Stochastic and Robust Optimization in Logistics

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Introductions

About Me

- At Georgia Tech for 9 years
- Research interests in dynamic and stochastic logistics optimization; routing and scheduling; logistics system resiliency
- www.isye.gatech.edu/~alerera
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If you don’t understand
Please interrupt me and ask questions...
What to remember

1. Stochastic and robust optimization are for *dynamic* decision problems
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2. Many ways to effectively incorporate parameter uncertainty in logistics optimization
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3. Modeling and treatment of *recourse* especially critical
What to remember

1. Stochastic and robust optimization are for *dynamic* decision problems
2. Many ways to effectively incorporate parameter uncertainty in logistics optimization
3. Modeling and treatment of *recourse* especially critical
4. Ensure that your model is useful (and interesting), then solve
Scope

What I will cover:

- Stochastic integer programming
- Chance constraints and integer programming
- Robust (worst-case) constraints and integer programming
- Primarily modeling, and solution heuristics
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- Stochastic integer programming
- Chance constraints and integer programming
- Robust (worst-case) constraints and integer programming
- Primarily modeling, and solution heuristics

What I will not cover:

- Dynamic programming (MDP)
- Approximate dynamic programming
Dynamic planning

Dynamic planning problems

A *dynamic planning problem* is one where planning decisions are made sequentially in time.
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Motivating Examples

- Each month, a load plan (freight routing plan) is determined for the following month.
Dynamic planning

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- Each month, a load plan (freight routing plan) is determined for the following month
- Each week, an empty trailer repositioning plan is determined for the following week
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- Each week, an empty trailer repositioning plan is determined for the following week
- Each evening, distribution vehicle routes are planned for tomorrow
Dynamic planning

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A *dynamic planning problem* is one where planning decisions are made sequentially in time.

Motivating Examples

- Each month, a load plan (freight routing plan) is determined for the following month.
- Each week, an empty trailer repositioning plan is determined for the following week.
- Each evening, distribution vehicle routes are planned for tomorrow.
- Each time a new customer request arrives, it is added to a vehicle route.
Are most logistics planning problems dynamic?
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Yes!
Are most logistics planning problems dynamic?

Yes!

- Many, if not most, quantitative decision problems in logistics are inherently dynamic, by my definition.
- Our focus: how to build and solve an appropriate optimization model for each such problem?
Planning and control

Control decisions

Between planning periods, *controls* are used to implement a plan feasibly and effectively.

Simple rules, or may result from *(recourse)* optimization problems.
Deterministic and Stochastic Planning Models

Two classes of optimization models for dynamic logistics planning:
Deterministic and Stochastic Planning Models

Two classes of optimization models for dynamic logistics planning:

**Deterministic Model**
A model in which all parameters are *assumed* to be *known* when planning
Two classes of optimization models for dynamic logistics planning:

**Deterministic Model**
A model in which all parameters are assumed to be known when planning.

**Stochastic Model**
A model in which one or more parameters are assumed to be uncertain when planning.
When Should Uncertainty Be Ignored?
When Should Uncertainty Be Ignored?

Parameter Availability

Are all model parameters known when planning?
When Should Uncertainty Be Ignored?

Parameter Availability
Are all model parameters known when planning?

Parameter Variability
If uncertain model parameters are replaced with nominal (expected) values when planning, does the model produce good results?
When Should Uncertainty Be Ignored?

**Parameter Availability**
Are all model parameters known when planning?

**Parameter Variability**
If uncertain model parameters are replaced with nominal (expected) values when planning, does the model produce good results?

The latter is another *engineering* decision.
How Should Uncertainty Be Incorporated?

1. Probabilistic programming models
2. Two-stage models
3. Multiple-stage models
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2. Two-stage models

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1. Probabilistic programming models
   - Chance-constrained programming and heuristics
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   - Two-stage stochastic integer programming and heuristics
   - Two-stage SIP with robust or chance constraints
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3. Multiple-stage models
   - Multi-stage SIP
   - Dynamic programming and approximate dynamic programming
How Should Uncertainty Be Incorporated?

Two-Stage Models

Imagine an environment with two decision stages:

1. **First (Planning) Stage**: Planning decisions are made, some parameters uncertain

2. **Second (Control, Recourse) Stage**: Control decisions are made, **all** uncertain parameters revealed (known)
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Many real-world problems can be modeled with precision in this way
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- Many real-world problems can be modeled with precision in this way
- **Even for those that cannot**, this is still frequently a reasonable first approximation for including uncertainty during planning
Two-stage Recourse Models

Explicit modeling of control decisions:

1. **First (Planning) Stage**: Planning decisions are made, some parameters uncertain

2. **Second (Control, Recourse) Stage**: Control decisions are made, **all** uncertain parameters revealed (known)
Two-stage Recourse Models

Explicit modeling of control decisions:

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A Key Modeling Issue

How are second stage (recourse) decisions to be modeled?
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1. Fixed operating rules, or
Two-stage Recourse Models

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**A Key Modeling Issue**

How are second stage (recourse) decisions to be modeled?

1. Fixed operating rules, or
2. Optimization problem for control
How Should Uncertainty Be Incorporated?

**Multi-Stage Models**

In many dynamic planning settings, uncertainty is revealed in multiple stages over time:

1. **First Stage**: Given \((0, I_0)\), decisions \(x_1\) are determined.
2. **nth Stage**: Given \((x_{n-1}, I_{n-1})\), decisions \(x_n\) are determined.
How Should Uncertainty Be Incorporated?

Multi-Stage Models

In many dynamic planning settings, uncertainty is revealed in multiple stages over time:

1. **First Stage**: Given \((0, I_0)\), decisions \(x_1\) are determined.
2. **nth Stage**: Given \((x_{n-1}, I_{n-1})\), decisions \(x_n\) are determined.

- Information pattern \(I_k\) summarizes known and uncertain information available for stage \(k + 1\) decision-making.
How Should Uncertainty Be Incorporated?

Probabilistic Programming

- Planning decisions are made using models that use probabilistic forms in the constraints or objective function
- Control or recourse decisions are not modeled explicitly
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Probabilistic Programming

- Planning decisions are made using models that use probabilistic forms in the constraints or objective function
- Control or recourse decisions are not modeled explicitly

- Not typically used directly today, but
- Ideas like *chance constraints* or *robust constraints* can be useful, and can be incorporated if necessary within explicit two-stage models
Remainder of Presentation

- Illustration of the ideas via examples
- References for more detailed information
VRP with Stochastic Demands (VRPSD)

Capacitated Vehicle Routing Problem

Given depot-based fleet of vehicles of capacity $Q$, travel cost matrix $\{c_{ij}\}$, and known customer demands $\{q_i\}$, find set of depot-based capacity feasible vehicle tours with minimum total travel cost.
VRP with Stochastic Demands (VRPSD)

Parameter Availability

- Customer demand values not known with certainty when planning
VRP with Stochastic Demands (VRPSD)

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- Different models use different assumptions about when they are known
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  - Before vehicle departure from depot each day
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**Parameter Availability**

- Customer demand values not known with certainty when planning
- Different models use different assumptions about when they are known
  1. Before vehicle departure from depot each day
  2. Upon arrival at customer location
VRP with Stochastic Demands (VRPSD)

Given depot-based fleet of vehicles of capacity $Q$, travel cost matrix $\{c_{ij}\}$, and uncertain customer demands $\{\tilde{q}_i\}$ independent with known distributions, find set of depot-based vehicle tours that (*)

**Vehicle Routing Problem with Stochastic Demands**
Vehicle Routing Problem with Stochastic Demands

Given depot-based fleet of vehicles of capacity $Q$, travel cost matrix $\{c_{ij}\}$, and uncertain customer demands $\{\tilde{q}_i\}$ independent with known distributions, find set of depot-based vehicle tours that

\[ (*) \]

