in Y_0(x_0)} d^T \nabla_x L(x_0, y, \alpha(y)).$$

Since under our assumptions the set $Y_0(x_0)$ is nonempty and compact, the set $Y_1(x_0, d)$ is a nonempty and compact subset of $Y_0(x_0)$.

(A7) (Strict complementarity condition) For every $\bar{y} \in Y_1(x_0, d)$ it follows that $\alpha_j(\bar{y}) > 0$ for all $j \in I(x_0, \bar{y})$, i.e., $I(x_0, \bar{y}) = I_+(x_0, \bar{y})$.

Under assumptions (A6) and (A7), the set $I(x_0, y)$ of active inequality constraints does not change for all $y \in Y_0(x_0)$ sufficiently close to $\bar{y}$ and, hence, the (active) inequality constraints can be treated (locally) as equality constraints. It follows that the set $Y_0(x_0)$, restricted to a neighborhood of $\bar{y}$, is a subset of $\Theta(x_0, \bar{y})$.

(A8) For every $\bar{y} \in Y_1(x_0, d)$, the set $Y_0(x_0)$, restricted to a neighborhood of $\bar{y}$, is a smooth submanifold of $\Theta(x_0, \bar{y})$.

If $\bar{y}$ is an isolated point of $Y_0(x_0)$, then $Y_0(x_0)$, restricted to a neighborhood of $\bar{y}$, is a smooth manifold of dimension 0.

Consider the tangent space

$$T_\Theta(\bar{y}) = \left\{y \in \mathbb{R}^k \mid y^T \nabla_y h_i(x_0, \bar{y}) = 0,\ i \in \{1, \dots, p\} \cup I(x_0, \bar{y})\right\}$$

of the smooth manifold $\Theta(x_0, \bar{y})$ at the point $\bar{y}$. Denote by $T_0(\bar{y})$ the tangent space of the smooth manifold $Y_0(x_0)$ at $\bar{y}$. Note that since $Y_0(x_0)$ is a submanifold of $\Theta(x_0, \bar{y})$, we have that $T_0(\bar{y}) \subset T_\Theta(\bar{y})$.

(A9) The following second-order sufficient conditions (for the lower level problem) hold at every $\bar{y} \in Y_1(x_0, d)$:

$$\zeta^T \nabla^2_{yy} L(x_0, \bar{y}, \alpha(\bar{y}))\zeta < 0, \quad \forall \zeta \in [T_0(\bar{y})]^\perp \cap T_\Theta(\bar{y}),\ \zeta \neq 0.$$

Note that, under our assumptions, the above second-order conditions are necessary and sufficient for the second-order growth condition (for the lower level problem) to hold near the point $\bar{y}$, [23].

4. Second-Order Optimality Conditions in Generalized Semi-Infinite Programming

In this section we derive second-order necessary and sufficient conditions for x_0 to be a locally optimal solution of the problem (1.1). It is possible to show that if (MFCQ) and a certain form of second-order sufficient conditions hold at every $y_0 \in Y_0(x_0)$, then y_0 is an isolated point of $Y_0(x_0)$ (and, thus, $Y_0(x_0)$ is finite under (A2)) and the optimal value function $v(x)$ is second-order directionally differentiable at x_0. The corresponding second-order directional derivatives are given then as optimal values of some min-max problems (see [3] and references therein). Together with (2.4) this leads to second-order necessary conditions for the generalized semi-infinite problem (1.1). In all their generality such conditions are too complicated. Moreover, it is not clear whether the optimal value function $v(x)$ is second-order epiregular under such conditions, and hence whether the corresponding second-order sufficient conditions, for the problem (1.1), hold. Therefore we deal in this paper only with the situations where the Lagrange multipliers of the lower level problem are *unique*, i.e., assumption (A4) (or (A6)) hold.

We suppose in the remainder of the paper that assumptions (A1) and (A2) hold, and consider first the case where assumptions (A3)–(A5) are satisfied.

