



On a time consistency concept in risk averse multistage stochastic programming

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We discuss time consistency of multistage risk averse stochastic programming problems. The concept of time consistency is approached from an optimization point of view. That is, at each state of the system optimality of a decision policy should not involve states which cannot happen in the future.

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1. Introduction

In recent years risk averse stochastic optimization attracted considerable attention from theoretical and application points of view (see, e.g., [1–6]). Starting with pioneering paper by Artzner et al. [7], coherent risk measures were investigated in numerous studies (see [8] and the references therein). It seems that there is a general agreement now of how to model static risk averse stochastic programming problems. A dynamical setting is more involved and several approaches to modeling dynamic risk measures were suggested by various authors (e.g., [1,9–11,12,13,5,6]).

A basic concept of multistage stochastic programming is the requirement of *nonanticipativity*. That is, our decisions should be a function of the history of the data process available at the time when decisions are made. This requirement is necessary for the designed policy to be implementable in the sense that at every state of the system it produces a decision based on available information. The nonanticipativity is a feasibility type constraint indicating which policies are implementable and which are not. On top of that one has to define a criterion for choosing an optimal policy. In the risk neutral setting the optimization is performed on average. Under the nonanticipativity constraint this allows us to write the corresponding dynamic programming equations for an optimal policy. In risk averse settings the situation is more subtle. This motivated an introduction of various concepts of time invariance and time consistency by several authors at various degrees of generality and abstraction (cf., [1,9,10,14–16,6]).

In this paper we discuss an approach to time consistency tailored to conceptual optimality of a decision policy. If we are currently at a certain state of the system, then we know the past, and hence

it is reasonable to require that our decisions should be based on that information. This is the nonanticipativity constraint. If we believe in the considered model, we also have an idea which scenarios could and which cannot happen in the future. Therefore it is also reasonable to consider the requirement that at every state of the system our “optimal” decisions should *not* depend on scenarios which we already know cannot happen in the future. We call this principle the *time consistency* requirement. This time consistency requirement is closely related to, although is not the same, as the so-called Bellman’s principle used to derive dynamic programming equations. The standard risk neutral formulation of multistage stochastic programming problems satisfies this principle. On the other hand, some approaches to risk averse stochastic programming satisfy while others do not satisfy this requirement. It should be mentioned that if the time consistency property does not hold, it does not mean that the corresponding policies are not implementable. We only would like to point out that there is an additional consideration, associated with a chosen optimality criterion, which is worthwhile to keep in mind.

This paper is organized as follows. In the next section we formalize the concept of time consistency. We also discuss its relation to deriving dynamic programming equations. In Section 3 we give examples and show that some approaches to risk averse multistage stochastic programming are time consistent while some others are not.

2. Basic analysis

Consider a T -stage scenario tree representing evolution of the corresponding data process. Such scenario tree represents a finite number of possibilities of what can happen in the future. Considering the case with a *finite* number of scenarios will allow us to avoid some technical complications and concentrate on conceptual issues. At stage (time) $t = 1$ we have one root node

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3. Examples and a discussion

Consider the classical (risk neutral) multistage stochastic programming problem written in the nested formulation form

$$\begin{aligned} \text{Min}_{x_1 \in \mathcal{X}_1} & f_1(x_1) + \mathbb{E} \left[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} f_2(x_2, \xi_2) \right. \\ & \left. + \mathbb{E} \left[\dots + \mathbb{E} \left[\inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} f_T(x_T, \xi_T) \right] \right] \right]. \end{aligned} \quad (9)$$

Here the process $\xi_1, \xi_2, \dots, \xi_T$ is equipped with a probability structure such that it becomes a stochastic process. Define

$$\mathbb{F}_t(Z_t, \dots, Z_T | \xi_{[1,t]}) := \mathbb{E}[Z_t + \dots + Z_T | \xi_{[1,t]}]. \quad (10)$$

Property (A2) follows here with

$$\begin{aligned} \phi_t(f_t(x_t, \xi_t), V_{t+1}(x_t, \xi_{[1,t+1]}) | \xi_{[1,t]}) \\ := \mathbb{E}[f_t(x_t, \xi_t) + V_{t+1}(x_t, \xi_{[1,t+1]}) | \xi_{[1,t]}]. \end{aligned} \quad (11)$$