- Probabilistic programming version
  - (*) Minimize total travel cost subject to chance constraints on the capacity feasibility of each vehicle tour
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Given depot-based fleet of vehicles of capacity $Q$, travel cost matrix $\{c_{ij}\}$, and uncertain customer demands $\{\tilde{q}_i\}$ independent with known distributions, find set of depot-based vehicle tours that

(*)

- Probabilistic programming version
  - (*) Minimize total travel cost subject to chance constraints on the capacity feasibility of each vehicle tour
- Tours must be planned before uncertainty revealed
Chance-constrained VRPSD Model

Stewart and Golden (1982)

\((m\text{-vehicle VRP})\)

\[
\min \sum_{k} \sum_{i,j} c_{ij} x_{ijk}
\]

\[
\sum_{i,j} q_i x_{ijk} \leq Q \quad \forall \ k
\]

\[
\{x_{ijk}\} \in S_m
\]

where \(S_m\) is set of all \(m\)-traveling salesman solutions
Chance-constrained VRPSD Model

Stewart and Golden (1982)

\((m\text{-vehicle chance-constrained VRPSD})\)

\[
\min \sum_k \sum_{i,j} c_{ij} x_{ijk}
\]

\[
P \left( \sum_{i,j} \tilde{q}_i x_{ijk} \leq Q \right) \geq 1 - \alpha \quad \forall \ k
\]

\(\{x_{ijk}\} \in S_m\)
Stewart and Golden (1982)

\[(m\text{-vehicle chance-constrained VRPSD})\]

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\min \sum_{k} \sum_{i,j} c_{ij} x_{ijk}
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P \left( \sum_{i,j} \tilde{q}_{i} x_{ijk} \leq Q \right) \geq 1 - \alpha \quad \forall k
\]

\[
\{x_{ijk}\} \in S_m
\]

- $\alpha$ is a tour failure probability

Chance-constrained VRPSD Model
Representing chance constraints

Deterministic equivalents

Can we find a equivalent deterministic representation of the set of all solutions satisfying chance constraints:

$$K = \cap_i K_i$$

where

$$K_i = \{x | P(A^i(\omega)x \geq h^i(\omega)) \geq \rho^i\}$$

Straightforward and linear when $A_i$ fixed, but more difficult when $A_i(\omega)$ varies (even if $h_i(\omega)$ fixed).
Representing chance constraints

Deterministic equivalents

Can we find an equivalent deterministic representation of the set of all solutions satisfying chance constraints:

\[ K = \cap_i K^i \]

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- Straightforward and linear when \( A^i \) fixed, but
- More difficult when \( A^i(\omega) \) varies (even if \( h^i(\omega) \) fixed)
Deterministic equivalent for capacity chance constraint

\[ P \left( \sum_{i,j} \tilde{q}_{ijk} x_{ijk} \leq Q \right) \geq 1 - \alpha \quad \forall \ k \]

Deterministic equivalent

\[ M_k + \tau S_k \leq Q \quad \forall \ k \]

where \( M_k = \sum_{i,j} \mu_i x_{ijk} \) and \( S_k = \sqrt{\sum_{i,j} \sigma_i^2 x_{ijk}} \), and

\[ P \left( \frac{\sum_{i,j} \tilde{q}_{ijk} x_{ijk} - M_k}{S_k} \leq \tau \right) = 1 - \alpha \]
Deterministic equivalent for capacity chance constraint

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\[ P \left( \frac{\sum_{i,j} \tilde{q}_i x_{ijk} - M_k}{S_k} \leq \tau \right) = 1 - \alpha \]

Approach works when \( \tilde{q}_i \) are independent, then there may exist a \( \tau \) that satisfies the expression (true for normal, Poisson, binomial random variables)
Normal distribution example

- Suppose \( \{\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n\} \) are independent and normally distributed
  - Means \( \mu_i \), variances \( \sigma_i^2 \)
Normal distribution example

- Suppose \( \{ \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n \} \) are independent and normally distributed
  - Means \( \mu_i \), variances \( \sigma_i^2 \)
- Then \( \tilde{q}(S) = \sum_{i \in S} \tilde{q}_i \) remains normal
  - Mean \( \mu(S) = \sum_{i \in S} \mu_i \), variance \( \sigma^2(S) = \sum_{i \in S} \sigma_i^2 \)
Normal distribution example

- Suppose \( \{q_1, q_2, ..., q_n\} \) are independent and normally distributed
  - Means \( \mu_i \), variances \( \sigma_i^2 \)
- Then \( q(S) = \sum_{i \in S} q_i \) remains normal
  - Mean \( \mu(S) = \sum_{i \in S} \mu_i \), variance \( \sigma^2(S) = \sum_{i \in S} \sigma_i^2 \)
- And \( \frac{q(S) - \mu(S)}{\sigma(S)} \) is \( N(0, 1) \)
Suppose \( \{\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n\} \) are independent and normally distributed

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- Mean \( \mu(S) = \sum_{i \in S} \mu_i \), variance \( \sigma^2(S) = \sum_{i \in S} \sigma_i^2 \)

And \( \frac{\tilde{q}(S) - \mu(S)}{\sigma(S)} \) is \( N(0, 1) \)

Therefore \( \tau = \Phi^{-1}(1 - \alpha) \)
Using deterministic equivalent

Heuristics

- Computing $M_k$ and $S_k^2$ for all routes not difficult
- Simple updating procedures when customers enter or leave routes in neighborhood search

Remember, variance of a sum of independent random variables is the sum of the variances of the individual random variables.
Laporte, Louveaux, and Mercure (1989): “subtour” elimination for 2-index formulation

- Consider customer set $U$
- Let $V_\alpha(U)$ be smallest integer s.t.
  \[ P \left( \sum_{i \in U} \tilde{q}_i > QV_\alpha(U) \right) \leq \alpha \]
- “Subtour” elimination cut:
  \[
  \sum_{i \in U, j \notin U} x_{ij} + \sum_{i \notin U, j \in U} x_{ij} \geq 2V_\alpha(U)
  \]
Using deterministic equivalent

Exact approaches

Laporte, Louveaux, and Mercure (1989): “subtour” elimination for 2-index formulation

- Consider customer set $U$
- Let $V_\alpha(U)$ be smallest integer s.t.
  $P \left( \sum_{i \in U} \tilde{q}_i > QV_\alpha(U) \right) \leq \alpha$
- “Subtour” elimination cut:

  $$\sum_{i \in U, j \in \bar{U}} x_{ij} + \sum_{i \in \bar{U}, j \in U} x_{ij} \geq 2V_\alpha(U)$$

- We can determine $V_\alpha(U)$ as follows:
  $$V_\alpha(U) - 1Q < M_U + \tau S_U \leq V_\alpha(U)Q$$
Two-stage models: fixed routes

- What are fixed routes?
Two-stage models: fixed routes

- What are fixed routes?
  - Delivery routes used essentially unchanged daily for some period of time
Two-stage models: fixed routes

- What are fixed routes?
  - Delivery routes used essentially unchanged daily for some period of time
- Why use fixed routes?