PROPOSITION 4.1. *Suppose that assumptions* (A1)–(A5) *hold. Then the optimal value function $v(x)$ is Lipschitz continuous in a neighborhood of x_0, second-order directionally differentiable and second-order epiregular at x_0 and*

$$v'(x_0, d) = \max_{\ell \in \{1,\dots,m\}} d^T \nabla_x L(x_0, y_\ell, \alpha_\ell), \tag{4.1}$$

$$v''(x_0; d, w) = \max_{\ell \in \imath(x_0,d)} \left\{ w^T \nabla_x L(x_0, y_\ell, \alpha_\ell) + \xi_d(x_0, y_\ell) \right\}, \tag{4.2}$$

where

$$\imath(x_0, d) := \arg\max_{\ell \in \{1,\dots,m\}} d^T \nabla_x L(x_0, y_\ell, \alpha_\ell),$$

and

$$\begin{aligned} &\xi_d(x_0, y_\ell) \\ &\quad := \sup_{\zeta \in \mathcal{Z}_d(x_0, y_\ell)} \big\{ \zeta^T \nabla^2_{yy} L(x_0, y_\ell, \alpha_\ell)\zeta + 2d^T \nabla^2_{xy} L(x_0, y_\ell, \alpha_\ell)\zeta + \\ &\qquad\qquad + d^T \nabla^2_{xx} L(x_0, y_\ell, \alpha_\ell) d \big\}, \end{aligned}$$

$$\begin{aligned} &\mathcal{Z}_d(x_0, y_\ell) \\ &\quad := \left\{ \zeta \;\middle|\; \begin{array}{l} \zeta^T \nabla_y h_i(x_0, y_\ell) + d^T \nabla_x h_i(x_0, y_\ell) = 0, \\ \quad i \in \{1, \dots, p\} \cup I_+(x_0, y_\ell), \\ \zeta^T \nabla_y h_j(x_0, y_\ell) + d^T \nabla_x h_j(x_0, y_\ell) \leqslant 0, \;\; j \in I_0(x_0, y_\ell) \end{array} \right\}. \end{aligned}$$

Formulas (4.1) and (4.2) are obtained by local analysis of the lower level problem at each y_ℓ, $\ell \in \{1, \dots, m\}$, and then by taking the maximum of the obtained directional derivatives. Under the assumptions (A1)–(A5), formula (4.1), for directional derivatives of the optimal value function, is well known (e.g., [5, 17]). The corresponding formula (4.2), for second-order directional derivatives under the assumption of uniqueness of the corresponding vector of Lagrange multipliers, was obtained in [22] (for a discussion of such results see [4, Section 5.2.3]). Second-order epiregularity of the optimal value function, under the assumptions (A1)–(A5), is shown in [4, Theorem 4.139].

Note that under conditions (K1) and (K2) given in Section 3, the set $\mathcal{Z}_d(x_0, y_\ell)$ is nonempty for any $d \in \mathbb{R}^n$. Moreover, the critical cone $Z(x_0, y_\ell)$ in (3.4) is the recession cone of $\mathcal{Z}_d(x_0, y_\ell)$ and, hence, it follows from second-order conditions (3.3) that $\xi_d(x_0, y_\ell)$ is finite valued.

Consider now the max-function

$$\psi(x) := \max\{f(x) - f(x_0), v(x)\}. \tag{4.3}$$

It is not difficult to see (and is well known) that if x_0 is a (locally) optimal solution of the problem (1.1), then x_0 is a (local) minimizer of the function $\psi(x)$. Conversely, if the second-order growth condition (2.3) holds, then the corresponding second-order growth condition holds for the problem (1.1) as well, i.e.,

$$f(x) \geqslant f(x_0) + c\|x - x_0\|^2, \quad \forall x \in S \cap N. \tag{4.4}$$

