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First-Order Optimality Conditions in Generalized Semi-Infinite Programming[1]

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Abstract. In this paper, we consider a generalized semi-infinite optimization problem where the index set of the corresponding inequality constraints depends on the decision variables and the involved functions are assumed to be continuously differentiable. We derive first-order necessary optimality conditions for such problems by using bounds for the upper and lower directional derivatives of the corresponding optimal value function. In the case where the optimal value function is directly differentiable, we present first-order conditions based on the linearization of the given problem. Finally, we investigate necessary and sufficient first-order conditions by using the calculus of quasidifferentiable functions.

Key Words. Generalized semi-infinite programming, necessary and sufficient first-order optimality condition, optimal value function, directional differentiability, quasidifferentiability.

1. Introduction

In the present paper, we consider the following optimization problem:

$$\min f(x), \qquad \text{s.t.} \quad x \in S, \tag{1}$$

where the feasible set S is defined as

$$S := \{x \in \mathbb{R}^n | g(x, y) \leq 0, y \in Y(x)\},$$

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with

$$Y(x) := \{y \in \mathbb{R}^k | h_i(x, y) = 0, i = 1, \dots, p, h_j(x, y) \le 0, j = p+1, \dots, q\}.$$

Here, to each $y \in Y(x)$, there corresponds an inequality constraint of the original problem (1). We refer to the above problem as a generalized semi-infinite problem, whose difference from a standard semi-infinite problem is the x-dependence of the index set of the inequality constraints $Y(x)$. In recent years, (generalized) semi-infinite optimization has become an active field of research in applied mathematics. Various engineering problems lead to optimization problems of type (1), e.g., design problems (Refs. 1 and 2), time-minimal heating or cooling of a ball (Ref. 3), and reverse Chebyshev approximation (Refs. 4 and 5).

The goal of this paper is to provide necessary and occasionally sufficient first-order optimality conditions for problem (1). The results to be presented can be reformulated easily in the case where a finite number of additional equality constraints are taken into the definition of the feasible set S. However, this will not change the complexity of the problem; therefore, we have omitted equality constraints for the sake of simplicity.

Note that the feasible set can be written in the following equivalent form:

$$S = \{x \in \mathbb{R}^n | v(x) \le 0\},$$

where $v(x)$ is the optimal value function

$$v(x) := \sup_{y \in Y(x)} g(x, y). \tag{2}$$

Our investigation is closely related to the differentiability properties of the optimal value function.

The paper is organized as follows. In Section 2, some basic assumptions and results are given. Section 3 contains necessary optimality conditions for the original problem (1) based on various bounds for the directional derivatives of the optimal value function $v(x)$. In Section 4, we discuss several optimality conditions assuming that $v(x)$ is directionally differentiable and using the calculus of quasidifferentiable functions.

We mention related papers (Refs. 6 and 7) on first-order and second-order optimality conditions for problems of the type (1), as well as Ref. 8. Furthermore, we mention two survey papers (Refs. 9 and 10) as well as a book (Ref. 11) which contains, in tutorial form, several survey papers on (generalized) semi-infinite programming and its relations to other topics such as semidefinite programming, optimal control, wavelets, and others. Finally, we mention the close relation between generalized problems and semi-infinite problems, which depend additionally on a real parameter; see e.g. Refs. 12–14.

We use the following notation and terminology throughout the paper. By $\nabla_y g(x^0, y^0)$ we denote the gradient of g at (x^0, y^0) with respect to y; similarly $\nabla g(x^0, y^0)$ denotes the gradient of g as a function of (x, y), etc. For a real-valued function $f(x)$, we denote by

$$f'_+(x, d) := \limsup_{t \to 0^+} [f(x+td) - f(x)]/t,$$
$$f'_-(x, d) := \liminf_{t \to 0^+} [f(x+td) - f(x)]/t$$

its upper and lower directional derivatives, respectively. The function f is said to be directionally differentiable at a point x in the direction d, iff

$$f'_+(x, d) = f'_-(x, d).$$

For a set $\Xi \subset \mathbb{R}^n$, we denote by $\sigma(\cdot, \Xi)$ its support function,

$$\sigma(d, \Xi) := \sup_{\xi \in \Xi} \xi^T d,$$

and we denote by $\mathrm{conv}(\Xi)$ its convex hull.

2. Basic Results

Now, let us introduce some basic assumptions which we will use throughout the paper.