Reduce costs
- Simplify picking/staging costs at distribution center
- Eliminate daily use of optimization software

Improve driver performance
- Develop familiarity with a delivery area and set of customers
- Improve customer service
- Driver develops relationship with customer
- Driver performs additional services for customer
Two-stage models: fixed routes

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S+R Optimization in Logistics
Vehicle Routing under Uncertainty
Two-stage Models

**VRP with Stochastic Demands (VRPSD)**

**Vehicle Routing Problem with Stochastic Demands (collection version)**

Given depot-based fleet of vehicles of capacity $Q$, travel cost matrix $\{c_{ij}\}$, and **uncertain** customer demands $\{\tilde{q}_i\}$ independent with known distributions, find set of depot-based vehicle tours that

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- Two-stage integer programming version
  - (*) Minimize expected total travel cost given a recourse policy (control decision strategy)
Two-stage Models

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- Two-stage integer programming version
  - (*) Minimize expected total travel cost given a recourse policy (control decision strategy)
- A priori tours must be planned before uncertainty revealed
  - Parameter availability: customer demands known upon vehicle arrival
Two-stage model for VRPSD

1st stage: *A priori* tours

Minimize expected cost

2nd stage: Operational tours

Use recourse (control) policy

Dror et al. (1989) Recourse Policy

- Follow *a priori* tour
- When vehicle capacity met or exceeded, *detour to depot* to unload
Two-stage stochastic integer program

$$\min_{x \in X} z = c^T x + E[\min_{y} \{q(\omega)^T y \mid Wy = h(\omega) - T(\omega)x, \ y \in Y\}]$$

s.t. \quad Ax = b

where \(X\) and/or \(Y\) impose integrality restrictions.
Two-stage stochastic integer program

\[
\min_{x \in X} z = c^T x + E[\min_{y \in Y} \{q(\omega)^T y \mid W y = h(\omega) - T(\omega) x, \ y \in Y\}]
\]

\[s.t. \ Ax = b\]

where \( X \) and/or \( Y \) impose integrality restrictions.

Deterministic equivalent form

\[
\min_{x \in X} z = c^T x + Q(x)
\]

\[s.t. \ Ax = b\]
Consider single tour with homogeneous discrete customer demand distributions, and recourse only initiated if observed customer demand would exceed remaining vehicle capacity

- Tour $T = \{1, 2, \cdots, n\}$
- $p_i(\delta)$ probability that customer $i$ demand value is $\delta$
- $\beta(i, s, q)$ probability of remaining capacity $q$ after serving customer $i$, $s = 1$ if recourse action occurred at $i$, 0 otherwise

\[
\beta(i, 0, q) = \sum_s \sum_{\bar{q} \in [q, Q]} \beta(i - 1, s, \bar{q}) \ p_i(\bar{q} - q)
\]

\[
\beta(i, 1, q) = \sum_s \sum_{\bar{q} \in [0, Q - q - 1]} \beta(i - 1, s, \bar{q}) \ p_i(Q - q)
\]
Consider single tour with homogeneous discrete customer demand distributions, and recourse only initiated if observed customer demand would exceed remaining vehicle capacity

- $\pi_i$ probability of a tour failure and recourse at customer $i$

$$\pi_i = \sum_q \beta(i, 1, q)$$

- Expected recourse cost

$$\sum_{i \in T} 2\pi_i (c_{i,0})$$
Example $Q(x)$ computation

Setting

- Suppose tour has three customers, and $Q = 2$
- Each customer demand distribution:

$$p(\delta) = \begin{cases} 
0.2 & \delta = 0 \\
0.8 & \delta = 1 
\end{cases}$$
Example $Q(x)$ computation

Computations

Customer 1
$\beta(1, s = 0, q = 0) = p(Q - q = 2) = 0$

Customer 2
Example $Q(x)$ computation

Computations

Customer 1

$\beta(1, s = 0, q = 0) = p(Q - q = 2) = 0$

$\beta(1, s = 0, q = 1) = p(1) = 0.8$

Customer 2
Example $Q(x)$ computation

Computations

Customer 1

$\beta(1, s = 0, q = 0) = p(Q - q = 2) = 0$

$\beta(1, s = 0, q = 1) = p(1) = 0.8$

$\beta(1, s = 0, q = 2) = p(0) = 0.2$

Customer 2
Example $Q(x)$ computation

Computations

Customer 1
\[
\beta(1, s = 0, q = 0) = p(Q - q = 2) = 0 \\
\beta(1, s = 0, q = 1) = p(1) = 0.8 \\
\beta(1, s = 0, q = 2) = p(0) = 0.2 \\
\beta(1, s = 1, q) = 0
\]

Customer 2

\[
\beta(2, s = 0, q = 0) = \beta(1, 0, 1) = \beta(0, 1, 2) = 0 \\
\beta(2, s = 0, q = 1) = \beta(1, 0, 2) = 0.8 \\
\beta(2, s = 0, q = 2) = \beta(1, 0) = 0.2 \\
\beta(2, s = 1, q) = 0
\]
Example $Q(x)$ computation

**Computations**

**Customer 1**
\[
\beta(1, s = 0, q = 0) = p(Q - q = 2) = 0
\]
\[
\beta(1, s = 0, q = 1) = p(1) = 0.8
\]
\[
\beta(1, s = 0, q = 2) = p(0) = 0.2
\]
\[
\beta(1, s = 1, q) = 0
\]

**Customer 2**
\[
\beta(2, s = 0, q = 0) = \beta(1, 0, 1)p(1) = 0.64
\]
Example $Q(x)$ computation

Computations

Customer 1
\[
\begin{align*}
\beta(1, s = 0, q = 0) &= p(Q - q = 2) = 0 \\
\beta(1, s = 0, q = 1) &= p(1) = 0.8 \\
\beta(1, s = 0, q = 2) &= p(0) = 0.2 \\
\beta(1, s = 1, q) &= 0
\end{align*}
\]

Customer 2
\[
\begin{align*}
\beta(2, s = 0, q = 0) &= \beta(1, 0, 1)p(1) = 0.64 \\
\beta(2, s = 0, q = 1) &= \beta(1, 0, 1)p(0) + \beta(1, 0, 2)p(1) = (0.8)(0.2) + (0.2)(0.8) = 0.32
\end{align*}
\]
Example $Q(x)$ computation

Computations

Customer 1
\[ \beta(1, s = 0, q = 0) = p(Q - q = 2) = 0 \]
\[ \beta(1, s = 0, q = 1) = p(1) = 0.8 \]
\[ \beta(1, s = 0, q = 2) = p(0) = 0.2 \]
\[ \beta(1, s = 1, q) = 0 \]

Customer 2
\[ \beta(2, s = 0, q = 0) = \beta(1, 0, 1)p(1) = 0.64 \]
\[ \beta(2, s = 0, q = 1) = \beta(1, 0, 1)p(0) + \beta(1, 0, 2)p(1) = (0.8)(0.2) + (0.2)(0.8) = 0.32 \]
\[ \beta(2, s = 0, q = 2) = \beta(1, 0, 2)p(0) = 0.04 \]
Example $Q(x)$ computation

Computations

Customer 1
\[
\begin{align*}
\beta(1, s = 0, q = 0) &= p(Q - q = 2) = 0 \\
\beta(1, s = 0, q = 1) &= p(1) = 0.8 \\
\beta(1, s = 0, q = 2) &= p(0) = 0.2 \\
\beta(1, s = 1, q) &= 0
\end{align*}
\]

Customer 2
\[
\begin{align*}
\beta(2, s = 0, q = 0) &= \beta(1, 0, 1)p(1) = 0.64 \\
\beta(2, s = 0, q = 1) &= \beta(1, 0, 1)p(0) + \beta(1, 0, 2)p(1) = (0.8)(0.2) + (0.2)(0.8) = 0.32 \\
\beta(2, s = 0, q = 2) &= \beta(1, 0, 2)p(0) = 0.04 \\
\beta(2, s = 1, q) &= 0
\end{align*}
\]
Example $Q(x)$ computation

Computations

Customer 3
Example $Q(x)$ computation

Computations

Customer 3

$\beta(3, s = 0, q = 0) = \beta(2, 0, 0)p(0) + \beta(2, 0, 1)p(1) = (0.64)(0.2) + (0.32)(0.8) = 0.384$
Example $Q(x)$ computation

Computations

Customer 3

$\beta(3, s = 0, q = 0) = \beta(2, 0, 0)p(0) + \beta(2, 0, 1)p(1) = (0.64)(0.2) + (0.32)(0.8) = 0.384$