We have by (4.1) that, under assumptions (A1)–(A4),

$$\psi'(x_0, d) = \max\big\{d^T \nabla f(x_0), d^T \nabla_x L(x_0, y_\ell, \alpha_\ell),\ \ell \in \{1, \dots, m\}\big\}. \tag{4.5}$$

Consequently, by duality arguments, first-order necessary conditions (2.1) are equivalent to the following first-order necessary conditions, for x_0 to be a locally optimal solution of the problem (1.1) (see [21, Theorem 2]):

$$\Lambda^*(x_0) \neq \emptyset, \tag{4.6}$$

where $\Lambda^*(x_0)$ is the set of Fritz–John type multipliers

$$\Lambda^*(x_0) := \left\{ \lambda \in \mathbb{R}^{m+1} \;\middle|\; \begin{array}{l} \lambda_0 \nabla f(x_0) + \sum_{\ell=1}^m \lambda_\ell \nabla_x L(x_0, y_\ell, \alpha_\ell) = 0 \\ \lambda_\ell \geqslant 0,\ \ell = 0, \dots, m,\ \sum_{\ell=0}^m \lambda_\ell = 1 \end{array} \right\}. \tag{4.7}$$

If a constraint qualification of the Mangasarian–Fromovitz type holds at x_0, i.e., there exists a vector $b \in \mathbb{R}^n$ such that $b^T \nabla_x L(x_0, y_\ell, \alpha_\ell) < 0$, $\ell = 1, \dots, m$, then for any $\lambda \in \Lambda^*(x_0)$ we have that $\lambda_0 \neq 0$. In that case we can take $\lambda_0 = 1$ and the following set of Lagrange multipliers is nonempty

$$\Lambda(x_0) := \left\{ \lambda \in \mathbb{R}^{m} \;\middle|\; \begin{array}{l} \nabla f(x_0) + \sum_{\ell=1}^m \lambda_\ell \nabla_x L(x_0, y_\ell, \alpha_\ell) = 0 \\ \lambda_\ell \geqslant 0,\ \ell = 1, \dots, m \end{array} \right\}.$$

Consider the critical cone $C(x_0)$ defined in (2.2). If x_0 is a locally optimal solution of the problem (1.1), then by (4.5), the critical cone can be written as

$$C(x_0) = \left\{d \mid d^T\nabla f(x_0) \leqslant 0,\ d^T\nabla_x L(x_0, y_\ell, \alpha_\ell) \leqslant 0,\ \ell \in \{1, \ldots, m\}\right\}.$$

Note that if there exists $\lambda \in \Lambda^*(x_0)$ such that $\lambda_0 \neq 0$, then for any $d \in C(x_0)$, $d^T\nabla f(x_0) \geqslant 0$ and, hence, $d^T\nabla f(x_0) = 0$.

THEOREM 4.1. *Suppose that assumptions* (A1)–(A5) *are satisfied, and that the set* $\Lambda^*(x_0)$ *is nonempty, i.e., Fritz–John type first-order necessary conditions are satisfied at* x_0. *Let* ψ *be the max-function given in* (4.3). *Then the following holds.*

(i) *Conditions*

$$\sup_{\lambda\in\Lambda^*(x_0)} \left\{\lambda_0 d^T\nabla^2 f(x_0)d + \sum_{\ell=1}^{m} \lambda_\ell \xi_d(x_0, y_\ell)\right\} \geqslant 0, \quad \forall d \in C(x_0), \tag{4.8}$$

are necessary for x_0 *to be a locally optimal solution of problem* (1.1).

(ii) *Conditions*

$$\sup_{\lambda\in\Lambda^*(x_0)} \left\{\lambda_0 d^T\nabla^2 f(x_0)d + \sum_{\ell=1}^{m} \lambda_\ell \xi_d(x_0, y_\ell)\right\} > 0,$$
$$\forall d \in C(x_0) \setminus \{0\}, \tag{4.9}$$

are necessary and sufficient for the second-order growth condition (2.3) *to hold at* x_0.

