Math. Program., Ser. B 108, 571–595 (2006)

Digital Object Identifier (DOI) 10.1007/s10107-006-0723-7

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Solving multistage asset investment problems by the sample average approximation method

Received: June 22, 2004 / Accepted: August 26, 2005

Abstract. The vast size of real world stochastic programming instances requires sampling to make them practically solvable. In this paper we extend the understanding of how sampling affects the solution quality of multistage stochastic programming problems. We present a new heuristic for determining good feasible solutions for a multistage decision problem. For power and log-utility functions we address the question of how tree structures, number of stages, number of outcomes and number of assets affect the solution quality. We also present a new method for evaluating the quality of first stage decisions.

Key words. Stochastic programming – Asset allocation – Monte Carlo sampling – SAA method – Statistical bounds

1. Introduction

To fully model the complex nature of decision problems, optimization models should in principle contain stochastic components. Extensive research have been done within the field of stochastic programming to design solvers that can handle problems where the uncertainty is described in a tree structure. Birge and Louveaux [3] and Rusczyński [16] give a good overview of different solution methods. More recent work includes, for example, [5–7, 19] where primal and primal-dual interior point methods have been developed that can solve problems with more than 2 stages and non-linear objectives.

The asset allocation problem is a frequently used stochastic programming model. For an introduction of the model see, e.g., [12]. An excellent overview of relevant research, where the model have been applied, can be found in [13], and more recent work can be found, e.g., in [8, 9, 2]. The model usually comes in two flavors, with and without transaction costs (the latter is a special case of the former). There may be also some other particularities, however, we only study the basic model.

We address several important aspects which are inherent in stochastic programming by studying the asset allocation model. We also test the practical applicability of determining upper and lower bounds for multistage problems. For power and log-utility functions, with and without transaction costs, and also for piecewise linear and exponential utility functions, we address how tree structures, number of stages, number of outcomes and number of assets affect the solution quality.

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2. Model

We consider an investment problem with set $\mathcal{A}$ of assets. For time periods $t = 1, \dots, T$, the investor would like to determine the optimal amount of units u_t^a of each asset $a \in \mathcal{A}$ to buy/sell. The total units x_t^a, of asset a at time t, are governed by the recursive equations $x_t^a = x_{t-1}^a + u_{t-1}^a$, $t = 2, \dots, T$, where u_{t-1}^a can be positive or negative depending on buying or selling asset a. By x_t^c and c_t^a we denote the amount in cash and the price of asset a, respectively, at time t, and by $R = 1 + r$ where r is the interest rate. Note that $c_t^a \geq 0$. We assume that $c_t = (c_t^a)_{a\in\mathcal{A}}$ forms a random process with a known probability distribution. For the sake of simplicity we assume that the interest rate r recived in each time stage is fixed.

Given the initial units of assets $a \in \mathcal{A}$, the initial amount in cash, and utility function $U(\cdot)$, the objective is to maximize the expected utility of wealth, at the final time stage T. Neither short selling assets nor borrowing is allowed. This can be formulated as the following optimization problem

$$\text{Max} \quad \mathbb{E}\left[U\left(x_T^c + \sum_{a\in\mathcal{A}} c_T^a x_T^a\right)\right] \tag{1}$$

$$\text{s.t.} \quad x_t^a = x_{t-1}^a + u_{t-1}^a, \quad t = 2, \dots, T, \quad a \in \mathcal{A}, \tag{2}$$

$$x_t^c = \left(x_{t-1}^c - \sum_{a\in\mathcal{A}} c_{t-1}^a u_{t-1}^a\right) R, \quad t = 2, \dots, T, \tag{3}$$

$$x_t^a \geq 0, \quad t = 2, \dots, T, \ a \in \mathcal{A}, \tag{4}$$

$$x_t^c \geq 0, \quad t = 2, \dots, T. \tag{5}$$

Note that constraints (4) and (5) correspond to "not short selling assets" and "not borrowing" policies, respectively, and ensure nonnegative wealth. This will be especially important later on when sampled versions of (1–5) will be considered. If these constraints were left out, then the optimal solution to a sample version of (1–5) might be infeasible in the original problem. We assume that the utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a continuous concave increasing function.

To allow for transaction costs in the model the buy/sell decision, u_t^a, have to be split into two variables; one for the buy decision, u_t^{ab}, and one for the sell decision u_t^{as}. The proportional transaction cost is denoted τ. Thus the income from selling an asset is now $(1-\tau)c_t^a$ and the cost of buying is $(1+\tau)c_t^a$ for some $\tau \in (0, 1)$. This gives the modified constraint (8) in the following formulation of the corresponding optimization problem

$$\text{Max} \quad \mathbb{E}\left[U\left(x_T^c + \sum_{a\in\mathcal{A}} c_T^a x_T^a\right)\right] \tag{6}$$

$$\text{s.t.} \quad x_t^a = x_{t-1}^a + u_{t-1}^{ab} - u_{t-1}^{as}, \quad t = 2, \dots, T, \quad a \in \mathcal{A}, \tag{7}$$

$$x_t^c = \left(x_{t-1}^c + \sum_{a\in\mathcal{A}} \left(\tau^s c_{t-1}^a u_{t-1}^{as} - \tau^b c_{t-1}^a u_{t-1}^{ab}\right)\right) R, \quad t = 2, \dots, T, \tag{8}$$

$$x_t^a \geq 0, \quad t = 2, \dots, T, \quad a \in \mathcal{A}, \tag{9}$$

$$x_t^c \geq 0, \quad t = 2, \dots, T, \tag{10}$$

$$u_t^{ab}, u_t^{as} \geq 0, \quad t = 1, \ldots, T-1, \quad a \in \mathcal{A}, \tag{11}$$

where $\tau^s := 1 - \tau$ and $\tau^b := 1 + \tau$.

Consider the asset investment model with transaction costs. By bold script, like $\boldsymbol{c}_t^a$, we denote random variables, while c_t^a denotes a particular realization of the corresponding random variable. For the sake of simplicity we assume that the random process $\boldsymbol{c}_t = (\boldsymbol{c}_t^a)_{a \in \mathcal{A}}$, $t = 2, \ldots, T$, is *Markovian*. We also assume that $x_t = (x_t^c, x_t^a)_{a \in \mathcal{A}}$ satisfy linear constraints $\ell_i(x_t) \geq 0$, $i \in \mathcal{I}$, where $\mathcal{I}$ is a finite index set and

$$\ell_i(x_t) := \alpha_i^c x_t^c + \sum_{a \in \mathcal{A}} \alpha_i^a x_t^a, \quad i \in \mathcal{I}.$$

For example, we can set $\ell_a(x_t) := x_t^a$ and $\ell_c(x_t) := x_t^c$, with $\mathcal{I} := \mathcal{A} \cup \{c\}$, which introduce constraints (9) and (10) into the problem.

Let us define the following cost-to-go functions. At the period $T-1$ the corresponding cost-to-go function $Q_{T-1}(x_{T-1}, c_{T-1})$ is given by the optimal value of the problem

$$\begin{aligned}
\underset{u_{T-1}, x_T}{\text{Max}} \quad & \mathbb{E}\left[U\left(x_T^c + \sum_{a \in \mathcal{A}} \boldsymbol{c}_T^a x_T^a \right) \middle| \boldsymbol{c}_{T-1} = c_{T-1} \right] \\
\text{subject to} \quad & x_T^a = x_{T-1}^a + u_{T-1}^{ab} - u_{T-1}^{as}, \;\; a \in \mathcal{A}, \\
& x_T^c = \left(x_{T-1}^c + \sum_{a \in \mathcal{A}} \left(\tau^s c_{T-1}^a u_{T-1}^{as} - \tau^b c_{T-1}^a u_{T-1}^{ab} \right) \right) R, \\
& u_{T-1}^{ab} \geq 0, \;\; u_{T-1}^{as} \geq 0, \;\; a \in \mathcal{A}, \\
& \ell_i(x_T) \geq 0, \;\; i \in \mathcal{I}.
\end{aligned} \tag{12}$$

Here $\mathbb{E}\left[\cdot \middle| \boldsymbol{c}_t = c_t \right]$ denotes the conditional expectation given $\boldsymbol{c}_t = c_t$.

For $t = T-2, \ldots, 1$, the corresponding cost-to-go function $Q_t(x_t, c_t)$ is defined as the optimal value of the problem

$$\begin{aligned}
\underset{u_t, x_{t+1}}{\text{Max}} \quad & \mathbb{E}\left[Q_{t+1}(x_{t+1}, \boldsymbol{c}_{t+1}) \middle| \boldsymbol{c}_t = c_t \right] \\
\text{subject to} \quad & x_{t+1}^a = x_t^a + u_t^{ab} - u_t^{as}, \;\; a \in \mathcal{A}, \\
& x_{t+1}^c = \left(x_t^c + \sum_{a \in \mathcal{A}} \left(\tau^s c_t^a u_t^{as} - \tau^b c_t^a u_t^{ab} \right) \right) R, \\
& u_t^{ab} \geq 0, \;\; u_t^{as} \geq 0, \;\; a \in \mathcal{A}, \\
& \ell_i(x_{t+1}) \geq 0, \;\; i \in \mathcal{I}.
\end{aligned} \tag{13}$$

The optimal decision vector $u_1 = (u_1^{ab}, u_1^{as})_{a \in \mathcal{A}}$ is obtained by solving the problem

$$\begin{aligned}
\underset{u_1, x_2}{\text{Max}} \quad & \mathbb{E}\left[Q_2(x_2, \boldsymbol{c}_2) \right] \\
\text{subject to} \quad & x_2^a = x_1^a + u_1^{ab} - u_1^{as}, \;\; a \in \mathcal{A}, \\
& x_2^c = \left(x_1^c + \sum_{a \in \mathcal{A}} \left(\tau^s c_1^a u_1^{as} - \tau^b c_1^a u_1^{ab} \right) \right) R, \\
& u_1^{ab} \geq 0, \;\; u_1^{as} \geq 0, \;\; a \in \mathcal{A}, \\
& \ell_i(x_2) \geq 0, \;\; i \in \mathcal{I}.
\end{aligned} \tag{14}$$

Note that at the first stage, vector x_1 is given and $(c_1^a)_{a \in \mathcal{A}}$ are known.