$\beta(3, s = 0, q = 1) = \beta(2, 0, 1)p(0) + \beta(2, 0, 2)p(1) = (0.32)(0.2) + (0.04)(0.8) = 0.096$
Example $Q(x)$ computation

Computations

Customer 3

$\beta(3, s = 0, q = 0) = \beta(2, 0, 0)p(0) + \beta(2, 0, 1)p(1) =
(0.64)(0.2) + (0.32)(0.8) = 0.384$

$\beta(3, s = 0, q = 1) = \beta(2, 0, 1)p(0) + \beta(2, 0, 2)p(1) =
(0.32)(0.2) + (0.04)(0.8) = 0.096$

$\beta(3, s = 0, q = 2) = \beta(2, 0, 2)p(0) = (0.04)(0.2) = 0.008$
Example $Q(x)$ computation

Computations

Customer 3

$\beta(3, s = 0, q = 0) = \beta(2, 0, 0)p(0) + \beta(2, 0, 1)p(1) = (0.64)(0.2) + (0.32)(0.8) = 0.384$

$\beta(3, s = 0, q = 1) = \beta(2, 0, 1)p(0) + \beta(2, 0, 2)p(1) = (0.32)(0.2) + (0.04)(0.8) = 0.096$

$\beta(3, s = 0, q = 2) = \beta(2, 0, 2)p(0) = (0.04)(0.2) = 0.008$

$\beta(3, s = 1, q = 1) = \beta(2, 0, 0)p(1) = (0.64)(0.8) = 0.512$
Example $Q(x)$ computation

Computations

Customer 3

$\beta(3, s = 0, q = 0) = \beta(2, 0, 0)p(0) + \beta(2, 0, 1)p(1) =$

$(0.64)(0.2) + (0.32)(0.8) = 0.384$

$\beta(3, s = 0, q = 1) = \beta(2, 0, 1)p(0) + \beta(2, 0, 2)p(1) =$

$(0.32)(0.2) + (0.04)(0.8) = 0.096$

$\beta(3, s = 0, q = 2) = \beta(2, 0, 2)p(0) = (0.04)(0.2) = 0.008$

$\beta(3, s = 1, q = 1) = \beta(2, 0, 0)p(1) = (0.64)(0.8) = 0.512$

Failure Probabilities $\pi_1 = \pi_2 = 0; \pi_3 = 0.512$

Recourse Cost = $2(0.512)c_{03} = 1.024c_{03}$
S+R Optimization in Logistics
Vehicle Routing under Uncertainty
Two-stage Models

Integer L-shaped method

Extension of Van Slyke and Wets (1969) for SP
- Bender’s decomposition approach
- Use cuts on first-stage decisions to:
  - Ensure second-stage \textit{feasibility} with feasibility cuts
  - Create a linear approximation of \( Q(x) \) with optimality cuts
S+R Optimization in Logistics
Vehicle Routing under Uncertainty
Two-stage Models

Integer L-shaped method

Extension of Van Slyke and Wets (1969) for SP
- Bender’s decomposition approach
- Use cuts on first-stage decisions to:
  - Ensure second-stage feasibility with feasibility cuts
  - Create a linear approximation of $Q(x)$ with optimality cuts

First-stage feasibility is trivial for VRPSD, so focus only on optimality cuts.
Integer L-shaped method for VRPSD


- Add constraint that expected demand of each \textit{a priori} tour cannot exceed vehicle capacity
Integer L-shaped method for VRPSD


- Add constraint that expected demand of each *a priori* tour cannot exceed vehicle capacity

**Relaxed Formulation**

\[
\begin{align*}
\min_{i<j} & \quad c_{ij}x_{ij} + \theta \\
\text{subject to constraints of two-index undirected CVRP problem, where customer demand } & \quad \tilde{q}_i = E[\tilde{q}_i] \text{ for “subtour elimination”}
\end{align*}
\]
Basic branch-and-cut approach

- Let $\theta^v$ be a lower bound on $Q(x)$, and fathom if $cx^v + \theta^v \geq \bar{z}$
- When integer $x^v$ found, compute $Q(x^v)$ and update best found solution
  - If $\theta^v \geq Q(x^v)$, fathom (branch is optimal)
  - Else, introduce cut to move away from this solution, and continue

Lower bounding of $Q(x)$; Laporte et al. (2002)

Add cuts that are valid lower bounds on $Q(x^v)$ (regardless of $x^v$ integer)

Detailed, but based on the idea of approximating expected recourse cost of failure by considering first tour failure only
**Integer L-shaped method for VRPSD**

**Basic branch-and-cut approach**
- Let $\theta^v$ be a lower bound on $Q(x)$, and fathom if $cx^v + \theta^v \geq \bar{z}$
- When integer $x^v$ found, compute $Q(x^v)$ and update best found solution
  - If $\theta^v \geq Q(x^v)$, fathom (branch is optimal)
  - Else, introduce cut to move away from this solution, and continue

**Lower bounding of $Q(x)$; Laporte et al. (2002)**
- Add cuts that are valid lower bounds on $Q(x^v)$ (regardless of $x^v$ integer)
  - Detailed, but based on the idea of approximating expected recourse cost of failure by considering first tour failure only
Computational results from Laporte, et al. (2002)

- Heterogeneous Poisson demands
- Number of customers $n \in \{25, 50, 75, 100\}$
- Number of vehicles $m \in \{2, 3, 4\}$, 2 only for $n \geq 75$
- “Fill rate” $0.9 = \frac{\text{Total expected demand}}{\text{Total capacity of all vehicles}}$
  - As fill rate approaches 1, evidence suggests computational difficulty increases much faster than linearly
Sample average approximation (SAA)

Kleywegt, Shapiro, and Homem-de-Mello (2002)

- An alternative approach for solving two-stage SIP
Sample average approximation (SAA)

Kleywegt, Shapiro, and Homem-de-Mello (2002)

- An alternative approach for solving two-stage SIP

**SAA**

- Generate a *sample* of all realizations of uncertain parameters
Sample average approximation (SAA)

Kleywegt, Shapiro, and Homem-de-Mello (2002)

- An alternative approach for solving two-stage SIP

**SAA**

- Generate a *sample* of all realizations of uncertain parameters
- Solve deterministic problem, explicitly satisfying all constraints for each realization in the sample
Sample average approximation (SAA)

Kleywegt, Shapiro, and Homem-de-Mello (2002)

- An alternative approach for solving two-stage SIP

**SAA**

- Generate a *sample* of all realizations of uncertain parameters
- Solve deterministic problem, explicitly satisfying all constraints for each realization in the sample
- Use many such samples to:
  - Identify best plan
  - Approximate the optimality gap
Sample average approximation (SAA)

Kleywegt, Shapiro, and Homem-de-Mello (2002)

- An alternative approach for solving two-stage SIP

SAA

- Generate a *sample* of all realizations of uncertain parameters
- Solve deterministic problem, explicitly satisfying all constraints for each realization in the sample
- Use many such samples to:
  - Identify best plan
  - Approximate the optimality gap

See Verweij, Ahmed, Kleywegt, Nemhauser, Shapiro (2003) for stochastic routing applications
Limitations of VRPSD models

Metaheuristics for practical instance sizes

- Gendreau et al. (1996) extend TABUROUTE to problems with stochastic demands: TABUSTOCH
- Reasonable performance, although solutions degrade as $n$ increases to 50
Limitations of VRPSD models

Detour-to-depot recourse policy is limited
Limitations of VRPSD models

Detour-to-depot recourse policy is limited

- Each vehicle operates its \textit{a priori} tour independently
Limitations of VRPSD models

Detour-to-depot recourse policy is limited

- Each vehicle operates its *a priori* tour independently
- Enables analysis, but does not provide any opportunity for *risk pooling*
Limitations of VRPSD models

Detour-to-depot recourse policy is limited

- Each vehicle operates its \textit{a priori} tour independently
- Enables analysis, but does not provide any opportunity for risk pooling