Proof. Consider a critical direction $d \in C(x_0)$. By Proposition 4.1 we have that the max-function ψ is second-order epiregular at x_0 and, moreover, if $d^T\nabla f(x_0) = 0$, then

$$\psi''(x_0; d, w) = \max\left\{\begin{array}{l} w^T\nabla f(x_0) + d^T\nabla^2 f(x_0)d, \\ w^T\nabla_x L(x_0, y_\ell, \alpha_\ell) + \xi_d(x_0, y_\ell),\ \ell \in \imath(x_0, d) \end{array}\right\},$$

and if $d^T\nabla f(x_0) < 0$, then

$$\psi''(x_0; d, w) = \max\left\{w^T\nabla_x L(x_0, y_\ell, \alpha_\ell) + \xi_d(x_0, y_\ell),\ \ell \in \imath(x_0, d)\right\}.$$

Note that if $\lambda \in \Lambda^*(x_0)$, then $\lambda_\ell = 0$ for all $\ell \in \{1, \ldots, m\} \setminus \imath(x_0, d)$ and $\lambda_0 = 0$ if $d^T\nabla f(x_0) < 0$.

We obtain that the problem

$$\operatorname*{Minimize}_{w\in\mathbb{R}^n} \psi''(x_0; d, w)$$

can be formulated as the following linear programming problem:

$$\operatorname*{Minimize}_{(\theta,w)\in\mathbb{R}\times\mathbb{R}^n} \theta$$
$$\text{subject to } \theta \geqslant w^T\nabla_x L(x_0, y_\ell, \alpha_\ell) + \xi_d(x_0, y_\ell),\ \ell \in \imath(x_0, d), \tag{4.10}$$
$$\text{and } \theta \geqslant w^T\nabla f(x_0) + d^T\nabla^2 f(x_0)d,\ \text{in case } d^T\nabla f(x_0) = 0,$$

where the last inequality constraint is not present in (4.10) if $d^T\nabla f(x_0) < 0$. The dual of the above linear programming problem is the problem

$$\operatorname*{Maximize}_{\lambda\in\Lambda^*(x_0)} \lambda_0 d^T\nabla^2 f(x_0)d + \sum_{\ell=1}^{m} \lambda_\ell \xi_d(x_0, y_\ell). \tag{4.11}$$

Recall that, by the basic theorem of linear programming, the optimal value of the problem (4.10) coincides with the optimal value of its dual (4.11). Consequently, the second-order necessary conditions (2.4) (applied to the function ψ defined in (4.3)) take the form (4.8), and the second-order conditions (2.5) take the form (4.9). This completes the proof. □

Remark 4.1. Under assumptions (A1)–(A5), second-order conditions (4.9) are necessary and sufficient for the second-order growth condition to hold for the unconstrained problem of minimization of the function ψ. It follows then that conditions (4.9) are also sufficient (although may be not necessary) for the second-order growth condition (4.4) for the problem (1.1) as well.

Remark 4.2. If the strict complementarity condition holds at y_ℓ, i.e., the set $I_0(x_0, y_\ell)$ is empty, then $\mathcal{Z}_d(x_0, y_\ell)$ becomes an affine space and $\xi_d(x_0, y_\ell)$ can be calculated explicitly.

Remark 4.3. If a constraint qualification of the Mangasarian–Fromovitz type holds at x_0, and hence the set $\Lambda(x_0)$ of Lagrange multipliers is nonempty, we have that $\lambda_0 \neq 0$ for any $\lambda \in \Lambda^*(x_0)$. Consequently in that case the set $\Lambda^*(x_0)$ can be replaced by the set $\Lambda(x_0)$ in the second-order conditions (4.8) and (4.9).

Let us discuss now the second case where assumptions (A6)–(A9) are satisfied. It is shown in [23] that under assumptions (A1), (A2) and (A6)–(A9), the optimal value function is second-order directionally differentiable at the point x_0. With some additional effort it is possible to show that the second-order epiregularity also holds in that case [4, Theorem 4.142]. That is, the following results hold.