Assumption A1. The functions $f(x)$, $g(x, y)$, $h_i(x, y)$, $i = 1, \ldots, q$, are real valued and continuously differentiable.

Since we deal with first-order optimality conditions, we concentrate on only first-order differentiability properties of the optimal value function $v(x)$. In particular, we will use bounds for the upper $v'_+(x, d)$ and lower $v'_-(x, d)$ directional derivatives of the optimal value function. Continuity and differentiability properties of optimal value functions, of the form (2), have been studied extensively; see e.g. Ref. 15. For a recent survey of such results, the reader is referred to Ref. 16.

Throughout the paper, let $x^0 \in S$ be a considered feasible point.

Assumption A2. The sets $Y(x)$ are uniformly bounded in a neighborhood of x^0; i.e., there exist a neighborhood N of x^0 and a bounded set $T \subset \mathbb{R}^k$ such that $Y(x) \subset T$ for all $x \in N$.

Since the functions h_i, $i=1,\ldots,q$, are continuous, the (set-valued) mapping $x \mapsto Y(x)$ is closed-valued. Then, by Assumption (A2), the optimal value function $v(x)$ is upper semicontinuous at x^0. Consequently, if $v(x^0)<0$, then x^0 is an interior point of S. In the latter case, if x^0 is a local minimizer of (1), then the standard first-order necessary condition takes the form $\nabla f(x^0)=0$. Therefore, we assume from now on that $v(x^0)=0$ at the considered point $x^0 \in S$.

2.1. Lower-Level Problem. By Assumptions (A1) and (A2), the sets $Y(x)$ are compact for all x near x^0. Therefore the supremum in the right-hand side of (2) is attained; since $v(x^0)=0$, we have that the so-called set of active constraints,

$$Y_0(x^0) := \{y \in Y(x^0) | g(x^0, y)=0\}, \tag{3}$$

is nonempty. Clearly, $Y(x^0)$ is just the set of (global) maximizers of the following lower-level problem:

$$\max g(x^0, y), \qquad \text{s.t.} \quad y \in Y(x^0). \tag{4}$$

We associate with this problem the Lagrangian

$$L(x^0, y, \alpha) := g(x^0, y) - \sum_{i=1}^{q} \alpha_i h_i(x^0, y),$$

where $\alpha \in \mathbb{R}^q$.

2.2. Mangasarian–Fromovitz Constraint Qualification. We recall from Ref. 17 that the Mangasarian–Fromovitz constraint qualification (MFCQ) is said to hold at $y^0 \in Y(x^0)$ if the following conditions are satisfied:

(i) The vectors $\nabla_y h_i(x^0, y^0)$, $i=1,\ldots,p$, are linearly independent.
(ii) There exists a vector $w \in \mathbb{R}^k$ satisfying

$$w^T \nabla_y h_i(x^0, y^0)=0, \qquad i=1,\ldots,p,$$

$$w^T \nabla_y h_j(x^0, y^0)<0, \qquad j \in \{v | h_v(x^0, y^0)=0,\ v=p+1,\ldots,q\}.$$

It is well known (Refs. 18 and 19) that, for $y^0 \in Y_0(x^0)$, the corresponding set of Lagrange multiplier vectors,

$$A(x^0, y^0) := \{\alpha \in \mathbb{R}^q | \nabla_y L(x^0, y^0, \alpha)=0,\ \alpha_j \geq 0,\ j=p+1,\ldots,q,\ \alpha_j h_j(x^0, y^0)=0,\ j=p+1,\ldots,q\},$$

is nonempty and compact iff (MFCQ) holds at y^0.

Proposition 2.1. Suppose that Assumptions (A1) and (A2) hold and that the (MFCQ) is satisfied at every point $y \in Y_0(x^0)$. Then, for all $d \in \mathbb{R}^n$, the following inequalities hold:

$$\sup_{y \in Y_0(x^0)} \inf_{\alpha \in A(x^0, y)} d^T \nabla_x L(x^0, y, \alpha) \leq v'_-(x, d), \tag{5}$$

$$v'_+(x, d) \leq \sup_{y \in Y_0(x^0)} \sup_{\alpha \in A(x^0, y)} d^T \nabla_x L(x^0, y, \alpha). \tag{6}$$

The lower bound (5) is well-known (e.g., Refs. 20–22). The upper bound (6) is derived in Refs. 20 and 23; for an extension to an infinite-dimensional setting and a simple proof of that result, see Ref. 21.