Multi-vehicle coordinated recourse policies

- Erera (2000): analysis of many policies using techniques of \textit{continuous approximation}
- Ak and Erera (2007): detailed analysis and tabu search heuristic for two-vehicle sharing policy
Two-vehicle sharing recourse policy

Paired locally-coordinated (PLC) recourse; Ak and Erera (2007)

- **Type I** capacity failure: unserved customers appended to type II tour
- **Type II** capacity failure: use detour-to-depot
Tabu search for PLC recourse strategy

PLC Tabu Search

Adapted from Gendreau et al. (1996)

- Exact recursive expected recourse cost $Q(x)$ computation for a given solution for homogeneous discrete demand distributions
  - Condition on the customer $\eta$ in the Type I tour where failure occurs, with probability $\bar{q}_\eta$
  - Insert customers $\eta + 1, \ldots$ into Type II tour after final customer, and use detour-to-depot
- Randomized neighborhood $N(p, r, q)$
  - Each of $q$ randomly selected customers is reinserted before or after one of $p$ randomly selected close neighbors from the list of $r$ nearest
Results for PLC recourse strategy

<table>
<thead>
<tr>
<th>Number of Customers</th>
<th>Center Depot</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DD 1</td>
</tr>
<tr>
<td>10</td>
<td>0.38%</td>
</tr>
<tr>
<td>25</td>
<td>2.04%</td>
</tr>
<tr>
<td>50</td>
<td>5.67%</td>
</tr>
<tr>
<td>100</td>
<td>8.42%</td>
</tr>
<tr>
<td>150</td>
<td>11.17%</td>
</tr>
</tbody>
</table>

Table: Average percent improvement in expected travel cost generated by the PLC recourse strategy for test problems; average over ten instances
## Results for PLC recourse strategy

<table>
<thead>
<tr>
<th>Number of Customers</th>
<th>Corner Depot</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DD 1</td>
</tr>
<tr>
<td>10</td>
<td>4.01%</td>
</tr>
<tr>
<td>25</td>
<td>5.85%</td>
</tr>
<tr>
<td>50</td>
<td>8.66%</td>
</tr>
<tr>
<td>100</td>
<td>15.46%</td>
</tr>
<tr>
<td>150</td>
<td>15.88%</td>
</tr>
</tbody>
</table>

Table: Average percent improvement in expected travel cost generated by the PLC recourse strategy for test problems; average over ten instances
Limitations of VRPSD models

Physical capacity $Q$ does not create a need for additional vehicles
Limitations of VRPSD models

Physical capacity $Q$ does not create a need for additional vehicles.

**Single vehicle feasibility**

There exists a feasible solution to VRPSDC problems in which a single tour is planned from the depot.
Limitations of VRPSD models

Physical capacity $Q$ does not create a need for additional vehicles.

**Single vehicle feasibility**

There exists a feasible solution to VRPSDC problems in which a single tour is planned from the depot.

- Remember, if capacity fails then we can always *detour-to-depot* to unload.
Limitations of VRPSD models

Using multiple vehicles to reduce expected cost

Any “vehicle” beyond the first used can be interpreted as a *pre-emptive* detour to the depot for vehicle one!
Limitations of VRPSD models

Using multiple vehicles to reduce expected cost

Any “vehicle” beyond the first used can be interpreted as a pre-emptive detour to the depot for vehicle one!

- $\tilde{q}_i = 1$ or 2 each with probability $\frac{1}{2}$, and $Q = 3$
- Expected cost (left) = 4.5, expected cost (right) = 4
Limitations of VRPSD models

*Time constraints* are the real reason why multiple vehicles are needed for stochastic routing problems

**Ad-hoc modeling**

- Insist on target fleet size $m$ that implicitly limits tour durations
Limitations of VRPSD models

*Time constraints* are the real reason why multiple vehicles are needed for stochastic routing problems.

**Ad-hoc modeling**
- Insist on target fleet size $m$ that implicitly limits tour durations.

**Explicit modeling**
- VRPSD with Duration Constraints (VRPSD-DC)
- VRPSDC with Time Window Constraints
Detour-to-depot adds to tour duration

Assumptions

- All travel times known with certainty
- Uncertain number and location of recourse actions creates uncertainty in tour duration
Using Robust Constraints for VRPSD-DC

Morales (2006); Erera, Morales, and Savelsbergh (2010, to appear)

Robust duration constraints
Using Robust Constraints for VRPSD-DC

Morales (2006); Erera, Morales, and Savelsbergh (2010, to appear)

Robust duration constraints

- Use a two-stage model to minimize expected tour costs under a recourse policy
Using Robust Constraints for VRPSD-DC

Morales (2006); Erera, Morales, and Savelsbergh (2010, to appear)

Robust duration constraints

- Use a two-stage model to minimize expected tour costs under a recourse policy
- No recourse for a tour requiring too much time!
  1. Chance constraint, or
  2. Objective function penalty, or
  3. Robust constraint
Modeling using robust constraints

- **Uncertainty space** $\mathcal{U}$: a subset (not necessarily strict) of the *support* of the random parameters
Modeling using robust constraints

- **Uncertainty space** $\mathcal{U}$: a subset (not necessarily strict) of the support of the random parameters
- We will say that a second-stage constraint is a **robust constraint** if it must hold for every parameter realization in $\mathcal{U}$
Modeling using robust constraints

- **Uncertainty space** $\mathcal{U}$: a subset (not necessarily strict) of the support of the random parameters
- We will say that a second-stage constraint is a robust constraint if it must hold for every parameter realization in $\mathcal{U}$
  - Note: if $\mathcal{U}$ contains all outcomes, this idea is covered by two-stage recourse model formulations
**VRPSD with Robust Duration Constraints**

- Set of $n$ customers, stochastic integer demand $\tilde{q}_i$
- Recourse policy $\mathcal{P}$, separable by tour
- Change in fixed tour duration due to recourse, $\phi(T, \mathcal{P}, q)$

$$\min_{T_1, \ldots, T_m} \sum_{k=1}^{m} t(T_k) + E_{\tilde{q}}[\phi(T_k, \mathcal{P}, \tilde{q})]$$

**Subject to**

Each customer on single tour

$$t(T_k) + \phi(T_k, \mathcal{P}, q) \leq D \quad \forall \ q \in \mathcal{U}$$
**VRPSD with Robust Duration Constraints**

- Set of $n$ customers, stochastic integer demand $\tilde{q}_i$
- Recourse policy $P$, separable by tour
- Change in fixed tour duration due to recourse, $\phi(T, P, q)$

\[
\begin{align*}
\min_{T_1, \ldots, T_m} & \sum_{k=1}^{m} t(T_k) + E_{\tilde{q}}[\phi(T_k, P, \tilde{q})] \\
\text{st} & \\
& \text{Each customer on single tour} \\
& t(T_k) + \max_{q \in U} \phi(T_k, P, q) \leq D
\end{align*}
\]
Adversarial problem

\[
\max_{q \in \mathcal{U}} \phi(T, \mathcal{P}, q)
\]

- Separability of recourse policy allows tour-by-tour evaluation
Adversarial problem

\[ \max_{q \in \mathcal{U}} \phi(T, \mathcal{P}, q) \]

- Separability of recourse policy allows tour-by-tour evaluation

Is adversarial problem challenging?

- Is \( q^* = \bar{q} \)?
Adversarial problem

\[ \max_{q \in U} \phi(T, P, q) \]

- Separability of recourse policy allows tour-by-tour evaluation

Is adversarial problem challenging?

- Is \( q^* = \overline{q} \)?
  - Not worst for most recourse policies \( P \)
Adversarial problem

\[ \max_{q \in \mathcal{U}} \phi(T, \mathcal{P}, q) \]

- Separability of recourse policy allows tour-by-tour evaluation

Is adversarial problem challenging?