PROPOSITION 4.2. *Suppose that assumptions* (A1), (A2) *and* (A6) *hold. Then the optimal value function $v(x)$ is Lipschitz continuous in a neighborhood of x_0, directionally differentiable at x_0 and*

$$v'(x_0, d) = \max_{y\in Y_0(x_0)} d^T\nabla_x L(x_0, y, \alpha(y)), \quad \forall d \in \mathbb{R}^n. \tag{4.12}$$

If, moreover, assumptions (A7)–(A9) *hold in a given direction $d \in \mathbb{R}^n$, then the optimal value function $v(x)$ is second-order directionally differentiable and second-order epiregular at x_0, in the direction d, and*

$$v''(x_0; d, w) = \max_{y\in Y_1(x_0,d)} \left\{ w^T\nabla_x L(x_0, y, \alpha(y)) + \xi_d(x_0, y) \right\}, \tag{4.13}$$

where

$$
\begin{aligned}
&\xi_d(x_0, y)\\
&\quad := \sup_{\zeta \in \mathcal{Z}_d(x_0,y)} \big\{\zeta^T \nabla^2_{yy} L(x_0, y, \alpha(y))\zeta + 2d^T \nabla^2_{xy} L(x_0, y, \alpha(y))\zeta + \\
&\qquad\qquad + d^T \nabla^2_{xx} L(x_0, y, \alpha(y))d\big\},\\
&\mathcal{Z}_d(x_0, y)\\
&\quad := \big\{\zeta \mid \zeta^T \nabla_y h_i(x_0, y) + d^T \nabla_x h_i(x_0, y) = 0,\ i \in \{1, \dots, p\} \cup I(x_0, y)\big\}.
\end{aligned}
$$

Note that, under the assumptions (A6) and (A7), by the second-order necessary conditions (for the lower level problem) at a point $\bar{y} \in Y_0(x_0)$, the Hessian matrix $\nabla^2_{yy} L(x_0, \bar{y}, \alpha(\bar{y}))$ is negative semidefinite on the linear space

$$\big\{\zeta \mid \zeta^T \nabla_y h_i(x_0, \bar{y}) = 0,\ i \in \{1, \dots, p\} \cup I(x_0, \bar{y})\big\}.$$

Moreover, for any $\bar{y} \in Y_1(x_0, d)$ we have by first-order necessary conditions applied to the corresponding optimization problem that

$$d^T \nabla^2_{xy} L(x_0, \bar{y}, \alpha(\bar{y}))\zeta = 0, \quad \forall \zeta \in T_0(\bar{y}).$$

It follows then by the second-order conditions of assumption (A9) that the maximum in the definition of $\xi_d(x_0, \bar{y})$ is attained. Also, since $\xi_d(x_0, \bar{y})$ is given by the maximum of a quadratic function over an affine space, it can be written in a closed form.

Consider the set $\Lambda^*(x_0)$, defined in (4.7), for some $y_\ell \in Y_0(x_0)$ and $\alpha_\ell := \alpha(y_\ell)$. It follows from (4.12) (see [21]) that this set is nonempty. By the arguments similar to those used in the previous case, it follows by (4.13) that the second-order necessary conditions, for the problem (1.1), can be formulated as follows: for any $d \in C(x_0)$ the optimal value of the following problem is nonnegative

$$
\begin{aligned}
&\operatorname*{Minimize}_{(\theta,w)\in\mathbb{R}\times\mathbb{R}^n} \theta\\
&\quad \text{subject to } \theta \geqslant w^T \nabla_x L(x_0, y, \alpha(y)) + \xi_d(x_0, y),\ y \in Y_1(x_0, d), \qquad (4.14)\\
&\quad \text{and } \theta \geqslant w^T \nabla f(x_0) + d^T \nabla^2 f(x_0)d, \text{ in case } d^T \nabla f(x_0) = 0,
\end{aligned}
$$