That is, in the considered sense the risk neutral problem (9) is time consistent. Note that problem (9) can be written in the equivalent form:

$$\begin{aligned} \text{Min}_{x_1, x_2, \dots, x_T} & \mathbb{E}[f_1(x_1) + f_2(x_2, \xi_2) + \dots + f_T(x_T, \xi_T)] \\ \text{s.t.} & x_1 \in \mathcal{X}_1, x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \quad t = 2, \dots, T, \end{aligned} \quad (12)$$

where the optimization is performed over policies $x_t = x_t(\xi_{[1,t]})$ satisfying w.p.1 the feasibility constraints (1).

Example 1 (Multiperiod Coherent Risk Measures). A way of extending risk neutral formulation (12) to a risk averse setting is the following. Let $\mathcal{Z} := \mathcal{Z}_1 \times \dots \times \mathcal{Z}_T$ and $\varrho : \mathcal{Z} \rightarrow \mathbb{R}$ be a coherent risk measure (cf., [7]), i.e., $\varrho(\cdot)$ is convex, monotone, positively homogeneous and $\varrho(Z + a) = \varrho(Z) + a$ for any $Z \in \mathcal{Z}$ and constant $a \in \mathbb{R}$ (compare with Eqs. (25) and (26) below). It is referred to as a *multiperiod coherent risk measure* since it is defined on the product of spaces $\mathcal{Z}_t, t = 1, \dots, T$. Consider the problem

$$\begin{aligned} \text{Min}_{x_1, x_2, \dots, x_T} & \varrho(f_1(x_1), f_2(x_2, \xi_2), \dots, f_T(x_T, \xi_T)) \\ \text{s.t.} & x_1 \in \mathcal{X}_1, x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \quad t = 2, \dots, T, \end{aligned} \quad (13)$$

where, similar to (12), the optimization is performed over policies $x_t = x_t(\xi_{[1,t]})$ satisfying the feasibility constraints. This framework, in various forms of abstraction, was considered by several authors (e.g., [1,3,6]).

Of course, for $\varrho(Z_1, \dots, Z_T) := \mathbb{E}(Z_1 + \dots + Z_T)$, formulation (13) coincides with (12), and satisfies the time consistency principle. Let now the multiperiod risk measure be defined as the absolute deviation risk measure:

$$\begin{aligned} \varrho(Z_1, \dots, Z_T) := & \mathbb{E}(Z_1 + \dots + Z_T) + \lambda \mathbb{E}|Z_1 + \dots + Z_T \\ & - \mathbb{E}(Z_1 + \dots + Z_T)|, \end{aligned} \quad (14)$$

where λ is a positive constant (for $\lambda \in [0, 1/2]$ this function $\varrho : \mathcal{Z} \rightarrow \mathbb{R}$ satisfies the axioms of coherent risk measures). For this multiperiod risk measure and $T > 2$ the corresponding problem (13) does not satisfy the time consistency principle and it is not clear how to write the associated dynamic programming equations.

As another example consider the following multiperiod risk measure

$$\varrho(Z_1, \dots, Z_T) := Z_1 + \rho_2(Z_2) + \dots + \rho_T(Z_T), \quad (15)$$

where $\rho_t : \mathcal{Z}_t \rightarrow \mathbb{R}, t = 2, \dots, T$, are (real valued) risk measures. In that case problem (13) takes the form

$$\begin{aligned} \text{Min}_{x_1, x_2, \dots, x_T} & f_1(x_1) + \rho_2[f_2(x_2, \xi_2)] + \dots + \rho_T[f_T(x_T, \xi_T)] \\ \text{s.t.} & x_1 \in \mathcal{X}_1, x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \quad t = 2, \dots, T. \end{aligned} \quad (16)$$