3. Necessary First-Order Optimality Condition

By using the upper bound (6), the following theorem provides a necessary first-order optimality condition of the Fritz John type (cf. Ref. 24) for the original problem (1), which is similar to that obtained in Ref. 6.

Theorem 3.1. Let x^0 be a local minimizer of (1), and suppose that the (MFCQ) holds at every point $y \in Y_0(x^0)$. Then, there exist $y^l \in Y_0(x^0)$, $l = 1, \ldots, m$, $\alpha^l \in A(x^0, y^l)$, $l = 1, \ldots, m$, and multipliers $\lambda_l \geq 0$, $l = 0, \ldots, m$, such that $\sum_{l=0}^m \lambda_l > 0$ and

$$\lambda_0 \nabla f(x^0) + \sum_{l=1}^m \lambda_l \nabla_x L(x^0, y^l, \alpha^l) = 0. \tag{7}$$

Proof. Since x^0 is a local minimizer of (1) and $v(x^0) = 0$, the function

$$\psi(x) := \max\{f(x) - f(x^0), v(x)\} \tag{8}$$

attains its local minimum at x^0 with $\psi(x^0) = 0$. From local optimality of x^0, it follows that

$$\psi'_-(x^0, d) \geq 0, \qquad \text{for all } d \in \mathbb{R}^n.$$

Moreover, we have that

$$\psi'_-(x^0, d) = \max\{d^T \nabla f(x^0), v'_-(x^0, d)\}.$$

By (6), this implies that

$$\psi'_-(x^0, \cdot) \leq \sigma(\cdot, \Omega),$$

where $\sigma(\cdot, \Omega)$ is the support function of the set

$$\Omega := \mathrm{conv}(\{\nabla f(x^0), \nabla_x L(x^0, y, \alpha) | y \in Y_0(x^0), \alpha \in A(x^0, y)\}).$$

Consequently

$$\sigma(d, \Omega) \geq 0, \qquad \text{for all } d \in \mathbb{R}^n.$$

Since (MFCQ) holds at every $y \in Y_0(x^0)$, the set $A(x^0, y)$ is compact; thus, Ω is also compact, and hence is closed. By the well-known Farkas lemma, it follows that

$$\sigma(d, \Omega) \geq 0, \text{ for all } d \in \mathbb{R}^n, \qquad \text{iff } 0 \in \Omega;$$

note that $0 \notin \Omega$ implies the existence of $d^* \in \mathbb{R}^n$ with $\xi^T d^* < 0$ for all $\xi \in \Omega$. The latter fact implies (7), and hence the proof is complete □

The proof of Theorem 3.1 is based on the upper estimate (6). For every $y \in Y_0(x^0)$, if the corresponding vector of Lagrange multipliers is unique [i.e., $A(x^0, y)$ is a singleton], then the lower and upper bounds given in (5) and (6) coincide, and hence the optimal value function $v(x)$ is directionally differentiable at x^0. In that case, condition (7) is in a sense the best possible as a first-order necessary condition.

However, if $A(x^0, y)$ is not a singleton for each $y \in Y_0(x^0)$, then the bound (6) may give a crude upper estimate, since in that case, quite often, the optimal value function $v(x)$ is still directionally differentiable at x^0 and

$$v'(x^0, d) = \sup_{y \in Y_0(x^0)} \inf_{\alpha \in A(x^0, y)} d^T \nabla_x L(x^0, y, \alpha). \tag{9}$$

In particular, it is known that (9) holds in two cases.

The first case occurs when the lower-level problem (4) is convex [i.e. $g(x^0, \cdot)$ is concave, $h_i(x^0, \cdot)$, $i = 1, \ldots, p$, are linear, and $h_j(x^0, \cdot)$, $j = p+1, \ldots, q$, are convex, and hence $Y(x^0)$ is a convex set] and the Slater condition holds (Ref. 25). Note that, in such convex case, the set $A(x^0, y) \equiv A(x^0)$ does not depend on $y \in Y_0(x_0)$ and coincides with the set of optimal solutions of the dual of the lower-level problem.

The second case occurs when the (MFCQ) holds at every $y \in Y_0(x^0)$ and a certain strong form of the second-order sufficient conditions is satisfied, which ensures the Lipschitzian stability of the corresponding minimizers. Results of this type were obtained by various authors (Refs. 20–23 and 26). Somewhat surprisingly, it was discovered in Ref. 27 that, in the case where a strong form of the second-order sufficient conditions is not satisfied, the first-order directional derivatives of the optimal value function involve explicitly second-order derivatives of the corresponding functions (see Refs. 16 and 26 for a discussion of results of this type).