where the last inequality constraint is not present in (4.14) if $d^T \nabla f(x_0) < 0$. The above optimization problem is a linear semi-infinite programming problem (unless the set $Y_1(x_0, d)$ is finite in which case it becomes a linear programming problem). It is not difficult to see that an (extended) constraint qualification of the Mangasarian–Fromovitz type always holds for (4.14). Therefore its optimal value is the same as the optimal value of its dual (see [6, 8] and references therein for a discussion of duality in semi-infinite programming). Then, by calculating the dual of (4.14), the second-order necessary and sufficient conditions take the form (4.8) and (4.9), respectively. We obtain the following result.

THEOREM 4.2. *Suppose that assumptions* (A1), (A2) *and* (A6) *hold, and that assumptions* (A7)–(A9) *hold for any* $d \in C(x_0)$. *Suppose, further, that Fritz–John type first-order necessary conditions hold at* x_0. *Then the following holds.*

(i) *The following conditions are necessary for* x_0 *to be a locally optimal solution of problem* (1.1): *for any* $d \in C(x_0)$ *there exist finitely many* $y_\ell \in Y_1(x_0, d)$, $\ell = 1, \ldots, m$, *and* $\lambda = (\lambda_0, \ldots, \lambda_m) \in \Lambda^*(x_0)$ *such that*

$$\lambda_0 d^T \nabla^2 f(x_0) d + \sum_{\ell=1}^{m} \lambda_\ell \xi_d(x_0, y_\ell) \geqslant 0. \tag{4.15}$$

(ii) *The following conditions are necessary and sufficient for the second-order growth condition* (2.3) *to hold at* x_0 *with* ψ *given in* (4.3): *for any* $d \in C(x_0) \setminus \{0\}$, *there exist finitely many* $y_\ell \in Y_1(x_0, d)$, $\ell = 1, \ldots, m$, *and* $\lambda = (\lambda_0, \ldots, \lambda_m) \in \Lambda^*(x_0)$ *such that*

$$\lambda_0 d^T \nabla^2 f(x_0) d + \sum_{\ell=1}^{m} \lambda_\ell \xi_d(x_0, y_\ell) > 0. \tag{4.16}$$

5. Examples

This final section contains two examples illustrating the second-order conditions presented in the previous section. These examples also give an insight into the possible topological structure of the feasible set S of a generalized semi-infinite optimization problem. The first example refers to the second-order necessary and sufficient conditions (4.8) and (4.9), respectively.

EXAMPLE 5.1. Consider the following problem of type (1.1):

$$\text{Minimize } f(x_1, x_2) = (4 + \tfrac{\varepsilon}{2})x_1^2 - x_2 - x_2^2 \quad \text{subject to } x \in S,$$

where $\varepsilon \geqslant 0$, and

$$S := \left\{ x \in \mathbb{R}^2 \mid x_2 - y_3 \leqslant 0,\ y \in Y(x) \right\},$$

$$Y(x) := \left\{ y \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} h_1(x, y) = y_1 - x_1 \leqslant 0 \\ h_2(x, y) = y_2 - x_1 \leqslant 0 \\ h_3(x, y) = (y_1 + y_2)^2 - y_3 \leqslant 0 \\ h_4(x, y) = y_3 - 16 \leqslant 0 \end{array} \right\}.$$

A short calculation shows that

$$\begin{aligned} S \;=\; & \left\{ x \in \mathbb{R}^2 \mid x_1 \geqslant 0, x_2 \leqslant 0 \right\} \cup \\ & \cup \left\{ x \in \mathbb{R}^2 \mid -2 \leqslant x_1 \leqslant 0, x_2 \leqslant 4x_1^2 \right\} \cup \left\{ x \in \mathbb{R}^2 \mid x_1 < -2 \right\}. \end{aligned}$$

Note that S is *not closed* (and not open) and it is the *union of sets* where each of them is defined by finitely many differentiable functions. Both topological properties cannot appear in standard semi-infinite (or finite) optimization problems. We refer to the related papers [11, 25] on investigations of the topological structure of the feasible set of a generalized semi-infinite optimization problem.