In the numerical experiments we assume that the asset prices $\boldsymbol{c}_t^a$ follow a geometric Brownian motion. That is,

$$\ln \boldsymbol{c}_t^a = \ln \boldsymbol{c}_{t-1}^a + \mu^a \Delta t + \sigma^a (\Delta t)^{1/2} \boldsymbol{\zeta}_t^a \ t = 2, \dots, T, \quad a \in \mathcal{A}, \tag{15}$$

where random vectors $\boldsymbol{\zeta}_t = (\boldsymbol{\zeta}_t^a)_{a \in \mathcal{A}}, t = 2, \dots, T$, have normal distribution $N(0, \Sigma)$ with $\operatorname{Var}(\boldsymbol{\zeta}_t^a) = 1, a \in \mathcal{A}$, and correlations $r_{a_1 a_2} = \mathbb{E}[\boldsymbol{\zeta}_t^{a_1} \boldsymbol{\zeta}_t^{a_2}]$, and the random process $\boldsymbol{\zeta}_t$ is *between stages independent* (i.e., random vectors $\boldsymbol{\zeta}_t, t = 2, \dots, T$, are mutually independent). Note that it follows from (15) and the between stages independence of $\boldsymbol{\zeta}_t$, that the process $\boldsymbol{\xi}_t^a := (\boldsymbol{c}_t^a / \boldsymbol{c}_{t-1}^a)_{a \in A}$ is also between stages independent.

3. Myopic policies

In practical applications quantities of interest usually are optimal values of *first* stage decision variables only. In some situations, in order to obtain optimal values of first stage decision variables, one does not really need to solve the corresponding *multi*-stage problem (see, e.g., [10, 1]). This is what we investigate in this section for this particular stochastic programming application.

Suppose that $\tau^s = \tau^b = 1$, i.e., that there are no transaction costs. In that case we can use control variables $u_t^a := u_t^{ab} - u_t^{as}$ and write the dynamic equations of (13) in the form

$$u_t^a = x_{t+1}^a - x_t^a \ \text{ and } \ R^{-1} x_{t+1}^c + \sum_{a \in \mathcal{A}} c_t^a x_{t+1}^a = W_t,$$

where $W_t := x_t^c + \sum_{a \in \mathcal{A}} c_t^a x_t^a$ is the wealth at stage t. Let us make the following change of variables:

$$y_{t+1}^a := c_t^a x_{t+1}^a, \quad y_{t+1}^c := R^{-1} x_{t+1}^c \ \text{ and } \ \xi_{t+1}^a := c_{t+1}^a / c_t^a.$$

Note that this change of variables transforms the functions $\ell_i(x_{t+1})$ into the functions

$$l_i\left(y_{t+1}, c_t\right) = \left(R\alpha_i^c\right) y_{t+1}^c + \sum_{a \in \mathcal{A}} \left(\alpha_i^a / c_t^a\right) y_{t+1}^a, \quad i \in \mathcal{I},$$

which are linear in y_{t+1}. We then can formulate problem (12) in the form

$$\begin{array}{ll} \operatorname*{Max}_{y_T} & \mathbb{E}\left[U\left(R y_T^c + \sum_{a \in \mathcal{A}} \boldsymbol{\xi}_T^a y_T^a \right) \middle| \boldsymbol{\xi}_{T-1} = \xi_{T-1} \right] \\ \text{subject to} & y_T^c + \sum_{a \in \mathcal{A}} y_T^a = W_{T-1}, \\ & l_i\left(y_T, c_{T-1}\right) \geq 0, \ i \in \mathcal{I}. \end{array} \tag{1}$$

Let us denote by $\widetilde{Q}_{T-1}(W_{T-1}, \xi_{T-1})$ the optimal value of problem (1). Note that

$$Q_{T-1}\left(x_{T-1}, c_{T-1}\right) = \widetilde{Q}_{T-1}\left(x_{T-1}^c + \sum_{a \in \mathcal{A}} c_{T-1}^a x_{T-1}^a, \xi_{T-1} \right). \tag{2}$$

By continuing this process backward in time, for $t = T - 2, \dots, 1$, we obtain that

$$Q_t\,(x_t, c_t) = \widetilde{Q}_t \left(x_t^c + \sum_{a \in \mathcal{A}} c_t^a x_t^a, \xi_t \right), \tag{3}$$

where $\widetilde{Q}_t\,(W_t, \xi_t)$ is the optimal value of the problem

$$\operatorname*{Max}_{y_{t+1}} \mathbb{E}\left[\widetilde{Q}_{t+1} \left(R y_{t+1}^c + \sum_{a \in \mathcal{A}} \boldsymbol{\xi}_{t+1}^a y_{t+1}^a, \boldsymbol{\xi}_{t+1} \right) \middle| \boldsymbol{\xi}_t = \xi_t \right] \tag{4}$$

$$\text{s.t.}\;\; y_{t+1}^c + \sum_{a \in \mathcal{A}} y_{t+1}^a = W_t, \tag{5}$$

$$l_i(y_{t+1}, c_t) \geq 0,\;\; i \in \mathcal{I}. \tag{6}$$

Note that at the first stage the wealth $W_1 := x_1^c + \sum_{a \in \mathcal{A}} c_1^a x_1^a$ and asset prices c_1^a are known.

Consider the set of vectors y_{t+1} satisfying constraints (5)–(6):

$$\mathcal{U}_t(W_t, \xi_t) := \left\{ y_{t+1} : y_{t+1}^c + \sum_{a \in \mathcal{A}} y_{t+1}^a = W_t,\; l_i(y_{t+1}, c_t) \geq 0,\; i \in \mathcal{I} \right\}. \tag{7}$$

Let us note that, since the constraints (5)–(6) are linear, the set $\mathcal{U}_t(W_t, \xi_t)$, $t = T - 1, \dots, 1$, is positively homogeneous with respect to W_t, i.e.,

$$\mathcal{U}_t(\alpha W_t, \xi_t) = \alpha\, \mathcal{U}_t(W_t, \xi_t) \quad \text{for any } \alpha > 0. \tag{8}$$

Note also that the feasible set of problem (4)–(6) should satisfy the implicit constraint

$$\mathbb{E}\left[\widetilde{Q}_{t+1} \left(R y_{t+1}^c + \sum_{a \in \mathcal{A}} \boldsymbol{\xi}_{t+1}^a y_{t+1}^a, \boldsymbol{\xi}_{t+1} \right) \middle| \boldsymbol{\xi}_t = \xi_t \right] > -\infty. \tag{9}$$

Consider now the log-utility function $U(z) := \log z$ if $z > 0$ and $U(z) := -\infty$ if $z \leq 0$. We then have that

$$U(\alpha z) = \log \alpha + U(z) \tag{10}$$

for any $\alpha > 0$ and $z > 0$. Since $\mathcal{U}_{T-1}(W_{T-1}, \xi_{T-1})$ is positively homogeneous with respect to W_{T-1} and because of (10), it follows that the set $\mathcal{S}_{T-1}(W_{T-1}, \xi_{T-1})$ of optimal solutions of (1) is also positively homogeneous with respect to W_{T-1}, and for any $W_{T-1} > 0$,

$$\widetilde{Q}_{T-1}(W_{T-1}, \xi_{T-1}) = \widetilde{Q}_{T-1}(1, \xi_{T-1}) + \log W_{T-1}. \tag{11}$$

Consequently,

$$\begin{aligned}\widetilde{Q}_{T-2}(W_{T-2}, \xi_{T-2}) = \mathbb{E}\big[\widetilde{Q}_{T-1}(1, \boldsymbol{\xi}_{T-1})\big|\boldsymbol{\xi}_{T-2} = \xi_{T-2}\big] \\ +\mathcal{Q}_{T-2}(W_{T-2}, \xi_{T-2}),\end{aligned} \tag{12}$$

where $\mathcal{Q}_{T-2}(W_{T-2}, \xi_{T-2})$ is the optimal value of the problem

$$\begin{array}{ll}\underset{y_{T-1}}{\text{Max}} & \mathbb{E}\left[\log\left(R y_{T-1}^c + \sum_{a\in\mathcal{A}} \boldsymbol{\xi}_{T-1}^a y_{T-1}^a\right)\middle|\boldsymbol{\xi}_{T-2} = \xi_{T-2}\right] \\ \text{subject to} & y_{T-1}^c + \sum_{a\in\mathcal{A}} y_{T-1}^a = W_{T-2}, \\ & l_i(y_{T-1}, c_{T-2}) \geq 0, \;\; i \in \mathcal{I}.\end{array} \tag{13}$$

Again we have that

$$\mathcal{Q}_{T-2}(W_{T-2}, \xi_{T-2}) = \mathcal{Q}_{T-2}(1, \xi_{T-2}) + \log W_{T-2}. \tag{14}$$

And so forth, for $W_t > 0$,

$$\widetilde{Q}_t(W_t, \xi_t) = \sum_{\tau=t}^{T-1} \mathbb{E}\big[\mathcal{Q}_\tau(1, \boldsymbol{\xi}_\tau)\big|\boldsymbol{\xi}_t = \xi_t\big] + \log W_t, \tag{15}$$

where $\mathcal{Q}_{T-1}(W_{T-1}, \xi_{T-1}) = \widetilde{Q}_{T-1}(W_{T-1}, \xi_{T-1})$, and $\mathcal{Q}_t(W_t, \xi_t)$ is the optimal value of

$$\begin{array}{ll}\underset{y_{t+1}}{\text{Max}} & \mathbb{E}\left[\log\left(R y_{t+1}^c + \sum_{a\in\mathcal{A}} \boldsymbol{\xi}_{t+1}^a y_{t+1}^a\right)\middle|\boldsymbol{\xi}_t = \xi_t\right] \\ \text{subject to} & y_{t+1}^c + \sum_{a\in\mathcal{A}} y_{t+1}^a = W_t, \\ & l_i(y_{t+1}, c_t) \geq 0, \;\; i \in \mathcal{I},\end{array} \tag{16}$$

for $t = T-2, \dots, 1$. Note that if random vectors $\boldsymbol{\xi}_t$ and $\boldsymbol{\xi}_{t+1}$ are independent, then $\mathcal{Q}_t(W_t, \xi_t)$ does not depend on ξ_t.