If each ρ_t is given by Conditional Value-at-Risk, i.e., $\rho_t := \text{CVaR}_{\alpha_t}, t = 2, \dots, T$, with

$$\text{CVaR}_{\alpha_t}(Z_t) := \inf_{r \in \mathbb{R}} \{r + \alpha_t^{-1} \mathbb{E}[Z_t - r]_+\}, \quad \alpha_t \in (0, 1], \quad (17)$$

then the corresponding multiperiod risk measure ϱ becomes a particular example of the polyhedral risk measures discussed in [2]. In that case we can write problem (16) as the multistage program

$$\begin{aligned} \text{Min}_{(r, x_1, x_2, \dots, x_T)} & r_2 + \dots + r_T + f_1(x_1) + \alpha_2^{-1} \mathbb{E}\{[f_2(x_2, \xi_2) - r_2]_+\} \\ & + \dots + \alpha_T^{-1} \mathbb{E}\{[f_T(x_T, \xi_T) - r_T]_+\} \\ \text{s.t.} & x_1 \in \mathcal{X}_1, x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \quad t = 2, \dots, T, \end{aligned} \quad (18)$$

where $r = (r_2, \dots, r_T)$. Problem (18) can be viewed as a standard multistage stochastic program with x_1, r_2, \dots, r_T being first stage decision variables.

Dynamic programming equations for problem (18) can be written as follows. At the last stage we solve problem

$$\begin{aligned} \text{Min}_{x_T} & \alpha_T^{-1} [f_T(x_T, \xi_T) - r_T]_+ \\ \text{s.t.} & x_T \in \mathcal{X}_T(x_{T-1}, \xi_T). \end{aligned} \quad (19)$$

Its optimal value is denoted $V_T(x_{T-1}, r_T, \xi_T)$. At stage $t = 2, \dots, T - 1$, the value function $V_t(x_{t-1}, r_t, \dots, r_T, \xi_{[1,t]})$ is given by the optimal value of problem

$$\begin{aligned} \text{Min}_{x_t} & \alpha_t^{-1} [f_t(x_t, \xi_t) - r_t]_+ \\ & + \mathbb{E}\{V_{t+1}(x_t, r_t, \dots, r_T, \xi_{[1,t+1]}) | \xi_{[1,t]}\} \\ \text{s.t.} & x_t \in \mathcal{X}_t(x_{t-1}, \xi_t). \end{aligned} \quad (20)$$

At the first stage we need to solve the problem

$$\text{Min}_{x_1 \in \mathcal{X}_1, r_2, \dots, r_T} r_2 + \dots + r_T + f_1(x_1) + \mathbb{E}[V_2(x_1, r_2, \dots, r_T, \xi_2)]. \quad (21)$$

Although it was possible to write dynamic programming equations for problem (18), please note that decision variables r_2, \dots, r_T are decided at the first stage and their optimal values depend on all scenarios starting at the root node at stage $t = 1$. Consequently optimal decisions at later stages depend on scenarios other than following a considered node, and hence formulation (18) (as well as (16)) is *not time consistent*.

As yet another example consider (cf., [1,10])

$$\varrho(Z_1, \dots, Z_T) := \rho \left(\max_{t=1, \dots, T} Z_t \right), \quad (22)$$

where $\rho : \mathcal{Z}_T \rightarrow \mathbb{R}$ is a coherent risk measure. It is not difficult to verify that the axioms of the coherent risk measure ρ imply the corresponding axioms for the multiperiod risk measure ϱ , and hence ϱ is a coherent risk measure as well. For this multiperiod risk measure ϱ the corresponding problem (13) does not satisfy the time consistency principle even if the coherent risk measure $\rho(\cdot) := \mathbb{E}(\cdot)$ is given by an expectation operator.

Example 2 (Conditional Risk Mappings). Let us discuss now an approach to risk averse formulation based on *conditional risk mappings* (cf., [11,13,5]). That is, consider the following nested formulation

$$\begin{aligned} \text{Min}_{x_1 \in \mathcal{X}_1} & f_1(x_1) + \rho_2 | \xi_{[1,1]} \left[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} f_2(x_2, \xi_2) + \dots \right. \\ & \left. + \rho_{T-1} | \xi_{[1, T-1]} \left[\inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} f_T(x_T, \xi_T) \right] \right], \end{aligned} \quad (23)$$

where $\rho_{t+1} | \xi_{[1,t]} : \mathcal{Z}_{t+1} \rightarrow \mathcal{Z}_t$ are conditional risk mappings (cf., [5]). Of course, for

$$\rho_{t+1} | \xi_{[1,t]}(Z_{t+1}) := \mathbb{E}[Z_{t+1} | \xi_{[1,t]}], \quad t = 1, \dots, T - 1, \quad (24)$$

the above nested problem (23) coincides with the risk neutral problem (9).