An interesting question is whether, for any choice of vectors $\alpha^l \in A(x^0, y^l)$, one can find multipliers λ_l such that the corresponding first-order optimality condition (7) holds. A counterexample to this is given in Ref. 6, so the answer to this question is negative. Interestingly enough, under an additional assumption of compactness of a certain set, such multipliers do exist.

Proposition 3.1. Let $x^0 \in S$ be a local minimizer of (1), and suppose that the optimal value function $v(x)$ is directionally differentiable at x^0 and that (9) holds. Furthermore, for each $y \in Y_0(x^0)$, choose an $\alpha(y) \in A(x^0, y)$ such that the set $\{\nabla_x L(x^0, y, \alpha(y)), y \in Y_0(x^0)\}$ is compact. Then, there exist $y^l \in Y_0(x^0)$, $l = 1, \ldots, m$, and multipliers $\lambda_l \leq 0$, $l = 0, \ldots, m$ such that $\sum_{l=0}^m \lambda_l > 0$ and

$$\lambda_0 \nabla f(x^0) + \sum_{l=1}^m \lambda_l \nabla_x L(x^0, y^l, \alpha(y^l)) = 0. \tag{10}$$

Proof. By (9), for any $d \in \mathbb{R}^n$, we obtain

$$0 \leq \psi'(x^0, d) = \max\left\{d^T \nabla f(x^0), \sup_{y \in Y_0(x^0)} \inf_{\alpha \in A(x^0, y)} d^T \nabla_x L(x^0, y, \alpha)\right\}$$
$$\leq \sup\{d^T \nabla f(x^0), d^T \nabla_x L(x^0, y, \alpha(y)), y \in Y_0(x^0)\}.$$

Since $\{\nabla_x L(x^0, y, \alpha(y)), y \in Y_0(x^0)\}$ is a compact set, its convex hull is also compact, and hence is closed. A similar argument as in the proof of Theorem 3.1 implies (10). □

The following example illustrates that, in Proposition 3.1, the assumption of compactness of the set $\{\nabla_x L(x^0, y, \alpha(y)), y \in Y_0(x^0)\}$ cannot be deleted without any substitute, although the (MFCQ) holds at every $y \in Y(x^0)$. The reason for such pathological behavior in the example is that the set $\{\nabla_x L(x^0, y, \alpha(y)), y \in Y_0(x^0)\}$ is bounded, but not closed.

Example 3.1. Consider the following generalized semi-infinite problem

$$\min \; f(x_1, x_2) = -x_1, \quad \text{s.t.} \quad x \in S, \tag{11}$$
$$S := \{x \in \mathbb{R}^2 | y_2 \leq 0, y \in Y(x)\},$$
$$Y(x) := \{y \in \mathbb{R}^2 | h_1(x, y) = y_2 - x_1 - x_2 y_1 \leq 0,$$
$$h_2(x, y) = y_2 - y_1^2 - x_2 \leq 0,$$
$$h_3(x, y) = y_1 \leq 2, h_4(x, y) = -y_1 \leq 2,$$
$$h_5(x, y) = y_2 \leq 4, h_6(x, y) = -y_2 \leq 4\}.$$

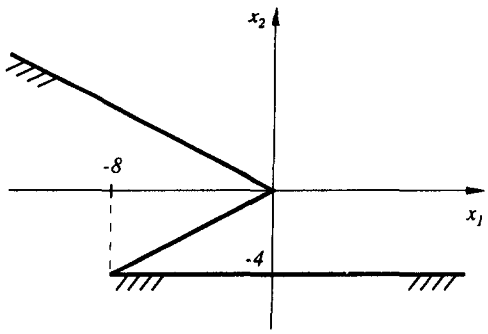

Fig. 1. The feasible set S.