It follows that the optimal value v^* of the corresponding (true) multistage problem is given by (recall that $\xi_1 = c_1$ and is not random)

$$v^* = \log W_1 + \sum_{t=1}^{T-1} \mathbb{E}\big[\mathcal{Q}_t(1, \boldsymbol{\xi}_t)\big], \tag{17}$$

and first stage optimal solutions are obtained by solving the problem

$$\begin{array}{ll} \underset{y_2}{\text{Max}} & \mathbb{E}\left[\log\left(R y_2^c + \sum_{a\in\mathcal{A}} \boldsymbol{\xi}_2^a y_2^a\right)\right] \\ \text{subject to} & y_2^c + \sum_{a\in\mathcal{A}} y_2^a = W_1, \\ & l_i(y_2, c_1) \geq 0,\ i \in \mathcal{I}. \end{array} \tag{18}$$

We obtain the following result.

Proposition 1. *Suppose that there are no transaction costs and let $U(\cdot)$ be the log-utility function. Then:* (i) *the optimal value v^*, of the multistage problem, is given by formula* (17)*,* (ii) *the set of optimal solutions of the first stage problem* (14) *depends only on the distribution of $\boldsymbol{c}_2$ (and is independent of realizations of the random data at the following stages $t = 3, \ldots, T$), and can be obtained by solving problem* (18)*.*

Proof. If $\bar{y}_2 = (\bar{y}_2^c, \bar{y}_2^a)_{a\in\mathcal{A}}$ is an optimal solution of problem (18), then

$$\bar{u}_1^a := (c_1^a)^{-1}\bar{y}_a^2 - x_a^1, \qquad a \in \mathcal{A}, \tag{19}$$

gives the corresponding optimal solution of the first stage problem (14). Clearly the set of optimal solutions of (18) does not depend on the distribution of $\boldsymbol{c}_3, \ldots, \boldsymbol{c}_T$. □

Remark 1. As it was mentioned above, if the process $\boldsymbol{\xi}_t$ is between stages independent, then the optimal value $\mathcal{Q}_t(W_t, \xi_t)$, of problem (16), does not depend on ξ_t and will be denoted $\mathcal{Q}_t(W_t)$. In that case formula (17) becomes

$$v^* = \log W_1 + \sum_{t=1}^{T-1} \mathcal{Q}_t(1). \tag{20}$$

Consider now the power utility function $U(z) \equiv z^\gamma/\gamma$, with $\gamma \leq 1$, $\gamma \neq 0$ (in that case $U(z) := -\infty$ for $z \leq 0$ if $\gamma < 0$, and $U(z) := -\infty$ for $z < 0$ if $0 < \gamma < 1$). Suppose that for $W_{T-1} = 1$ problem (1) has an optimal solution $\bar{y}_T$. (The following equation (22) can be proved without this assumption by considering an ε-optimal solution, we assumed existence of the optimal solution in order to simplify the presentation.) Because of the positive homogeneity of $\mathcal{U}_{T-1}(\cdot, \xi_{T-1})$ and since $U(\alpha z) = \alpha^\gamma U(z)$ for $\alpha > 0$, we then have that $W_{T-1}\bar{y}_T$ is an optimal solution of (1) for any $W_{T-1} > 0$. Then

$$\begin{aligned} \widetilde{Q}_{T-1}(W_{T-1}, \xi_{T-1}) &= \mathbb{E}\left[U\left(W_{T-1}\left(R\bar{y}_T^c + \sum_{a\in\mathcal{A}} \boldsymbol{\xi}_T^a \bar{y}_T^a\right)\right)\middle|\boldsymbol{\xi}_{T-1} = \xi_{T-1}\right] \\ &= W_{T-1}^\gamma \mathbb{E}\left[U\left(R\bar{y}_T^c + \sum_{a\in\mathcal{A}} \boldsymbol{\xi}_T^a \bar{y}_T^a\right)\middle|\boldsymbol{\xi}_{T-1} = \xi_{T-1}\right] \\ &= W_{T-1}^\gamma \widetilde{Q}_{T-1}(1, \xi_{T-1}). \end{aligned} \tag{21}$$

Suppose, further, that the random process $\boldsymbol{\xi}_t$ is between stages independent. Then, because of the independence of $\boldsymbol{\xi}_T$ and $\boldsymbol{\xi}_{T-1}$, we have that the conditional expectation in (1) is independent of $\boldsymbol{\xi}_{T-1}$, and hence $\widetilde{Q}_{T-1}(1, \xi_{T-1})$ does not depend on ξ_{T-1}. Consequently, we obtain by (21) that for any $W_{T-1} > 0$,

$$\widetilde{Q}_{T-1}\left(W_{T-1}, \xi_{T-1}\right) = W_{T-1}^{\gamma} \widetilde{Q}_{T-1}(1), \tag{22}$$

where $\widetilde{Q}_{T-1}(1)$ is the optimal value of (1) for $W_{T-1} = 1$. And so forth for $t = T - 2, \ldots, 1$ and $W_t > 0$,

$$\widetilde{Q}_t(W_t, \xi_t) = W_t^{\gamma} \widetilde{Q}_t(1). \tag{23}$$

Consider problems

$$\begin{aligned} \underset{y_{t+1}}{\text{Max}} \quad & \mathbb{E}\left[U\left(Ry_{t+1}^c + \sum_{a \in \mathcal{A}} \boldsymbol{\xi}_{t+1}^a y_{t+1}^a\right)\right] \\ \text{subject to} \quad & y_{t+1}^c + \sum_{a \in \mathcal{A}} y_{t+1}^a = W_t, \\ & l_i(y_{t+1}, c_t) \geq 0, \ i \in \mathcal{I}. \end{aligned} \tag{24}$$

We obtain the following results.

Proposition 2. *Suppose that there are no transaction costs and the random process* $(\boldsymbol{\xi}_t^a = \boldsymbol{c}_t^a / \boldsymbol{c}_{t-1}^a)_{a \in \mathcal{A}}$, $t = 2, \ldots, T$, *is between stages independent, and let* $U(\cdot)$ *be the power utility function for some* $\gamma \leq 1, \gamma \neq 0$. *Then the set of optimal solutions of the first stage problem* (14) *depends only on the distribution of* $\boldsymbol{\xi}_2$ *(and is independent of realizations of* $\boldsymbol{\xi}_3, \ldots, \boldsymbol{\xi}_T$*) and can be obtained by solving problem* (24) *for* $t = 1$, *and*

$$v^* = W_1^{\gamma} \prod_{t=1}^{T-1} \mathcal{Q}_t(1), \tag{25}$$

where $\mathcal{Q}_t(W_t)$ *is the optimal value of the problem* (24).

Remark 2. Formula (25) shows a ‘multiplicative’ behavior of the optimal value when a power utility function is used. This can be compared with an ‘additive’ behavior (see (17) and (20)) for the log-utility function. Let us also remark that the assumption of the between stages independence of the process $\boldsymbol{\xi}_t$ is essential in the above Proposition 2. It is possible to give examples where the myopic properties of optimal solutions do not hold for the power utility functions (even for $\gamma = 1$) for stage dependent processes $\boldsymbol{\xi}_t$. This is in contrast with the log-utility function where the between stages independence of $\boldsymbol{\xi}_t$ is not needed. Let us also note that the assumption of “no transaction costs” is essential for the above myopic properties to hold.

4. Solving MSP by Monte Carlo sampling

We use the following approach of conditional Monte Carlo sampling (cf., [17]). Let $\mathcal{N} = \{N_1, \ldots, N_{T-1}\}$ be a sequence of positive integers. At the first stage, N_1 replications of the random vector $\boldsymbol{c}_2$ are generated. These replications do not need to be (stochastically) independent, it is only required that each replication has the same probability distribution as $\boldsymbol{c}_2$. Then conditional on every generated realization of $\boldsymbol{c}_2$, N_2 replications of $\boldsymbol{c}_3$ are generated, and so forth for the following stages. In that way a scenario tree is generated with the total number of scenarios $N = \prod_{t=1}^{T-1} N_t$. Once such scenario tree is generated, we can view this scenario tree as a random process with N possible realizations (sample paths), each with equal probability $1/N$. Consequently, we can associate with a generated scenario tree the optimization problem (6–11). We refer to the obtained problem, associated with a generated sample, as the (multi-stage) sample average approximation (SAA) problem.

Provided that the sample size N is not too large, the generated SAA problem can be solved to optimality. The optimal value, denoted $\hat{v}_{\mathcal{N}}$, and *first stage* optimal solutions of the generated SAA problem give approximations for their counterparts of the "true" problem (6–11). (By "true" we mean the corresponding problem with the originally specified distribution of the random data). Note that the optimal value $\hat{v}_{\mathcal{N}}$ and optimal solutions of the SAA problem depend on the generated random sample, and therefore are random. It is possible to show that, under mild regularity conditions, the SAA estimators are consistent in the sense that they converge with probability one to their true counterparts as the sample sizes N_t, $t = 1, \ldots, T-1$, tend to infinity (cf., [17]).