- Is \( q^* = \overline{q} \)?
  - Not worst for most recourse policies \( \mathcal{P} \)

- Is \( q^* \in \{ q \in \mathbb{Z}^n_+ : q_i \in \{q_i, \overline{q}_i\} \ \forall \ i \} \)?
Adversarial problem

\[
\max_{q \in \mathcal{U}} \phi(T, \mathcal{P}, q)
\]

- Separability of recourse policy allows tour-by-tour evaluation

Is adversarial problem challenging?

- Is \( q^* = \overline{q} \)?
  - Not worst for most recourse policies \( \mathcal{P} \)
- Is \( q^* \in \{q \in \mathbb{Z}_+^n : q_i \in \{q_i, \overline{q}_i\} \ \forall \ i\} \)?
  - Again, no.
Solving adversarial problem for detour-to-depot recourse

Conceptual idea

Given a tour $T$, adversary can choose the demands $q_i$ of each customer to \textbf{maximize} the additional duration of the tour due to recourse actions.
Solving adversarial problem for detour-to-depot recourse

Conceptual idea

Given a tour $T$, adversary can choose the demands $q_i$ of each customer to maximize the additional duration of the tour due to recourse actions.

Polynomial longest-path problem

For a tour $T$ with $n$ customers, the maximum duration can be computed in $O(n^4)$ by generating an acyclic network and solving a longest-path problem.
Previous-recourse Network $G_1$

Cost of arcs into node $(r, i/j)$ is the additional travel time: $2t_{0i}$
Previous-recourse network $G_1$

Recourse conditions define which arcs exist

**Observation**

Given a demand realization $q$ such that a recourse action occurs at customer $i$, then remaining vehicle capacity when departing $i$ is $Q - q(i)$. 
Previous-recourse network $\mathcal{G}_1$

Recourse conditions define which arcs exist

**Observation**

Given a demand realization $q$ such that a recourse action occurs at customer $i$, then remaining vehicle capacity when departing $i$ is $Q - q(i)$.

**Observation**

If a recourse action occurs at $i$, and the prior recourse occurred at $j$, then there exists a minimum demand $\underline{q}(i/j)$ at $i$ that can cause recourse:

$$\underline{q}(i/j) = \max \left\{ 1, Q + 1 - \sum_{\ell=j}^{i-1} \bar{q}(\ell) \right\}$$
Theorem

Assume there exists \( q \in \mathcal{U} \) such that the \((r - 1)\)-th recourse occurs at \( j \) and the \( r \)-th recourse occurs at \( i > j \). The \((r + 1)\)-th recourse can occur at \( k > i \) if and only if

\[
\max \{q(i/j), q(i)\} + \sum_{\ell=i+1}^{k-1} q(\ell) \leq Q \quad \text{and} \quad \sum_{\ell=i}^{k} \overline{q}(\ell) \geq Q + 1
\]
How do duration constraints affect the solution?

Table 3: Metrics for the best solution found for the constrained version (with a total expected travel time of 352.32)

<table>
<thead>
<tr>
<th>Tour</th>
<th>Fixed Duration</th>
<th>Expected Duration</th>
<th>Max Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,20,9,3,12,4}</td>
<td>109.90</td>
<td>118.73</td>
<td>159.90</td>
</tr>
<tr>
<td>{10,11,19,7,8,18,6}</td>
<td>99.93</td>
<td>115.03</td>
<td>152.43</td>
</tr>
<tr>
<td>{5,17,16,14,15,2,13}</td>
<td>102.40</td>
<td>118.56</td>
<td>163.23</td>
</tr>
</tbody>
</table>

Observe that two out of the three tours in this solution would therefore exceed this duration limit.

Table 3 summarizes the solution of the constrained version when \( \alpha = 0.95 \), and Figure 7 depicts the solutions to both the unconstrained and constrained versions graphically.

Figure 7: Best unconstrained solution and best constrained solution (\( \alpha = 0.95 \)).
How do duration constraints affect the solution?

**Table:** Unconstrained version (total expected time 345.84)

<table>
<thead>
<tr>
<th>Tour</th>
<th>{4, ..., 1}</th>
<th>{10, ..., 18}</th>
<th>{2, ..., 13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed (L)</td>
<td>109.90</td>
<td>93.39</td>
<td>107.42</td>
</tr>
<tr>
<td>Expected (L_E)</td>
<td>115.73</td>
<td>99.34</td>
<td>130.77</td>
</tr>
<tr>
<td>Max (L)</td>
<td>173.14</td>
<td>145.88</td>
<td>168.25</td>
</tr>
</tbody>
</table>

**Table:** Constrained version (total expected time 352.32)

<table>
<thead>
<tr>
<th>Tour</th>
<th>{1, ..., 4}</th>
<th>{10, ..., 6}</th>
<th>{5, ..., 13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed (L)</td>
<td>109.90</td>
<td>99.93</td>
<td>102.40</td>
</tr>
<tr>
<td>Expected (L_E)</td>
<td>118.73</td>
<td>115.03</td>
<td>118.56</td>
</tr>
<tr>
<td>Max (L)</td>
<td>159.90</td>
<td>152.43</td>
<td>163.23</td>
</tr>
</tbody>
</table>
Impact of robust duration constraints

<table>
<thead>
<tr>
<th>$(N, Q, \sigma, m)$</th>
<th>$\alpha$</th>
<th>$\Delta \mathcal{L}_E$</th>
<th>$m + z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(100, Q_1, \text{High, 6})$</td>
<td>0.95</td>
<td>0.69%</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>1.60%</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>-0.31%</td>
<td>7</td>
</tr>
<tr>
<td>$(100, Q_1, \text{Med, 6})$</td>
<td>0.95</td>
<td>1.01%</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>2.86%</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>-0.41%</td>
<td>7</td>
</tr>
<tr>
<td>$(100, Q_1, \text{Low, 6})$</td>
<td>0.95</td>
<td>1.18%</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>1.18%</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>5.22%</td>
<td>7</td>
</tr>
<tr>
<td>$(100, Q_2, \text{High, 3})$</td>
<td>0.95</td>
<td>0.67%</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>1.28%</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>-1.66%</td>
<td>4</td>
</tr>
<tr>
<td>$(100, Q_2, \text{Med, 3})$</td>
<td>0.95</td>
<td>0.17%</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>0.61%</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>-1.15%</td>
<td>4</td>
</tr>
<tr>
<td>$(100, Q_2, \text{Low, 3})$</td>
<td>0.95</td>
<td>0.17%</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>0.89%</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>-0.74%</td>
<td>4</td>
</tr>
</tbody>
</table>
Recourse policies for time-constrained routing problems

Parameter availability

Assume that all customers to be served, and their demands, are known prior to vehicle loading
Parameter availability

Assume that all customers to be served, and their demands, are known prior to vehicle loading.