Let $x_0 = (0, 0)$ be the point under consideration. Then, assumptions (A1), (A2), (A3) and (A4) are satisfied with $Y_0(0) = \{y_1\}$, $y_1 = (0, 0, 0)$, $I(x_0, y_1) = \{1, 2, 3\}$, $A(x_0, y_1) = \{\alpha_1\}$, $\alpha_1 = (0, 0, 1)$, $I_+(x_0, y_1) = \{3\}$ and $I_0(x_0, y_1) = \{1, 2\}$. We obtain

$$\nabla^2_{yy} L(x_0, y_1, \alpha_1) = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the critical cone at y_1:

$$Z(x_0, y_1) = \left\{\zeta \in \mathbb{R}^3 \mid \zeta_1 \leqslant 0,\ \zeta_2 \leqslant 0,\ \zeta_3 = 0\right\}.$$

Thus, assumption (A5) is satisfied, too. However, note that (SSOSC) is not satisfied at x_0 and the *reduction approach is not applicable at* x_0 since for any $x_1 > 0$ near $x_{01} = 0$ the set $Y_0(x_1, 0) = \{(y_1, -y_1, 0) \mid |y_1| \leqslant x_1\}$ contains two *different* points; i.e., there is a bifurcation in the index set $Y_0(x)$ of active inequality constraints and its cardinality is changing as x varies.

Moreover, for $d, w \in \mathbb{R}^2$ we have

$$\mathcal{Z}_d(x_0, y_1) = \left\{\zeta \in \mathbb{R}^3 \mid \zeta_1 - d_1 \leqslant 0,\ \zeta_2 - d_2 \leqslant 0,\ \zeta_3 = 0\right\},$$

$$\xi_d(x_0, y_1) = \begin{cases} 0, & \text{if } d_1 \geqslant 0, \\ -8d_1^2, & \text{if } d_1 < 0, \end{cases}$$

$$v'(x_0, d) = d_2 \quad \text{and}$$

$$v''(x_0; d, w) = \begin{cases} w_2, & \text{if } d_1 \geqslant 0, \\ w_2 - 8d_1^2, & \text{if } d_1 < 0. \end{cases}$$

Furthermore,

$$\nabla f(x_0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad \nabla^2 f(x_0) = \begin{bmatrix} 8 + \varepsilon & 0 \\ 0 & -2 \end{bmatrix}$$

and the critical cone is

$$C(x_0) = \{(d_1, 0) \mid d_1 \in \mathbb{R}\}.$$

Hence, necessary conditions (4.6) hold with

$$\Lambda^*(x_0) = \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\},$$

and we obtain for $d \in C(x_0)$, $d_1 < 0$:

$$\tfrac{1}{2} d^T \nabla^2 f(x_0) d + \tfrac{1}{2}\xi_d(x_0, y_1) = \tfrac{1}{2}\varepsilon d_1^2.$$

Now, we distinguish two cases. If $\varepsilon = 0$, then necessary conditions (4.8) hold at x_0. However, sufficient conditions (4.9) do not hold and x_0 is not a locally optimal solution of (1.1) since f is monotonically decreasing along the boundary of

$$\left\{x \in \mathbb{R}^2 \mid 4x_1^2 - x_2 = 0, x_1 \leqslant 0\right\}$$

for decreasing x_1. If $\varepsilon > 0$, then x_0 is a locally optimal solution of (1.1), sufficient conditions (4.9) hold at x_0 and the second order growth condition holds at x_0 with $c < \varepsilon/2$.

In the final example the assumptions of Theorem 9 are fulfilled and, in particular, the set $Y_0(x_0)$ is a one-dimensional manifold. Furthermore, this example illustrates the sufficient optimality conditions presented in Theorem 9(ii).