4.1. Upper statistical bounds

It is well known that

$$v^* \leq \mathbb{E}[\hat{v}_{\mathcal{N}}], \tag{1}$$

where v^* denotes the optimal value of the true problem (recall that here we solve a maximization rather than a minimization problem). This gives a possibility of calculating an upper statistical bound for the true optimal value v^*. This idea was suggested in Norkin, Pflug and Ruszczyński [14], and developed in Mak, Morton and Wood [11] for two-stage stochastic programming.

That is, SAA problems are solved (to optimality) M times for independently generated samples each of size $\mathcal{N} = \{N_1, \ldots, N_{T-1}\}$. Let $\hat{v}_{\mathcal{N}}^1, \ldots, \hat{v}_{\mathcal{N}}^M$ be calculated optimal values of the generated SAA problems. We then have that

$$\bar{v}_{\mathcal{N},M} := M^{-1} \sum_{j=1}^{M} \hat{v}_{\mathcal{N}}^j \tag{2}$$

is an unbiased estimator of $\mathbb{E}[\hat{v}_{\mathcal{N}}]$, and hence $v^* \leq \mathbb{E}[\bar{v}_{\mathcal{N},M}]$. The sample variance of $\bar{v}_{\mathcal{N},M}$ is

$$\hat{\sigma}^2_{\mathcal{N},M} := \frac{1}{M(M-1)} \sum_{j=1}^{M} \left(\hat{v}_{\mathcal{N}}^j - \bar{v}_{\mathcal{N},M} \right)^2. \tag{3}$$

This leads to the following (approximate) $100(1-\alpha)\%$ confidence upper bound on $\mathbb{E}[\hat{v}_{\mathcal{N}}]$, and hence (because of (1)) for v^*:

$$\bar{v}_{\mathcal{N},M} + t_{\alpha,\nu}\hat{\sigma}_{\mathcal{N},M}, \tag{4}$$

where $\nu = M - 1$. It should be noted that there is no reason to believe that random numbers $\hat{v}^j_{\mathcal{N}}$ have a normal (or even symmetric) distribution, even approximately, for large values of the sample size N. Of course, by the Central limit Theorem, the distribution of the average $\bar{v}_{\mathcal{N},M}$ approaches normal as M tends to infinity. Since the sample size M in the following experiments is not large, we use in (4) more conservative critical values from Student's t, rather than standard normal, distribution. One can even take slightly larger critical values in (4) to make a correction for possibly nonsymmetrical distribution of $\hat{v}^j_{\mathcal{N}}$.

Suppose now that for a given (feasible) first stage decision vector $\bar{u}_1$, and the corresponding vector $\bar{x}_2$ satisfying the equations of problem (14), we want to evaluate the value $\mathbb{E}[Q_2(\bar{x}_2, \boldsymbol{c}_2)]$ of the true problem. By using the developed methodology we can calculate an upper statistical bound for $\mathbb{E}[Q_2(\bar{x}_2, \boldsymbol{c}_2)]$ in two somewhat different ways. One, rather simple, approach is to add the constraint $x_2 = \bar{x}_2$ to the corresponding optimization problem and to use the above methodology.

Another approach can be described as follows. First, generate random sample $c_2^1, \dots, c_2^{N_1}$, of size N_1, of the random vector $\boldsymbol{c}_2$. For $\bar{x}_2$ and each c_2^j, $j = 1, \dots, N_1$, approximate the corresponding $(T-1)$-stage problem by independently generated, conditionally on $\boldsymbol{c}_2 = c_2^j$, (with a chosen sample size $(N_2, \dots, N_{T-1})$) SAA problems M times. Let $\hat{v}^{j,m}$, $j = 1, \dots, N_1$, $m = 1, \dots, M$, be the optimal values of these SAA problems, and

$$\bar{\bar{v}}_{N_1,M} := \frac{1}{MN_1}\sum_{j=1}^{N_1}\sum_{m=1}^{M}\hat{v}^{j,m}. \tag{5}$$

We have that

$$Q_2(\bar{x}_2, c_2^j) \le \mathbb{E}\left[\hat{v}^{j,m} \middle| \boldsymbol{c}_2 = c_2^j\right], \quad j = 1, \dots, N_1, \ m = 1, \dots, M, \tag{6}$$

and hence (viewing c_2^j as random variables)

$$\mathbb{E}[Q_2(\bar{x}_2, \boldsymbol{c}_2)] = N_1^{-1}\sum_{j=1}^{N_1}\mathbb{E}\left[Q_2(\bar{x}_2, c_2^j)\right] \le \mathbb{E}[\bar{\bar{v}}_{N_1,M}]. \tag{7}$$

We can estimate the variance of $\bar{\bar{v}}_{N_1,M}$ as follows. Recall that if X and Y are random variables, then

$$\mathrm{Var}(Y) = \mathbb{E}[\mathrm{Var}(Y|X)] + \mathrm{Var}[\mathbb{E}(Y|X)], \tag{8}$$

where $\mathrm{Var}(Y|X) = \mathbb{E}[(Y - \mathbb{E}(Y|X))^2|X]$. By applying this formula we can write

$$\mathrm{Var}(\hat{v}^{j,m}) = \mathbb{E}\left[\mathrm{Var}(\hat{v}^{j,m}|c_2^j)\right] + \mathrm{Var}\left[\mathbb{E}(\hat{v}^{j,m}|c_2^j)\right]. \tag{9}$$

Consequently, we can estimate the variance of $\bar{\bar{v}}_{N_1,M}$ by

$$\hat{\sigma}^2_{N_1,M} := \frac{1}{N_1 M(M-1)} \sum_{j=1}^{N_1} \sum_{m=1}^{M} \left(\hat{v}^{j,m} - \bar{\hat{v}}^j\right)^2 + \frac{1}{N_1(N_1-1)} \sum_{j=1}^{N_1} \left(\bar{\hat{v}}^j - \bar{\bar{v}}_{N_1,M}\right)^2, \tag{10}$$

where $\bar{\hat{v}}^j := M^{-1} \sum_{m=1}^{M} \hat{v}^{j,m}$.

This leads to the following (approximate) $100(1-\alpha)\%$ confidence upper bound on $\mathbb{E}[Q_2(\bar{x}_2, \boldsymbol{c}_2)]$:

$$\bar{\bar{v}}_{N_1,M} + z_\alpha \hat{\sigma}_{N_1,M}. \tag{11}$$

Note that here we use the critical value z_α from standard normal, rather than t, distribution since the total number $N_1 M$ of used variables is large.

At the first glance it seems that the second approach could be advantageous since there we need to solve $(T-1)$-stage problems as compared with solving T-stage problems in the first approach. It turned out, however, in our numerical experiments that the second approach involved too large variances to be practically useful.

4.2. *First stage solutions*

Consider the model without transaction costs and with log-utility function. In that case the problem is myopic, and optimal first stage decision variables $\bar{u}^a_1$ are given by $\bar{u}^a_1 = \bar{x}^a_2 - \bar{x}^a_1$ and $\bar{x}^a_2 = \bar{y}^a_2 / c^a_1$, $a \in \mathcal{A}$, where $\bar{y}^a_2$ are optimal solutions of the problem (18). Therefore, if one is interested only in optimal first stage decisions, the corresponding multistage problem effectively is reduced to a two-stage problem. Consequently the accuracy (rate of convergence) of the SAA estimates of optimal first stage decision variables depends on the sample size N_1 while is independent of the following sample sizes $N_2, \ldots, N_{T-1}$. Similar conclusions hold in the case of a power utility function and between stages independence of the process $\boldsymbol{\xi}_t$.

4.3. *Statistical properties of the upper bounds*

In this section we discuss statistical properties of the upper bounds introduced in section 4.1. By (1) we have that $\hat{v}_{\mathcal{N}}$ is a biased upwards estimator of the optimal value v^* of the true problem. In particular, we investigate how the corresponding bias behaves for different sample sizes and number of stages.

Let us consider the case without transaction costs and with log-utility function. Recall that conditional on a sample point ξ_t, at stage t, we generate a random sample $\xi^j_{t+1} = (\xi^{a,j}_{t+1})_{a\in\mathcal{A}}$, $j = 1, \ldots, N_t$, of size N_t, of $\boldsymbol{\xi}_{t+1}$. We have then that, for

$W_t = 1$, the optimal value $\mathcal{Q}_t(1, \xi_t)$, of problem (16) is approximated by the optimal value $\hat{\mathcal{Q}}_{t,N_t}(1, \xi_t)$ of the problem

$$\begin{array}{ll} \underset{y_{t+1}}{\text{Max}} & \dfrac{1}{N_t} \displaystyle\sum_{j=1}^{N_t} U\left(R y_{t+1}^c + \sum_{a \in \mathcal{A}} \xi_{t+1}^{a,j} y_{t+1}^a \right) \\ \text{subject to} & y_{t+1}^c + \displaystyle\sum_{a \in \mathcal{A}} y_{t+1}^a = 1, \\ & l_i(y_{t+1}, c_t) \geq 0, \ i \in \mathcal{I}, \end{array} \tag{12}$$

with $U(z) \equiv \log z$. The difference

$$B_{t,N_t}(\xi_t) := \mathbb{E}\left[\hat{\mathcal{Q}}_{t,N_t}(1, \xi_t)\right] - \mathcal{Q}_t(1, \xi_t) \tag{13}$$

represents the bias of this sample estimate conditional on $\boldsymbol{\xi}_t = \xi_t$. We have that

$$\mathbb{E}\left[\hat{\mathcal{Q}}_{t,N_t}(1, \xi_t)\right] \geq \mathcal{Q}_t(1, \xi_t), \tag{14}$$

and hence $B_{t,N_t}(\xi_t) \geq 0$.