From the point of view of the nested formulation (23), two main properties of conditional risk mappings are the *monotonicity*: for any $Z_{t+1}, Z'_{t+1} \in \mathcal{Z}_{t+1}$ such that $Z_{t+1} \leq Z'_{t+1}$ it holds that

$$\rho_{t+1|\xi_{[1,t]}}(Z_{t+1}) \leq \rho_{t+1|\xi_{[1,t]}}(Z'_{t+1}), \tag{25}$$

and the *translation equivariance*: if $Z_{t+1} \in \mathcal{Z}_{t+1}$ and $Z_t \in \mathcal{Z}_t$, then

$$\rho_{t+1|\xi_{[1,t]}}(Z_{t+1} + Z_t) = \rho_{t+1|\xi_{[1,t]}}(Z_{t+1}) + Z_t. \tag{26}$$

Apart from these two properties, conditional risk mappings are assumed to satisfy conditions of convexity and positive homogeneity. However, the above monotonicity and translation equivariance properties alone allow us to write the corresponding dynamic programming equations (see below).

We have here

$$\mathbb{E}_t(Z_t, \dots, Z_T | \xi_{[1,t]}) := \tilde{\rho}_t[Z_t + \dots + Z_T | \xi_{[1,t]}], \tag{27}$$

where

$$\tilde{\rho}_t[\cdot | \xi_{[1,t]}] := \rho_{t+1|\xi_{[1,t]}} \circ \dots \circ \rho_{T|\xi_{[1,T-1]}}[\cdot] \tag{28}$$

are composite risk measures. Moreover, condition (A2) holds with

$$\begin{aligned} \phi_t(f_t(x_t, \xi_t), V_{t+1}(x_t, \xi_{[1,t+1]} | \xi_{[1,t]})) \\ := \rho_{t+1|\xi_{[1,t]}}[f_t(x_t, \xi_t) + V_{t+1}(x_t, \xi_{[1,t+1]})]. \end{aligned} \tag{29}$$

This formulation is time consistent and the corresponding dynamic programming equations are (see [5])

$$V_t(x_{t-1}, \xi_{[1,t]}) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \rho_{t+1|\xi_{[1,t]}}[f_t(x_t, \xi_t) + V_{t+1}(x_t, \xi_{[1,t+1]})]. \tag{30}$$

Nested formulation (23) can be written in the form

$$\begin{aligned} \text{Min}_{x_1, x_2, \dots, x_T} \quad & \tilde{\rho}_1[f_1(x_1) + f_2(x_2, \xi_2) + \dots + f_T(x_T, \xi_T)] \\ \text{s.t.} \quad & x_t \in \mathcal{X}_t, x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), t = 2, \dots, T, \end{aligned} \tag{31}$$

where $\tilde{\rho}_1 = \rho_{2|\xi_{[1,1]}} \circ \dots \circ \rho_{T|\xi_{[1,T-1]}}$ and optimization is performed over feasible policies $x_t = x_t(\xi_{[1,t]})$. If the conditional risk mappings are given as conditional expectations, of the form (24), then $\tilde{\rho}_1(\cdot) = \mathbb{E}(\cdot)$ is the expectation operator. Unfortunately, in general it is quite difficult to write the composite function $\tilde{\rho}_1(\cdot)$ explicitly.

Example 3 (Portfolio Selection). We discuss below an example of portfolio selection. Nested formulation of multistage portfolio selection can be written as

$$\begin{aligned} \text{Min} \quad & \rho_1[\dots \rho_{T-1}[\rho_T[W_T]]] \\ \text{s.t.} \quad & W_{t+1} = \sum_{i=1}^n \xi_{i,T+1} x_{i,t}, \sum_{i=1}^n x_{i,t} = W_t, x_t \geq 0, \\ & t = 0, \dots, T-1, \end{aligned} \tag{32}$$

where ρ_t are conditional risk mappings. Note that in order to formulate this as a minimization problem we changed the sign of ξ_{it} . Suppose that the random process ξ_t is *stagewise independent*, i.e., ξ_{t+1} is independent of $\xi_{[1,t]}$, $t = 1, \dots, T-1$. Let us write dynamic programming equations. At the last stage we have to solve problem

$$\begin{aligned} \text{Min}_{x_{T-1} \geq 0, W_T} \quad & \rho_T[W_T | W_{T-1}] \\ \text{s.t.} \quad & W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \sum_{i=1}^n x_{i,T-1} = W_{T-1}. \end{aligned} \tag{33}$$