Question

Can we create a plan that preserves most of the benefits of traditional *a priori* routes, but can be used for problems with hard time constraints?
VRP with Stochastic Demand and Customers and Time Windows

Definition (VRPSDC-TW)

- Set of $n$ possible customers
- Stochastic integer demand $\{\tilde{q}_i\}$, non-zero demand probability $\{p_i\}$
- Time windows $[e_{1i}^1, l_{1i}^1]$ and $[e_{2i}^2, l_{2i}^2]$ 
- Recourse (control) strategy

Find:

- Set of fixed routes such that
  - Each customer served by exactly one fixed route
  - Control strategy yields actual routes that are capacity and time feasible
  - Total expected travel costs given $\mathcal{P}$ minimized
Traditional recourse policy for VRPSDC

1st stage: *A priori* tours

- Minimize expected cost

2nd stage: Operational tours

- Use fixed recourse policy

**Bertsimas (1992) Type 'B' Recourse Policy**

- Follow *a priori* tour, skipping customers with no demand
- When vehicle capacity met or exceeded, *detour to depot* to unload
Recourse Strategy for Time Hedging

Erera, Savelsbergh, Uyar (2009)
Introduce backup, or secondary, vehicles

- Each customer assigned to at most 2 fixed “routes”
Recourse Strategy for Time Hedging

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  - A “route” now is simply an unordered set of customers
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- Each customer assigned to at most 2 fixed “routes”
  - Primary and secondary
  - A “route” now is simply an unordered set of customers
- Recourse decisions determined by problem of finding set of actual routes such that:
  - Each customer served by either its primary or its secondary vehicle
  - All actual routes time and capacity feasible
  - Total travel cost of actual routes is minimized

Features

1. Preserves benefits of traditional fixed routes
2. Allows “flexibility to restore feasibility and reduce costs
Recourse Strategy for Time Hedging

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Features
1. Preserves benefits of traditional fixed routes
2. Allows flexibility to restore feasibility and reduce costs
Primary + Secondary Recourse

Primary assignments
Primary + Secondary Recourse

Primary + secondary assignments
Primary + Secondary Recourse

Actual operational routes
United Distributors, Atlanta, USA

- Distributor of beer, wine, and spirits
- Serve northern Georgia: (150 by 150 km)
- Customer set
  - approximately 2500 known customer locations
  - Wide variation in $p_i$
  - Moderate variation in $\tilde{q}_i$
- Single depot
- Fleet of approximately 50 homogeneous vehicles
S+R Optimization in Logistics

Vehicle Routing under Uncertainty

Using Chance Constraints

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- Serve northern Georgia: (150 by 150 km)
- Customer set
  - approximately 2500 known customer locations
  - Wide variation in $p_i$
  - Moderate variation in $\tilde{q}_i$
- Single depot
- Fleet of approximately 50 homogeneous vehicles

Conjecture

Problem size indicates that heuristic approach appropriate!
Customer classification

- Company prefers different fixed routes for each delivery day
  - Mon, Tue, Wed, Thu, Fri
- Large fraction of customers have very low probability of delivery

<table>
<thead>
<tr>
<th>Probability Range</th>
<th>M</th>
<th>Tu</th>
<th>W</th>
<th>Th</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_i &lt; 10%$</td>
<td>522</td>
<td>621</td>
<td>820</td>
<td>870</td>
<td>925</td>
</tr>
<tr>
<td>$p_i \geq 10%$</td>
<td>330</td>
<td>1453</td>
<td>1366</td>
<td>1647</td>
<td>1573</td>
</tr>
</tbody>
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<td>1366</td>
<td>1647</td>
<td>1573</td>
</tr>
</tbody>
</table>

Customer partition

- High probability customers assigned to primary and secondary routes
- Low probability customers only added dynamically to operational routes during recourse
## Time Window Characteristics

<table>
<thead>
<tr>
<th>Earliest 1</th>
<th>Latest 1</th>
<th>Earliest 2</th>
<th>Latest 2</th>
<th>Customers</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00 AM</td>
<td>11:00 AM</td>
<td>2:00 PM</td>
<td>6:00 PM</td>
<td>1240</td>
<td>28.47</td>
</tr>
<tr>
<td>8:00 AM</td>
<td>4:00 PM</td>
<td>/</td>
<td>/</td>
<td>902</td>
<td>20.71</td>
</tr>
<tr>
<td>6:00 AM</td>
<td>11:00 AM</td>
<td>/</td>
<td>/</td>
<td>359</td>
<td>8.24</td>
</tr>
<tr>
<td>11:00 AM</td>
<td>6:00 PM</td>
<td>/</td>
<td>/</td>
<td>234</td>
<td>5.37</td>
</tr>
<tr>
<td>10:00 AM</td>
<td>6:00 PM</td>
<td>/</td>
<td>/</td>
<td>177</td>
<td>4.06</td>
</tr>
<tr>
<td>2:00 PM</td>
<td>6:00 PM</td>
<td>/</td>
<td>/</td>
<td>114</td>
<td>2.62</td>
</tr>
<tr>
<td>6:00 AM</td>
<td>1:00 PM</td>
<td>/</td>
<td>/</td>
<td>96</td>
<td>2.20</td>
</tr>
<tr>
<td>8:00 AM</td>
<td>1:00 PM</td>
<td>/</td>
<td>/</td>
<td>80</td>
<td>1.84</td>
</tr>
<tr>
<td>8:00 AM</td>
<td>12:00 PM</td>
<td>/</td>
<td>/</td>
<td>75</td>
<td>1.72</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>TOTAL=</strong></td>
<td><strong>3277</strong></td>
</tr>
</tbody>
</table>
Using two-stage model with chance constraints

Find primary assignments:

- Set of $n$ possible customers
- Stochastic integer demand $\{\tilde{q}_i\}$, non-zero demand probability $\{p_i\}$
- Time windows $[e^1_i, l^1_i]$ and $[e^2_i, l^2_i]$
- Skipping policy $\mathcal{P}$

Find:

- Set of fixed routes such that
  - Each customer served by exactly one fixed route
  - Policy $\mathcal{P}$ creates feasible actual routes with high probability
  - Total expected travel costs given $\mathcal{P}$ minimized
Primary routes heuristic

Main Ideas

- Construct primary routes via sequential insertion
- Periodic calls to local search improvement routine
- Evaluate feasibility and expected travel cost via sampling, assuming operational routes will be constructed by skipping recourse strategy only
Primary routes heuristic

Main Ideas

- Construct primary routes via sequential insertion
- Periodic calls to local search improvement routine
- Evaluate feasibility and expected travel cost via sampling, assuming operational routes will be constructed by skipping recourse strategy only

Primary routes planned as traditional fixed routes
Insertion feasibility

Capacity Feasibility

- Central limit theorem for tour demand normality
- Use traditional chance constraint form: $M_k + \tau S_k \leq Q$, with $i$ added
- $\tau$ corresponds to $\alpha = 0.90$
Insertion feasibility

Capacity Feasibility
- Central limit theorem for tour demand normality
- Use traditional chance constraint form: $M_k + \tau S_k \leq Q$, with $i$ added
- $\tau$ corresponds to $\alpha = 0.90$

Time Window Feasibility
- Using $\{p_j\}$, generate Monte Carlo sample of $N$ customer realizations
- Customer $i$ in all realizations (conditional sample)
- Time windows must be satisfied in fraction $\beta$ of realizations ($\beta = 0.80$)
Secondary routes heuristic

- Generate a sample of realizations
- Solve a simple control problem for each realization
  - First apply simple skipping strategy to each primary route
  - Select customer on infeasible route to eject at random and find feasible reinsert location that minimizes change in route quality
  - Repeat until all routes feasible
  - Apply improvement local search to improve route quality
  - Record route serving each customer
- Most frequent route serving each customer, excluding the primary route, is secondary assignment
Operational routes heuristic

- Given a single actual realization of customers and their actual demands
- First apply simple skipping strategy to each primary route
- Restore feasibility using secondary assignments
  - Select customer on infeasible route to eject at random and find feasible reinsert location on secondary route that minimizes change in route quality
  - Repeat until all routes feasible
- Apply improvement local search to improve route quality
- Insert all low-probability customers arriving which do not have primary+secondary assignments
- Apply improvement local search to improve route quality
- Apply route elimination, respecting primary+secondary assignments
### Thursdays: Comparison with History

<table>
<thead>
<tr>
<th>Day</th>
<th>Routes</th>
<th>Customers</th>
<th>Total Miles</th>
<th>Travel Min</th>
<th>% in Miles</th>
<th>% in Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.H</td>
<td>43</td>
<td>856</td>
<td>5360</td>
<td>8428</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>1.GT</td>
<td>40</td>
<td>856</td>
<td>4459</td>
<td>7080</td>
<td>16.80</td>
<td>15.99</td>
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<tr>
<td>2.H</td>
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<td>3.H</td>
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</tr>
<tr>
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<td>14.47</td>
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Route assignments:
- 60-65% of customers served by primary route
- 7% of customers are dynamic (not on planned routes)