At stage t there are $\mathcal{N}_t = \prod_{\tau=1}^{t} N_\tau$ realizations of $\boldsymbol{\xi}_t$, denoted ξ_t^j, $j \in \mathcal{J}_t$, with $|\mathcal{J}_t| = \mathcal{N}_t$. We then have (compare with (17)) that

$$\hat{v}_{\mathcal{N}} = W_1 + \sum_{t=1}^{T-1} \left(\frac{1}{\mathcal{N}_t} \sum_{j \in \mathcal{J}_t} \hat{\mathcal{Q}}_{t,N_t}(1, \xi_t^j) \right). \tag{15}$$

The bias of $\bar{v}_{\mathcal{N},M}$ is equal to the bias of $\hat{v}_{\mathcal{N}}$ and is given by

$$\mathbb{E}[\bar{v}_{\mathcal{N},M}] - v^* = \sum_{t=1}^{T-1} \left(\frac{1}{\mathcal{N}_t} \sum_{j \in \mathcal{J}_t} \mathcal{B}_{t,N_t}(1, \xi_t^j) \right). \tag{16}$$

The situation simplifies further if we assume that the process $\boldsymbol{\xi}_t$ is between stages *independent*. Then the optimal values $\mathcal{Q}_t(1, \xi_t)$ do not depend on ξ_t, $t = 1, \ldots, T-1$, and hence $B_{t,N_t}(\xi_t) = B_{t,N_t}$ also do not depend on ξ_t. Consequently in such case

$$\mathbb{E}[\bar{v}_{\mathcal{N},M}] - v^* = \sum_{t=1}^{T-1} B_{t,N_t}. \tag{17}$$

It follows that under the above assumptions and for constant sample sizes N_t, the bias $\mathbb{E}[\bar{v}_{\mathcal{N},M}] - v^*$ grows *linearly* with the number of stages.

Also because of the additive structure of the bias, given by the right hand side of (17), it is possible (in the considered case) to study asymptotic behavior of the bias by investigating asymptotics of each component B_{t,N_t} with increase of the sample size N_t. This reduces such analysis to a two-stage situation. We may refer to [18] for a discussion of asymptotics of statistical estimators in two-stage stochastic programming.

The variance of $\bar{v}_{\mathcal{N},M}$ depends on a way how conditional samples are generated. Suppose that the process $\boldsymbol{\xi}_t$ is between stages independent. Under this assumption, we can use the following two strategies. We can use the same sample ξ_{t+1}^j, $j = 1, \ldots, N_t$, for every sample point ξ_t at stage t. Alternatively, we can generate independent samples conditional on sample points at stage t. In both cases the bias $\mathbb{E}[\hat{v}_{\mathcal{N}}] - v^*$ is the same, and is equal to the right hand side of (17). Because of the between stages independence assumption, the variances $\mathrm{Var}\left(\hat{\mathcal{Q}}_{t,N_t}(1, \xi_t^j)\right)$ do not depend on $j \in \mathcal{J}_t$, and will be denoted $\mathrm{Var}\left[\hat{\mathcal{Q}}_{t,N_t}\right]$. For independently generated samples, we have that all $\hat{\mathcal{Q}}_{t,N_t}\left(1, \xi_t^j\right)$, $j \in \mathcal{J}_t$, are mutually independent and hence

$$\mathrm{Var}\left(\hat{v}_{\mathcal{N}}\right) = \sum_{t=1}^{T-1} \left(\frac{\mathrm{Var}\left[\hat{\mathcal{Q}}_{t,N_t}\right]}{\mathcal{N}_t} \right). \tag{18}$$

On the other hand for conditional samples which are generated the same, we have

$$\mathrm{Var}\left(\hat{v}_{\mathcal{N}}\right) = \sum_{t=1}^{T-1} \mathrm{Var}\left[\hat{\mathcal{Q}}_{t,N_t}\right]. \tag{19}$$

Consider now the power utility function $U(z) \equiv z^\gamma/\gamma$, with $\gamma \leq 1$, $\gamma \neq 0$. Assume the "no transaction costs" model and the between stages independence condition. By Proposition 2 we have that

$$\hat{v}_{\mathcal{N}} = W_1^\gamma \prod_{t=1}^{T-1} \left(\frac{1}{\mathcal{N}_t} \hat{\mathcal{Q}}_{t,N_t}(1, \xi_t^j) \right), \tag{20}$$

where $\hat{\mathcal{Q}}_{t,N_t}(1, \xi_t^j)$ is the optimal value of problem (12) for the considered utility function. Also because of the between stages independence condition we have that

$$\mathbb{E}\left[\hat{v}_{\mathcal{N}}\right] = W_1^\gamma \prod_{t=1}^{T-1} \mathbb{E}\left[\frac{1}{\mathcal{N}_t} \hat{\mathcal{Q}}_{t,N_t}\left(1, \xi_t^j\right) \right] = W_1^\gamma \prod_{t=1}^{T-1} \left(\mathcal{Q}_t(1) + B_{t,N_t} \right), \tag{21}$$

where B_{t,N_t} is defined the same way as in the above. It follows that

$$\begin{aligned} \mathbb{E}\left[\bar{v}_{\mathcal{N},M}\right] - v^* &= W_1^\gamma \prod_{t=1}^{T-1} \left(\mathcal{Q}_t(1) + B_{t,N_t} \right) - W_1^\gamma \prod_{t=1}^{T-1} \mathcal{Q}_t(1) \\ &= v^* \prod_{t=1}^{T-1} \left(1 + \frac{B_{t,N_t}}{\mathcal{Q}_t(1)} \right). \end{aligned} \tag{22}$$

For the power utility function, the above formula suggests a 'multiplicative' behavior of the bias with growth of the number of stages. Of course, for 'small' $B_{t,N_t}/\mathcal{Q}_t(1)$ and 'not too' large T, we can use the approximation

$$\prod_{t=1}^{T-1} \left(1 + \frac{B_{t,N_t}}{\mathcal{Q}_t(1)} \right) \approx 1 + \sum_{t=1}^{T-1} \frac{B_{t,N_t}}{\mathcal{Q}_t(1)},$$

which suggests an approximately additive behavior of the bias for a small number of stages T.

4.4. Lower statistical bounds

In order to compute a valid lower statistical bound one needs to construct an implementable and feasible policy. Given a policy of feasible decisions yielding the wealth W_T, we have that

$$\mathbb{E}\left[U\left(W_T\right)\right] \le v^*. \tag{23}$$

(Note that the expectation in the left hand side of (23) is taken with respect to the considered policy. We suppress this in the notation for the sake of notational simplicity.) By using Monte Carlo simulations, it is straightforward to construct an *unbiased* estimator of $\mathbb{E}\left[U\left(W_T\right)\right]$. That is, a random sample of N' realizations of the considered random process is generated and $\mathbb{E}\left[U\left(W_T\right)\right]$ is estimated by the corresponding average

$$\underline{v}_{N'} := \frac{1}{N'}\sum_{j=1}^{N'} U\left(W_T^j\right) \tag{24}$$

(cf., [18, p. 403]). Since $\mathbb{E}\left[\underline{v}_{N'}\right] = \mathbb{E}\left[U\left(W_T\right)\right]$, we have that $\underline{v}_{N'}$ gives a valid lower statistical bound for v^*. Of course, quality of this lower bound depends on the quality of the corresponding feasible policy. The sample variance of $\underline{v}_{N'}$ is

$$\underline{\sigma}_{N'}^2 = \frac{1}{N'(N'-1)}\sum_{j=1}^{N'}\left[U\left(W_T^j\right) - \underline{v}_{N'}\right]^2. \tag{25}$$

This leads to the following (approximate) $100(1-\alpha)\%$ confidence lower bound on $\mathbb{E}\left[U\left(W_T\right)\right]$:

$$\underline{v}_{N'} - z_\alpha \underline{\sigma}_{N'}. \tag{26}$$

The sample size N' used in numerical experiments is large, therefore we use the critical value z_α from the standard normal distribution.

We will now study two different approaches to determine feasible decisions. The SAA counterpart of the "true" optimization problem (6–11) can be formulated as

$$\operatorname{Max} \sum_{i\in\mathcal{I}} U_i(x_i, u_i) \tag{27}$$

$$\text{s.t. } x_i = A_i x_{i\_} + B_i u_{i\_} + b_i \tag{28}$$

$$C_i x_i + D_i u_i = d_i \tag{29}$$

$$E_i x_i + F_i u_i \ge e_i. \tag{30}$$

Denote $\{x_i^*, u_i^*\}_{i\in\mathcal{I}}$ as the optimal solution, and let $\mathcal{I}_t$ denote the nodes that correspond to stage t. In node i the state of the stochastic parameters is $\xi_t \in \mathbb{R}^{|A|}$. We want to find a feasible decision, u_j, to node $j \notin \mathcal{I}$ with the state x_j, ξ_j.

It is difficult to find a decision u_j that is both good and feasible. We therefore divide the heuristics into two steps. First, we determine a target solution that is assumed to be good u^t_j, then a feasible solution, u_j, is determined by solving

$$\operatorname*{Min}_{u_j} \ \tfrac{1}{2}\|u^t_j - u_j\|^2 \tag{31}$$

$$\text{s.t.} \ x^l_{j+} \leq A_j x_j + B_j u_j + b_j \leq x^u_{j+} \tag{32}$$

$$C_j x_j + D_j u_j = d_j \tag{33}$$

$$E_j x_j + F_j u_j \geq e_j, \tag{34}$$

where x^l_{j+} and x^u_{j+} is the lower and upper bound for the state in the next stage and $\|\cdot\|$ denotes the Euclidean norm. We will next describe two heuristics for determining the target decision.

A common idea in stochastic programming is to reduce a scenario tree by merging nodes with similar states of the stochastic parameters (an approach to such scenario reduction in a certain optimal way is discussed in [4], for example), thus giving the same decision in these merged nodes. In a similar fashion we will use a decision from a similar node in the new node. To get a good decision we will however also have to consider the state of the variables, x_j. Define a distance between nodes in the optimal tree and the new node as $c_i = \|x^*_i - x_j\|^2 + \|\xi_i - \xi_j\|^2$. The closest decision is now $u^t_j = u^*_k$, where $k = \arg\min_{i\in\mathcal{I}_t}\{c_i\}$. We denote this as the *closest state*.