EXAMPLE 5.2. Consider the following problem of type (1):

$$\text{Minimize } f(x_1, x_2) = -\tfrac{1}{2}x_1^4 + 2x_1x_2 - 2x_1^2 \quad \text{subject to } x \in S$$

where

$$S = \left\{x \in \mathbb{R}^2 \mid y_1^2 + y_2^2 - x_1 + x_1^2 - x_2 \leqslant 0,\ y \in Y(x)\right\}$$

and

$$Y(x) = \left\{y \in \mathbb{R}^3 \mid h_1(x, y) = y_1^2 + y_2^2 + y_3^2 - x_1 \leqslant 0\right\}.$$

As in Example 5.1., S is the union of an open set and a closed set and it can be written as

$$S = \left\{x \in \mathbb{R}^2 \mid x_1 < 0\right\} \cup \left\{x \in \mathbb{R}^2 \mid x_1 \geqslant 0,\ x_2 \geqslant x_1^2\right\}.$$

Let $x_0 = (1, 1)$ be the point under consideration. Obviously, assumptions (A1) and (A2) hold. Furthermore, it is

$$Y_0(x_0) = \left\{y \in \mathbb{R}^3 \mid y_3 = 0,\ y_1^2 + y_2^2 = 1\right\}$$

and for each $\bar{y} \in Y_0(x_0)$ we have $I_0(x_0, \bar{y}) = \{1\}$, $\nabla_y h_1(x_0, \bar{y}) \neq 0$ and $\alpha(\bar{y}) = 1$. Thus, $Y_0(x_0)$ is a one-dimensional smooth submanifold of

$$\Theta(x_0, \bar{y}) = \left\{y \in \mathbb{R}^3 \mid y_1^2 + y_2^2 + y_3^2 = 1\right\}$$

and, consequently, assumptions (A6), (A7) and (A8) are fulfilled, too (note that $Y_0(x_0)$ is an *infinite* set and, therefore, the reduction approach is not applicable at x_0).

Moreover, a short calculation shows for each $\bar{y} \in Y_0(x_0)$:

$$[T_0(\bar{y})]^{\perp} \cap T_{\Theta}(\bar{y}) = \{(0, 0, \gamma) \mid \gamma \in \mathbb{R}\},$$

$$\nabla^2_{yy} L(x_0, \bar{y}, \alpha(\bar{y})) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

an, thus, assumption (A9) also holds.

Then, we have for $d \in \mathbb{R}^2$:

$$\mathcal{Z}_d(x_0, \bar{y}) = \{\zeta \in \mathbb{R}^3 \mid 2\zeta_1\bar{y}_1 + 2\zeta_2\bar{y}_2 = d_1\},$$
$$\xi_d(x_0, \bar{y}) = 2d_1^2$$

and the critical cone is

$$C(x_0) = \{(d_1, d_2) \mid d_1 \in \mathbb{R},\ d_2 = 2d_1\}.$$

For any $d \in C(x_0) \setminus \{0\}$ and any $\bar{y} \in Y_0(x_0)$ we choose $\lambda_0 = 1/3$, $\lambda_1 = 2/3$ and obtain

$$\lambda_0 \nabla f(x_0) + \lambda_1 \nabla_x L(x_0, \bar{y}, \alpha(\bar{y})) = \frac{1}{3}\begin{bmatrix} -4 \\ 2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0$$

as well as

$$\frac{1}{3}d^T\nabla^2 f(x_0)d + \frac{2}{3}\xi_d(x_0, \bar{y})$$
$$= \frac{1}{3}(d_1\ 2d_1)\begin{bmatrix} -10 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} d_1 \\ 2d_1 \end{bmatrix} + \frac{4}{3}d_1^2 = \frac{2}{3}d_1^2 > 0.$$

The latter inequality just represents relation (25) from Theorem 9(ii).

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