Since W_{T-1} is a function of $\xi_{[T-1]}$, by the stagewise independence we have that ξ_T is independent of W_{T-1} . It follows by positive homogeneity of ρ_T that the optimal value of (33) is $V_{T-1}(W_{T-1}) = W_{T-1} \nu_{T-1}$, where ν_{T-1} is the optimal value of

$$\begin{aligned} \text{Min}_{x_{T-1} \geq 0, W_T} \quad & \rho_T[W_T] \\ \text{s.t.} \quad & W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \sum_{i=1}^n x_{i,T-1} = 1, \end{aligned} \tag{34}$$

and an optimal solution of (33) is $\bar{x}_{T-1}(W_{T-1}) = W_{T-1} x_{T-1}^*$, where x_{T-1}^* is an optimal solution of (34). And so on we obtain that the optimal policy here is myopic. Note that the composite risk measure $\rho_1[\dots \rho_{T-1}[\rho_T[\cdot]]]$ can be quite complicated.

The alternative approach is to write problem

$$\begin{aligned} \text{Min} \quad & \rho[W_T] \\ \text{s.t.} \quad & W_{t+1} = \sum_{i=1}^n \xi_{i,t+1} x_{i,t}, \sum_{i=1}^n x_{i,t} = W_t, x_t \geq 0, \\ & t = 0, \dots, T-1, \end{aligned} \tag{35}$$

for an explicitly defined (real valued) risk measure ρ . In particular, let $\rho := \text{CVaR}_\alpha$. Then problem (35) becomes

$$\begin{aligned} \text{Min} \quad & r + \alpha^{-1} \mathbb{E}[W_T - r]_+ \\ \text{s.t.} \quad & W_{t+1} = \sum_{i=1}^n \xi_{i,t+1} x_{i,t}, \sum_{i=1}^n x_{i,t} = W_t, x_t \geq 0, \\ & t = 0, \dots, T-1, \end{aligned} \tag{36}$$

where $r \in \mathbb{R}$ is the (additional) first stage decision variable.

The respective dynamic programming equations become as follows. The last stage value function $V_{T-1}(W_{T-1}, r)$ is given by the optimal value of problem

$$\begin{aligned} \text{Min}_{x_{T-1} \geq 0, W_T} \quad & \alpha^{-1} \mathbb{E}[W_T - r]_+ \\ \text{s.t.} \quad & W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \sum_{i=1}^n x_{i,T-1} = W_{T-1}. \end{aligned} \tag{37}$$

And so on, at stage $t = T-2, \dots, 1$, we consider problem

$$\begin{aligned} \text{Min}_{x_t \geq 0, W_{t+1}} \quad & \mathbb{E}\{V_{t+1}(W_{t+1}, r)\} \\ \text{s.t.} \quad & W_{t+1} = \sum_{i=1}^n \xi_{i,t+1} x_{i,t}, \sum_{i=1}^n x_{i,t} = W_t, \end{aligned} \tag{38}$$

whose optimal value is denoted $V_t(W_t, r)$. Finally, at stage $t = 0$ we solve the problem

$$\begin{aligned} \text{Min}_{x_0 \geq 0, W_1, r} \quad & r + \mathbb{E}[V_1(W_1, r)] \\ \text{s.t.} \quad & W_1 = \sum_{i=1}^n \xi_{i1} x_{i0}, \sum_{i=1}^n x_{i0} = W_0. \end{aligned} \tag{39}$$

In formulation (37) the objective function $\alpha^{-1} \mathbb{E}[W_T - r]_+$ can be viewed as a utility function (or rather disutility function since we formulate this as a minimization rather than maximization problem). However, note again that r there is a first stage decision variable and formulation (36) is not time consistent.

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