A short calculation shows that (see Fig. 1)

$$S=\{x\in\mathbb{R}^2 | x_2\leq -4\}\cup\{x\in\mathbb{R}^2 | -4<x_2\leq 0,\ x_2\geq (1/2)x_1\}\\ \cup\{x\in\mathbb{R}^2 | x_2\geq 0,\ x_2\leq -(1/2)x_1\}$$

and that $x^0=(0,0)$ is a local minimizer of (11). Furthermore, we have that

$$Y_0(x_0)=\{(y_1,0) | -2\leq y_1\leq 2\},$$

$$A(x^0,(y_1,0))=\begin{cases}\{(\mu,1-\mu,0,0,0,0),\ \mu\in[0,1]\}, & \text{for } y_1=0,\\ \{(1,0,0,0,0,0)\}, & \text{for } y_1\in[-2,2]\backslash\{0\},\end{cases}$$

and for $y\in Y_0(x^0)$,

$$\nabla_x L(x^0,y,\alpha)=\begin{cases}(\mu,1-\mu), & \text{for } y=(0,0),\ \alpha=(\mu,1-\mu,0,0,0,0),\\ (1,y_1), & \text{for } y=(y_1,0),\ y_1\neq 0,\ \alpha=(1,0,0,0,0,0).\end{cases}$$

Now, it is easy to see that only for

$$y^1=(0,0)\quad\text{and}\quad \alpha(y^1)=(1,0,0,0,0,0),$$

i.e., for

$$\mu=1\quad\text{and}\quad \alpha(y)=(1,0,0,0,0,0),\ y\in Y_0(x^0)),$$

there exists a combination as in (10),

$$\lambda_0\nabla f(x^0)+\lambda_1\nabla_x L(x^0,y^1,\alpha(y^1))=\lambda_0\begin{pmatrix}-1\\0\end{pmatrix}+\lambda_1\begin{pmatrix}1\\0\end{pmatrix}=0.$$

Note that the (MFCQ) is satisfied here at every $y\in Y_0(x^0)$ and that $v(\cdot)$ is directionally differentiable at x^0 with (9) holding.

Clearly, if the set $Y_0(x^0)$ of active constraints is finite, then the compactness assumption of Proposition 3.1 always holds. This implies the following result.

Proposition 3.2. Let x^0 be a local minimizer of (1). Suppose that $Y_0(x^0)=\{y^1, \dots, y^m\}$ is finite and that $v(\cdot)$ is directionally differentiable and (9) holds. Then:

(i) For any choice of $\alpha^l \in A(x^0, y^l)$, $l=1, \dots, m$, there exist multipliers $\lambda_l \geq 0$, $l=0, 1, \dots, m$, such that $\sum_{l=0}^{m} \lambda_l > 0$ and

$$\lambda_0 \nabla f(x^0) + \sum_{l=1}^{m} \lambda_l \nabla_x L(x^0, y^l, \alpha^l) = 0. \tag{12}$$

(ii) If $Y_0(x^0) = \{y_0\}$ is a singleton and there exist $\alpha^1, \alpha^2 \in A(x^0, y^0)$ such that $\nabla_x L(x^0, y^0, \alpha^1)$ and $\nabla_x L(x^0, y^0, \alpha^2)$ are linearly independent, then $\nabla f(x^0) = 0$.

Proof. Assertion (i) follows from Proposition 3.1. Moreover, if $Y_0(x^0)$ is a singleton, then (12) holds for both α^1 and α^2. Since $\nabla_x L(x^0, y^0, \alpha^1)$ and $\nabla_x L(x^0, y^0, \alpha^2)$ are linearly independent, and hence nonzero, we have that the corresponding multiplier of $\nabla f(x^0)$ is nonzero in both cases. The conclusion then follows from (12). □

Propositions 3.1 and 3.2 show that it is possible to derive a family of first-order necessary conditions of the form (10), each one for an appropriate choice of multipliers of the lower-level problem. In the next section, we approach the problem from a somewhat different point of view by using the calculus of quasidifferentiable functions.

4. First-Order Conditions in the Form of Set Inclusion

In the remainder of this paper, we assume that the optimal value function $v(\cdot)$ is directionally differentiable at the considered point x^0 and that formula (9) holds. Let us consider the following linearization of (1) at x^0:

$$\min \quad d^T \nabla f(x^0), \qquad \text{s.t.} \quad d \in S^1, \tag{13}$$

$$S^1 := \{d \in \mathbb{R}^n \mid v'(x^0, d) \leq 0\}.$$

Note that, by (9), we have

$$S^1 = \left\{ d \in \mathbb{R}^n \,\middle|\, \inf_{\alpha \in A(x^0, y)} d^T \nabla_x L(x^0, y, \alpha) \le 0, \ \forall y \in Y_0(x^0) \right\}.$$

Then, under a constraint qualification, a first-order necessary optimality condition for (1) takes the following form:

$d^0 = 0$ is a local minimizer of (13).