By only using the closest node to determine the decision, much of the information in the optimal decisions is lost. There usually exist many nodes that are on approximately the same distance. The quality of the decision in each node can also be very bad, since nodes in later stages usually have relatively few successors. Considering these two properties we will determine the target decision as an affine combination of the decisions in the other nodes in the same time period t, $u_j = \sum_{i\in\mathcal{I}_t}\lambda_i u^*_i$. λ is determined by solving,

$$\operatorname*{Min}_{\lambda_i} \ \sum_{i\in\mathcal{I}_t} c_i\lambda_i^2 \tag{35}$$

$$\text{s.t.} \ \sum_{i\in\mathcal{I}_t}\lambda_i\xi_i = \xi_j, \tag{36}$$

$$\sum_{i\in\mathcal{I}_t}\lambda_i x^*_i = x_j, \tag{37}$$

$$\sum_{i\in\mathcal{I}_t}\lambda_i = 1, \tag{38}$$

where $c_i = \|x^*_i - x_j\|^2 + \|\xi_i - \xi_j\|^2$. This problem can be reformulated as

$$\operatorname*{Min}_{\lambda} \ \tfrac{1}{2}\lambda^T C\lambda \tag{39}$$

$$\text{s.t.} \ A\lambda = b, \tag{40}$$

where C is a diagonal matrix. The optimal solution $\lambda^* = C^{-1}A^T(A^TC^{-1}A)^{-1}b$ can be determined with $O(nm^2+m^3)$ operations, where $n = |\mathcal{I}_t|$ and $m = m_\xi + m_x + 1$. The target decision is defined as $u_j^t = \sum_{i \in \mathcal{I}_t} \lambda_i^* u_i^*$. This method is denoted *affine interpolation*.

5. Numerical results

We will study three different types of utility functions namely the logarithmic, piecewise linear and the exponential, Figure 1. Solving multistage optimization problems where the logarithmic utility function is used gives us the possibility to study the results in a setting where the true optimum can be estimated by solving a two-stage model (section 3). The multistage problems are solved with the primal interior point solver developed in [6].

To generate outcomes for one particular node, $\bar{\xi}_t^a$ is sampled with Latin Hypercube sampling. With the cholesky factorization of the correlation matrix $C = LL^T$, the correlated stochastic parameter can be determined as $\xi_t = L\bar{\xi}_t$, where $\bar{\xi}_t = \left(\bar{\xi}_t^a\right)_{a \in \mathcal{A}}$ and $\xi_t = \left(\xi_t^a\right)_{a \in \mathcal{A}}$. Given ξ_t^a and initial asset prices c_i^a, asset prices $c_{i_+}^a$ are computed with (15). The scenario tree is generated by applying this approach to generate asset prices recursively, starting from the root node.

In the numerical experiments it has been assumed that all assets are uncorrelated, and that they have the same expected return, $\mu^a = 0.1$, and standard deviation, $\sigma^a = 0.2$. To justify that this assumption does not have any major impact on the results, we conclude the tests with an experiment where the expected return, volatility, and correlation are random. The yearly interest rate is 2% and each time period is 6 months ($\Delta t = 0.5$). The settings for the different tests are summarized in the following table:

problem	assets	stages	outcomes	scenarios
tree structure	10	3	(10,300)–(300,10)	3000
stages	10	2–5	20	20–160000
outcomes	10	3	40-100	1600–10000
assets	1-20	3	80	6400

The second column contains the number of assets excluding the risk free asset. In the fourth column (10,300) represents 10 outcomes in the first stage and 300 in the second stage. For all the tests we solve the multistage stochastic programming problem 20 times to estimate the upper bound and use 10000 Monte Carlo simulations to determine the lower bound.

5.1. Choice of heuristic and tree structure

To numerically study how the tree structure affect the ability to solve a stochastic programming problem we have used a 3-staged problem instance with 10 assets and fixed the number of scenarios to 3000. The possible combinations that we have used range from 10 outcomes in the first stage and 300 in the second stage to 300 in the first stage and 10 in the second stage. For the power utility functions, and in particular the logarithmic, following the results in section (4.3), it is well understood how the scenario tree should be structured to give good upper bounds. We know from (17) that the bias for

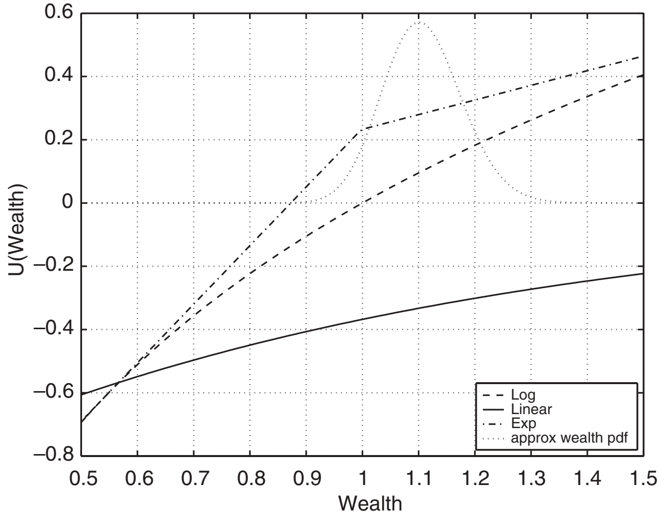

Fig. 1. Objective functions used in numerical results

a logarithmic utility function when the process is between stages independent depends additively on the bias for each stage. The minimal bias is achieved when we have the same number of outcomes in each stage. This result is verified from numerical experiments as is shown in Figure (2) where the expected upper bound (ubd) have a minimal value when we have an equal number of outcomes in both the first and second stage. This holds not only for the case of the myopic logarithmic utility function, but also for the other optimization problems. We also know from (18) that in order to get a good statistical upper bound more scenarios should be allocated to the first stage to reduce the variance. Figure (3) confirm this finding for all optimization problems. To minimize the variance we should have up to ten times more outcomes in the first period compared to the second. When both these effects are taken into account (the 95% ubd in Figure 2) it can be seen that the effect of the bias dominates that of the variance. Thus we should have approximately the same number outcomes in stage one and two in order to get a good statistical upper bound.

Concerning the heuristics it shows that the affine heuristic produce the best lower bounds (Figure 2). As can be seen in the myopic case, where we know the optimal objective function value v^*, the lower bound lies very close to v^* when there are 10 times more outcomes in the first stage compared to the second. To get a good first stage decision it is important to allocate as many as 10 times more scenarios to the first stage compared to the second. A good quality in the second stage decisions can be achieved by averaging close decisions of lower quality. In the following simulations the affine heuristic is used to estimate the lower bound.

5.2. Number of stages

To test how the number of stages impact the quality of the optimal solution, the number of assets (10) and outcomes in each stage (20) was kept constant. The number of stages

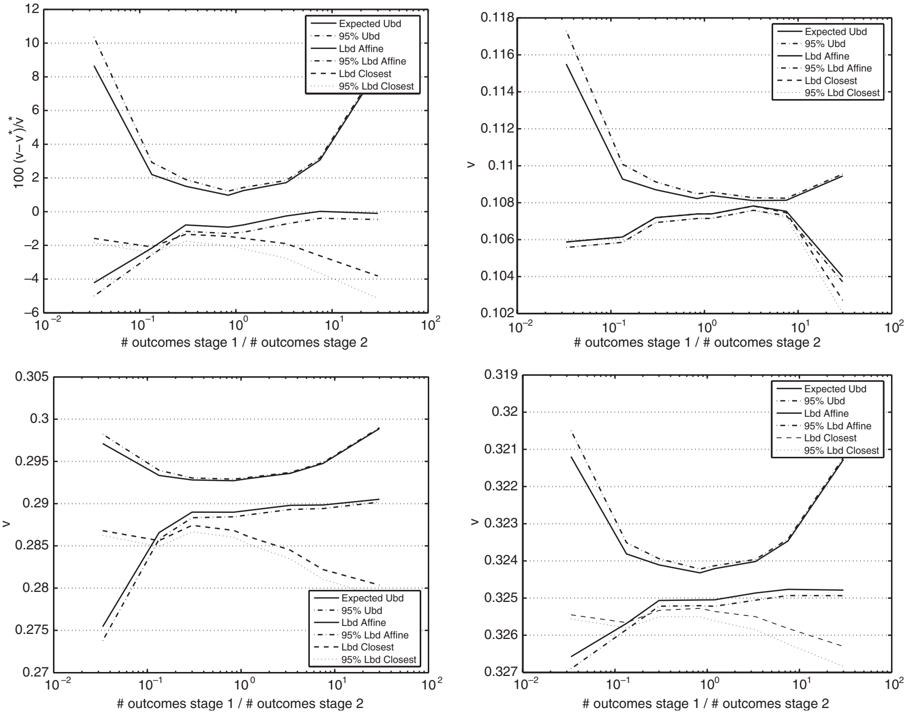

Fig. 2. Objective function values for different tree structures and heuristics for determining Lbd. Upper left: Scaled logarithmic utility function. Upper right: Logarithmic utility function with transaction costs. Lower left: Piecewise linear utility function. Lower right: Exponential utility function