For example, one can use the constraint qualification

$$\{d \in \mathbb{R}^n | v'(x^0, d) \le 0\} = \mathrm{cl}\{d \in \mathbb{R}^n | v'(x^0, d) < 0\}, \tag{14}$$

where cl denotes the topological closure.

Another approach, which does not require a constraint qualification, is to linearize the function ψ defined in (8). That is, if x^0 is a local minimizer of (1), then

$$\psi'(x^0, d) \ge 0, \qquad \text{for all } d \in \mathbb{R}^n.$$

Since

$$\psi'(x^0, d) = \max\{d^T \nabla f(x^0), v'(x^0, d)\},$$

this condition is equivalent to the property that either the optimal value of the problem

$$\inf d^T \nabla f(x^0), \qquad \text{s.t.} \quad d \in S^2, \tag{15}$$

is zero, where

$$\begin{aligned} S^2 &:= \{d \in \mathbb{R}^n | v'(x^0, d) < 0\} \\ &= \left\{ d \in \mathbb{R}^n \,\middle|\, \inf_{\alpha \in A(x^0, y)} d^T \nabla_x L(x^0, y, \alpha) < 0, \ \forall y \in Y_0(x^0) \right\}, \end{aligned}$$

or $S^2 = \emptyset$. If $v'(x^0, \cdot)$ is continuous, problems (13) and (15) are equivalent if the constraint qualification (14) holds.

The principal difference between standard and generalized semi-infinite problems is that the feasible sets S^1 and S^2 of the above linearizations of the generalized problem (1) need not be convex. In order to see this difference, suppose for a moment that $Y_0(x^0) = \{y^0\}$ is a singleton, while the corresponding set $A(x^0, y^0)$ is not a singleton. Then, S^1 is the union of half spaces,

$$S^1 = \bigcup_{\alpha \in A(x^0, y^0)} \{d \in \mathbb{R}^n | d^T \nabla_x L(x^0, y^0, \alpha) \le 0\}.$$

Consequently, if $d^0=0$ is a local minimizer of (13), then d^0 is also a local minimizer of the following problem:

$$\min d^T\nabla f(x^0), \qquad \text{s.t.} \quad \{d\in\mathbb{R}^n | d^T\nabla_x L(x^0, y^0, \alpha)\leq 0\},$$

for each $\alpha\in A(x^0, y^0)$; therefore, $\nabla f(x^0)$ and $\nabla_x L(x^0, y^0, \alpha)$ have to be linearly dependent for each $\alpha\in A(x^0, y^0)$. It follows that either $\nabla f(x^0)=0$ or $\nabla_x L(x^0, y^0, \alpha)$ is a multiple of $\nabla f(x^0)$ for each $\alpha\in A(x^0, y^0)$. This is another way of deriving statement (ii) of Proposition 3.2.

Example 4.1. Consider the feasible set

$$S:=\{x\in\mathbb{R}^2 | y_2\leq 0, y\in Y(x)\}$$

of problem (1) with

$$Y(x):=\{y\in\mathbb{R}^2 | h_1(x, y)=y_2+\epsilon x_1-x_2\leq 0,$$
$$h_2(x, y)=y_2+y_1^2-x_2\leq 0, h_3(x, y)=-y_2\leq 4\}$$

and $\epsilon\in\mathbb{R}$. Obviously, S is the union of two half spaces (cf. Fig. 2) and can be written as

$$S=\{x\in\mathbb{R}^2 | \min\{x_2, x_2-\epsilon x_1\}\leq 0\}$$
$$=\{x\in\mathbb{R}^2 | x_2\leq 0\}\cup\{x\in\mathbb{R}^2 | x_2-\epsilon x_1\leq 0\}.$$

For $x^0=0$, we obtain

$$Y_0(x^0)=\{y^0\}, \qquad y^0=(0, 0),$$

$$A(x^0, y^0)=\{(\mu, 1-\mu, 0), \mu\in[0, 1]\},$$

$$\nabla_x L(x^0, y^0, (\mu, 1-\mu, 0))=(-\mu\epsilon, 1),$$

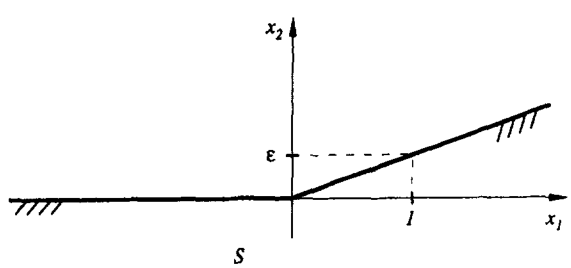

Fig. 2. S is the union of two half spaces.

and the feasible set

$$S^1 = \left\{ d \in \mathbb{R}^2 \,\middle|\, \inf_{\mu \in [0,1]} (-\mu\epsilon, 1)d \leq 0 \right\}$$

of the corresponding linearized problem (13), where we have $S = S^1$ (after substituting the coordinates d by x).

Now, it is easily seen from Fig. 2 that, if $\epsilon \neq 0$ and $\nabla f(x^0) \neq 0$, then there exists a $d \in S$ with $d^T \nabla f(0) < 0$; hence, $x^0 = 0$ cannot be a local minimizer of (1). This is not so surprising in view of Proposition 3.2(ii): if $\epsilon \neq 0$, μ^1, $\mu^2 \in [0, 1]$, and $\mu^1 \neq \mu^2$, then the vectors

$$\nabla_x L(x^0, y^0, (\mu^1, 1-\mu^1, 0)) = (-\mu^1\epsilon, 1),$$
$$\nabla_x L(x^0, y^0, (\mu^2, 1-\mu^2, 0)) = (-\mu^2\epsilon, 1)$$

are linearly independent.

Finally, note that (MFCQ) holds at y^0 and $v(\cdot)$ is directionally differentiable at x^0 with formula (9) holding.

Examples 3.1 and 4.1 illustrate the fact that the feasible set S of a generalized semi-infinite problem can be the union of sets having the property that each of them can be described as a feasible set of a standard problem. This is why, in many cases, a generalized problem can be written as a so-called disjunctive optimization problem; for more details, we refer to Ref. 28, where a critical point theory and optimality conditions for such problems are investigated.

4.1. Optimality Conditions Based on the Calculus of Quasidifferentiable Functions. In general, it is possible to derive first-order conditions based on linearizations (13) or (15) by using the calculus of quasidifferentiable functions (see Ref. 29). Suppose that the set $Y_0(x^0)$ is finite, say

$$Y_0(x^0) = \{y^1, \ldots, y^m\},$$

where y^l, $l = 1, \ldots, m$, are pairwise disjoint, and that (MFCQ) holds at each y^l, $l = 1, \ldots, m$. By (9), we obtain

$$v'(x^0, d) = \max\{-\sigma(d, B_l), l = 1, \ldots, m\},$$

$$B_l = -\text{conv}\left(\bigcup_{\alpha \in A(x^0, y^l)} \{\nabla_x L(x^0, y^l, \alpha)\} \right), \qquad l = 1, \ldots, m.$$

Since the (MFCQ) holds at y^l, the sets $A(x^0, y^l)$ are compact; hence, the sets B_l are compact. It follows that (cf. Ref. 29)

$$v'(x^0, d) = \sigma(d, C_1) - \sigma(d, C_2),$$

where

$$C_1 = \operatorname{conv}\left(\bigcup_{l=1}^{m}\left(\sum_{\nu\neq l} B_\nu\right)\right), \qquad C_2 = B_1 + \cdots + B_m.$$

Then, we obtain the following necessary and sufficient first-order optimality conditions.

Theorem 4.1. See Refs. 29 and 30. Suppose that, for some $x^0 \in S$, $Y_0(x^0) = \{y^1, \ldots, y^m\}$, that (9) holds, and that the (MFCQ) is satisfied at each y^l, $l = 1, \ldots, m$. Then:

(i) If x^0 is a local minimizer of (1),

$$C_2 \subset \operatorname{conv}(C_1 \cup (\nabla f(x^0) + C_2)).$$

(ii) If

$$C_2 \subset \operatorname{int}(\operatorname{conv}(C_1 \cup (\nabla f(x^0) + C_2))), \tag{16}$$

then x^0 is a local minimizer of (1), where int denotes the set of interior points.

Note again that, under the assumptions of Theorem 4.1, both sets C_1 and C_2 are compact; therefore, we do not need to take the closure of the above convex hulls. Furthermore, the first-order sufficient condition (16) simply means that

$$\psi'(x^0, d) > 0, \qquad \text{for all } d \in \mathbb{R}^n.$$

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