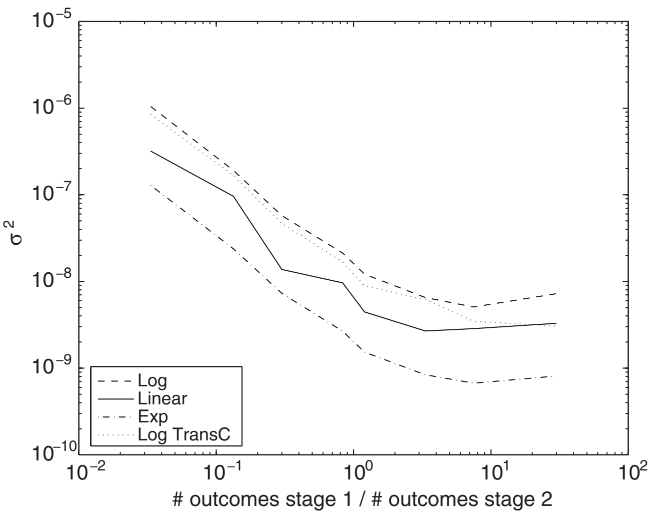

Fig. 3. Standard deviation of upper bounds for different tree structures

varied from 2–5, giving scenario trees with up to 160.000 scenarios. The upper left diagram in Figure (4) can be understood fairly well from the theory. The ratio between the average value of the upper bound and the optimal objective function value is essentially on the same level, since both the objective function value and the bias grow linearly with the number of stages (section 4.3). Considering that the contribution to the variance of the upper bound is equally weighted between the number of stages the large variance from the first stage will have decreasing impact when the number of stages increase, thus increasing the quality of the upper bound. A similar mechanism also improves the lower bound. The quality of the first stage decision is bad (there are only 20 outcomes), but the relative importance to the total objective function value decrease with an increase in the number of stages. With this limited scenario tree one can solve a 5-staged problem and get a policy that is 3% from the optimal policy, and with a total duality gap of 9%. Figure (5) shows that the gap decrease with the number of stages for the logarithmic utility function both with and without transaction costs. For the case with transaction costs we use the closest policy to generate the feasible decisions in the lbd heuristic. Creating an affine combination of decisions lead to decisions with to high transaction costs, since in the interpolated solution both the buy and sell decisions are usually nonzero.

Overall it does not seem that the number of stages decrease the quality of the multistage stochastic programming problem too much for this model, and that reasonable solutions can be found for problems with up to 5 stages when the number of outcomes is increased.

5.3. *Number of outcomes*

To increase the quality of the decisions the number of outcomes in each stage have to be increased. As is shown in this section, the asset investment problem can be solved to a relatively good precision, even though the samples are sparse in the 10-dimensional space of the stochastic parameters. First we investigate the behavior when solving the 3-staged problem with logarithmic utility function (Figure 6). Both the average upper bound and the variance are decreasing as the number of outcomes increase (see the discussion of section 4.3). At 100 outcomes in each stage, the upper bound is only 0.5% from the optimal objective function value, and the quality of the lower bound is even better. The rate of improvement for the other utility functions behave similarly, and the quality of the bounds also behave similarly (Figure 7). Each optimization problem has been solved to a relative precision less then 1%, using only 100 outcomes in each stage.

5.4. *Number of assets*

When the number of assets increase, the samples will become more and more sparse, indicating that the quality of the optimal decisions and the upper bound will decrease significantly. As Figures (8) and (9) show, this is not the case. Since the number of outcomes (80) in each stage is always the same, it is expected that the gap between the upper and lower bound will increase. The increase in both absolute and relative (Figure 9) value is however limited.

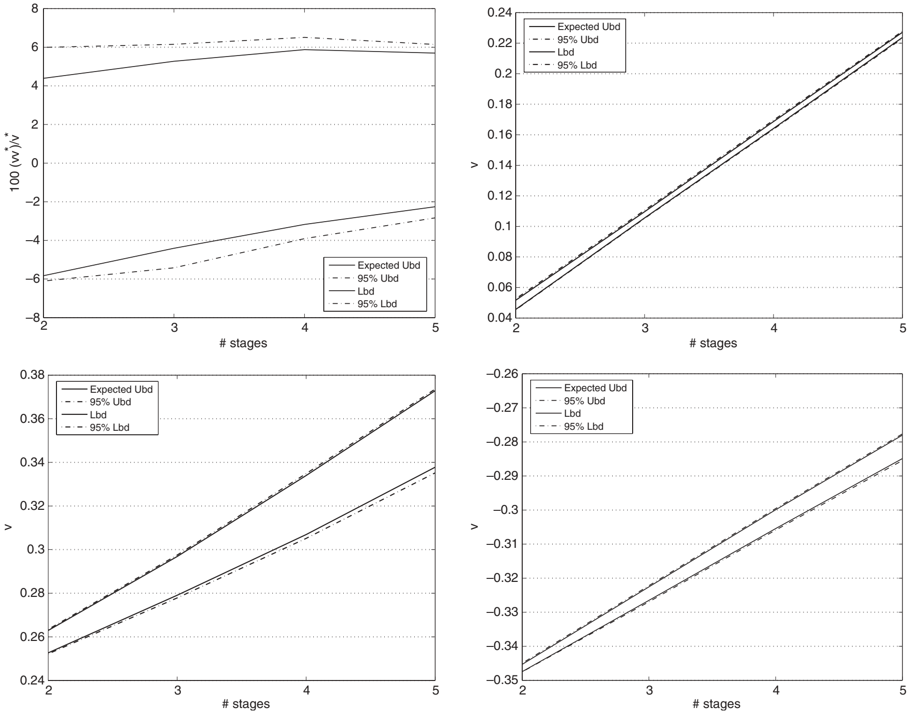

Fig. 4. Objective function values for varying number of stages. Upper left: Scaled logarithmic utility function. Upper right: Logarithmic utility function with transaction costs. Lower left: Piecewise linear utility function. Lower right: Exponential utility function

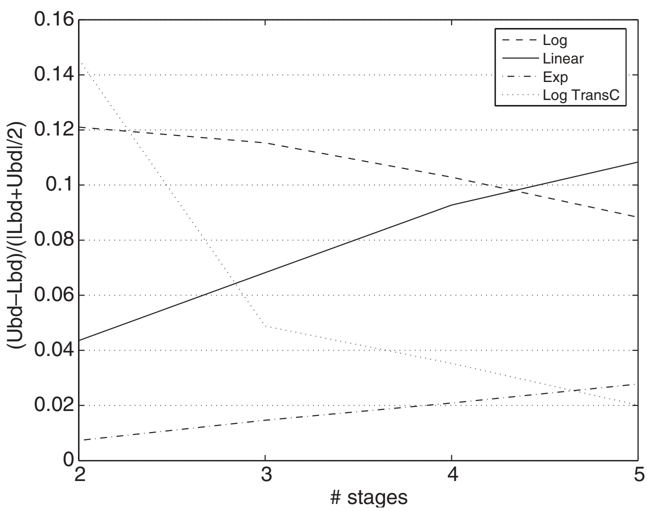

Fig. 5. Gap between 95% upper bound and 95% lower bound for varying number of stages

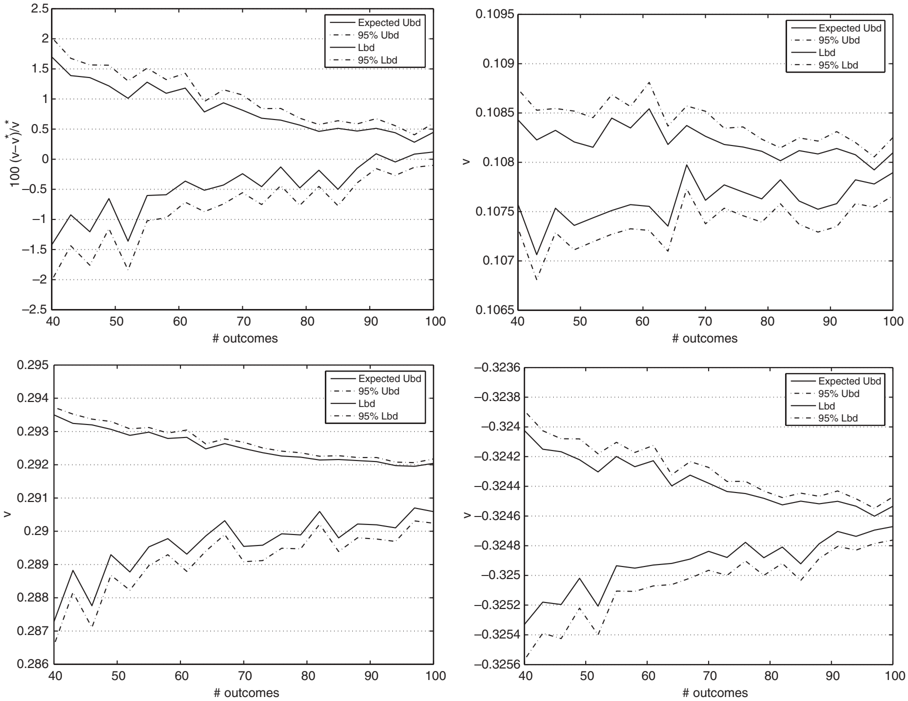

Fig. 6. Objective function values for varying number of outcomes. Upper left: Scaled logarithmic utility function. Upper right: Logarithmic utility function with transaction costs. Lower left: Piecewise linear utility function. Lower right: Exponential utility function

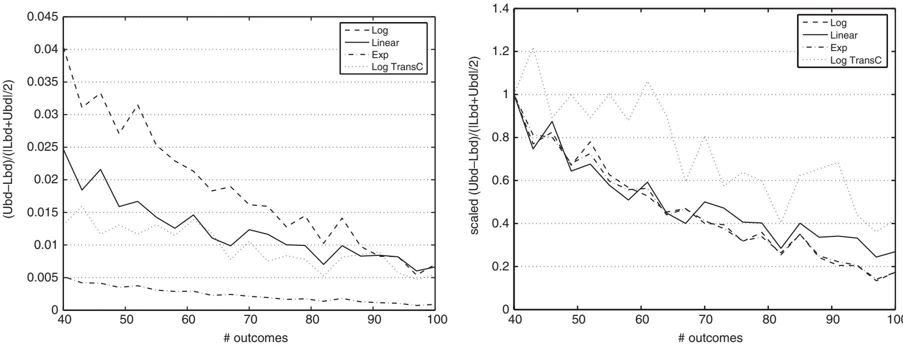

Fig. 7. Left: Gap between 95% upper bound and 95% lower bound for varying number of outcomes. Right: Same as left but scaled

5.5. *Stability of results*

In all the previous tests the expected return and volatility of the assets have been equal for all assets. With this setting many problems have been solved to a precision of a few percent. To further validate our findings the stability of the quality of solutions will be studied by randomly generating the parameters. The expected return and the

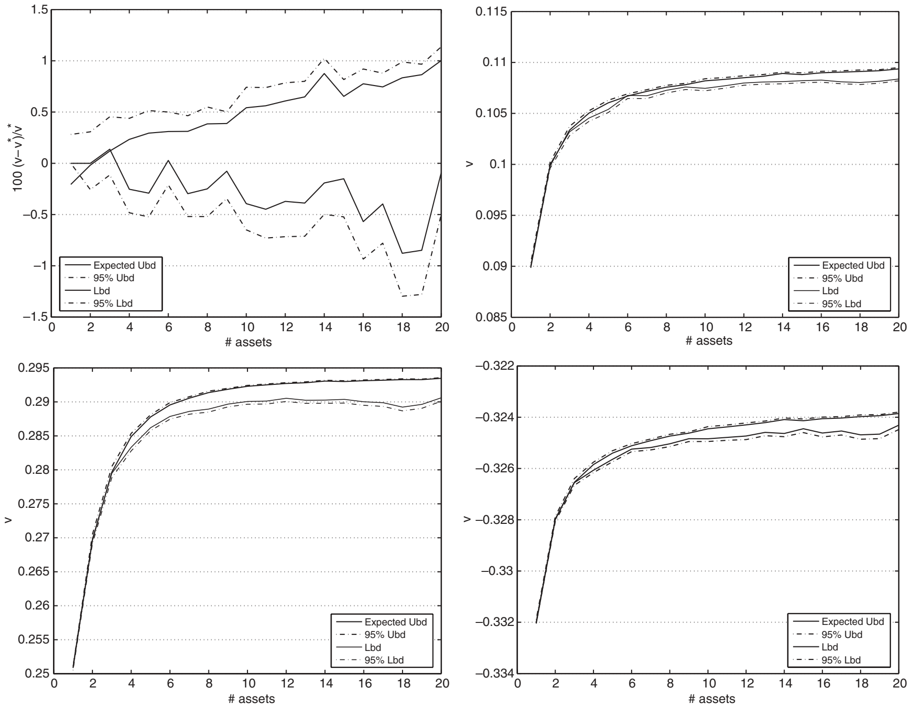

Fig. 8. Objective function values for varying number of assets. Upper left: Scaled logarithmic utility function. Upper right: Logarithmic utility function with transaction costs. Lower left: Piecewise linear utility function. Lower right: Exponential utility function

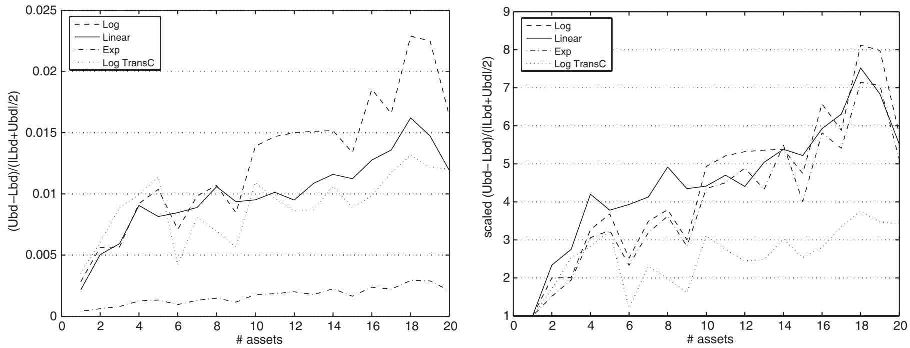

Fig. 9. Left: Gap between 95% upper bound and 95% lower bound for varying number of assets. Right: Same as left but scaled

volatility for each asset will be sampled from a rectangular probability distribution, $\mu^a \sim Rect(0.05, 0.25)$ and $\sigma^a \sim Rect(0.1, 0.4)$. The correlation for all assets is the same, it is sampled from a rectangular distribution, $c \sim Rect(0, 0.9)$. For each parameter setting a 3-staged optimization problem is solved with 10 random assets and 100 outcomes in each stage. This procedure is repeated 100 times, using the logarithmic

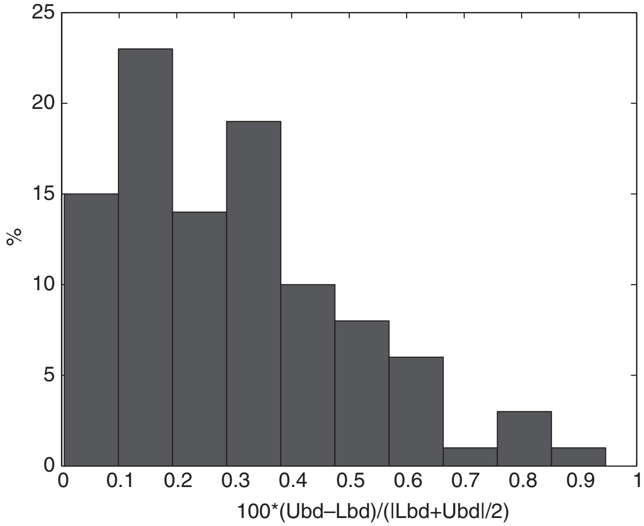

Fig. 10. Histogram of gap for randomly generated μ^a, σ^a, and $c_{a_1 a_2}$

utility function. For all these optimization problems, the quality of the solution seems to be very stable (Figure 10). The gap between the upper and lower bound is never above 1%, and the for most of the problems the gap is close to 0.7% or smaller. With the original parameters the gap was 0.6% (Figure 7). Considering this it seems reasonable to believe that the choice of parameter values has not had any major effect on the results, and that the results can be assumed to hold for any asset investment problem with reasonable parameter values.

6. Evaluating the quality of first stage decisions

Suppose that we want to evaluate the quality of a given first stage decision, $\bar{x}_2$, in a T-staged decision problem. The quality of the decision can be measured by determining the objective function value for a T-staged optimization problem with the additional constraint $x_2 = \bar{x}_2$ (see Section 4.1). To determine a statistical upper bound, sampled problem instances have to be solved. The additional constraint fix the first stage decision thus decomposing each problem instance into N_1 times $(T - 1)$-staged subproblems. To determine a statistical lower bound several estimates are made of the total objective function value. For each estimate $\mathbf{c}_2$ is sampled N_1 times. Each resulting $(T - 1)$-staged problem is solved and the outcomes contribution to the total objective function value is determined by simulation and affine interpolation of the decisions.

We have evaluated decisions for a 3-staged investment problem with 10 risky assets. The upper bound has been determined by 20 estimates of the objective function value and $N_1 = 100$, $N_2 = 100$. To estimate the lower bound, again, 20 estimates and $N_1 = 100$ are used. In each second stage node a 2-staged problem with $N_2 = 100$ is solved and 1000 simulations are made to determine the second stage objective function value. As

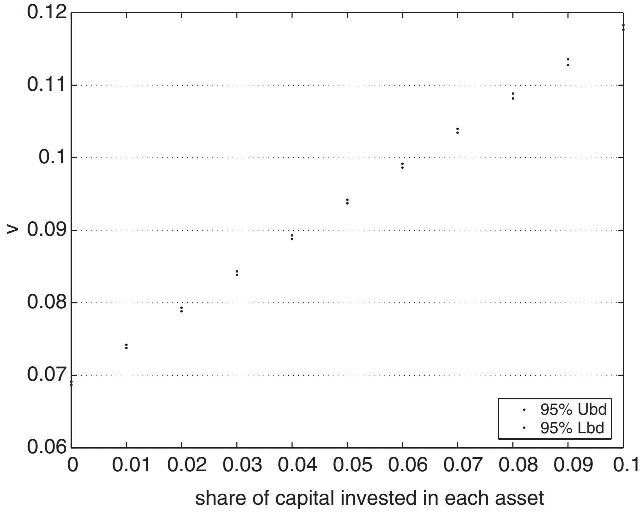

Fig. 11. The quality of different first stage decisions

Figure 11 shows, the quality of the first stage decision can be determined to a very high precision. Each decision can be ordered in relation to the others in terms of quality.

7. Conclusions

For the multistage asset investment problem it is necessary to solve multistage stochastic programming problems whenever at least one of the following properties does not hold:

- the returns are independent
- the transaction costs are zero
- the utility function is of the type $U(w) = w^\gamma/\gamma$.

For up to 5–6 stages, the multistage asset investment problem can be successfully solved by estimating upper and lower bounds for the objective function value. This conclusion is based on a number of observations. The behavior of the upper bound for power utility functions, in terms of average value and variance, is theoretically analyzed and gives a good understanding of how multistage scenario trees should be structured to provide a good upper bound. Based on the numerical experiments, it is reasonable to extend these characteristics also to the other utility functions used (exponential, piecewise linear, and logarithmic with transaction costs). By using a new heuristic to transform the decisions in a multistage stochastic programming tree to a policy, high quality lower bounds can be estimated. This new heuristic performs better than choosing the decision from the "closest" node.

Based on the results for generating upper and lower bounds extensive tests are made to test how well an optimization problem with continuous random variables can be solved with multistage stochastic programming techniques. From the tests it can be concluded that neither the number of stages nor the number of assets have a serious impact on the quality of the solution. It can also be concluded that the number of necessary outcomes in each time stage is rather small, in many instances a precision of 0.5% was achieved by using only 100 outcomes in each stage. The results also seems to be stable with respect to parameter choices. The major drawback with multistage stochastic programming is however still present, the exponential growth of scenarios. Thus limiting the number of stages to may be 5 or 6. These are encouraging results for users who solve multistage asset investment problems by stochastic programming. The stochastic programming solution will be reasonably close to the optimal solution, even though the number of scenarios are relatively small.

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