70/78
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</tr>
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</table>

### Route assignments

- **60-65 % of customers served by primary route**
- **7 % of customers are dynamic (not on planned routes)**
Impact of Sample Size $N$

Table: Fixed Route Results for Different Sample Size Parameter Values

<table>
<thead>
<tr>
<th>$N$</th>
<th>Run Time (hours)</th>
<th>Avg. Time Feasibility</th>
<th>Final Number of Routes</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>1.70</td>
<td>0.839</td>
<td>43</td>
</tr>
<tr>
<td>1000</td>
<td>3.24</td>
<td>0.860</td>
<td>43</td>
</tr>
<tr>
<td>2000</td>
<td>10.03</td>
<td>0.878</td>
<td>43</td>
</tr>
<tr>
<td>3000</td>
<td>15.80</td>
<td>0.882</td>
<td>42</td>
</tr>
</tbody>
</table>
### Impact of Sample Size $N$

**Table: Daily Route Results for Different Sample Size Parameter Values**

<table>
<thead>
<tr>
<th>$N$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of infeasible days</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Avg. travel time</td>
<td>4,411</td>
<td>4,503</td>
<td>4,534</td>
<td>4,485</td>
</tr>
<tr>
<td>Avg. number of vehicles</td>
<td>39.83</td>
<td>40.25</td>
<td>40.50</td>
<td>39.25</td>
</tr>
<tr>
<td>Max. number of vehicles</td>
<td>42</td>
<td>42</td>
<td>42</td>
<td>41</td>
</tr>
<tr>
<td>Percentage of customers visited by primary vehicle</td>
<td>62%</td>
<td>64%</td>
<td>63%</td>
<td>64%</td>
</tr>
<tr>
<td>Run times (secs.)</td>
<td>14.75</td>
<td>14.17</td>
<td>14.83</td>
<td>14.50</td>
</tr>
</tbody>
</table>
Approximating multiple stage problems

- When are multi-stage models appropriate?
  - When decisions made during each stage impact the initial state during the next (and future) stages
Approximating multiple stage problems

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- Multi-stage models capture the true process of uncertain information becoming known during stages
Approximating multiple stage problems

- When are multi-stage models appropriate?
  - When decisions made during each stage impact the initial state during the next (and future) stages

- Multi-stage models capture the true process of uncertain information becoming known during stages

- A reasonable approximation, however, is to assume that all uncertainty is revealed after the first (planning) stage
  - A direct extension of rolling horizon models that assume all uncertainty is revealed during the planning stage
Approximating multiple stage problems

Rolling horizon two-stage approximation
Approximating multiple stage problems

Rolling horizon two-stage approximation

- Specify a planning horizon of a number of time periods
Approximating multiple stage problems

Rolling horizon two-stage approximation

- Specify a planning horizon of a number of time periods
- Partition planning horizon into two stages
  - Stage 1 periods: little to no uncertainty in parameters, and not modeled
  - Stage 2 periods: some parameters modeled with uncertainty
Rolling horizon two-stage approximation

- Specify a planning horizon of a number of time periods
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- Solve two-stage model, assuming that all uncertainty is revealed in the second stage

Approximating multiple stage problems
Approximating multiple stage problems

Rolling horizon two-stage approximation

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- Partition planning horizon into two stages
  - Stage 1 periods: little to no uncertainty in parameters, and not modeled
  - Stage 2 periods: some parameters modeled with uncertainty
- Solve two-stage model, assuming that all uncertainty is revealed in the second stage
- Implement some decisions, roll horizon forward, and repeat
Two-stage robust repositioning problem

Erera, Morales, Savelsbergh (2009)
Repositioning in transportation

✓ Carriers earn revenue moving loads
✓ *Example:* Tank container management
Repositioning in transportation

- Seattle
- Oakland
- LA/LB

- NY/NJ
- Norfolk
- Savannah

- Southampton
- Hamburg
- Rotterdam

- 491
- 585
- 249

- Algeciras
- Gioia Tauro

- 246

- Busan
- Shanghai
- Hong Kong

- Port Klang
- Singapore

- Rio
- Buenos Aires

- 127
- 838
- 148
- 458

✓ Loads not balanced
✓ 6 month loaded flows

Net surplus
Net deficit
Repositioning in transportation

- Repositioning plan
- Static regional policy in this example
Repositioning in transportation

• System dynamics…
  – Short-term, long-term seasonality
  – Add new customers, lose old customers
  – Add new demand flows, lose old demand flows

• … and system uncertainties
  – Where and when will customer base change?
  – Where and when will demand flows change?
  – Business cycles
Repositioning in transportation

• Planning question
  – How should I move equipment this period?
Dynamic repositioning practice

Weekly depot-to-depot via network flow

- *Initial* number of resources at each depot
- *Point forecasts* of the net supply of resources at each depot during each week

\[
\begin{array}{cccccc}
\text{w=0} & \text{w=1} & \text{w=2} & \text{w=3} & \text{w=4} \\
\text{Initial inventory} & 14 & 44 & 23 & 37 & \text{Forecast Outflow} \\
\text{Net forecast} & -30 & -23 & -25 & \text{Forecast Inflow} \\
\end{array}
\]
Network flow problem

$w=0$  $w=1$  $w=2$  $w=3$  $w=4$  $w=5$

$+5$  $-10$  $-32$  $-27$  $-12$  $-18$

$+15$  $+10$  $-5$  $-13$  $+7$  $-3$

$+40$  $+35$  $+20$  $+14$  $+5$  $+17$

$I = \text{Inventory arcs}$

$R = \text{Repositioning arcs}$

$b = \text{Net estimated supply}$
Network flow problem

\[ G = (N, A) \]

\[ A = I \cup R \]

Nominal repositioning problem

\[ \text{NP} \min_{x \in \mathbb{Z}^n} \{ cx : A x = b, \; x \geq 0 \} \]

\[ A = \text{node-arc incidence matrix} \]

\[ b = \text{point forecast supply vector, nominal values} \]
Distribution forecasts

- Estimation difficulties
  - Blending known and unknown information
  - Signal may be evolving rapidly, or unpredictably
Alternative robust approach

• Expected value minimization limitations
  – Estimation of distribution forecasts
  – Risk-neutrality
  – Computability for large-scale problems

• Robust approach goals
  – Simpler input requirements
  – Computability with off-the-shelf optimization software
  – Focus on service
    • Ensure ability to serve future customer requirements
    • Parametric control of conservatism
Robust optimization

Adjustable robust counterpart

\[
\text{ARC} \min_{x \in \mathbb{R}^n} \left\{ cx : \forall \tilde{b} \in \mathcal{Z} \ \exists y(\tilde{b}) : Ax + By(\tilde{b}) \leq \tilde{b} \right\}
\]

Erera, et al. (2009)
Transformable robust problem
Related work

• Atamturk and Zhang (2007)
  – Two-stage network flow and design with uncertain demand
  – Complexity of separation problem
  – Tractable special cases
    • Lot-sizing problems
• Bertsimas and Sim (2003)
  – Robust network flow with uncertain costs
Robust repositioning framework

Symmetric interval forecasts

\[ \tilde{b} \in [b - \bar{b}, b + \bar{b}] \quad \text{where} \quad \bar{b} \geq 0 \]

Point forecast \( b \)

<table>
<thead>
<tr>
<th>w=0</th>
<th>w=1</th>
<th>w=2</th>
<th>w=3</th>
<th>w=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>-30</td>
<td>-23</td>
<td>-25</td>
<td>-50</td>
<td></td>
</tr>
</tbody>
</table>

Intervals

\([-32,-28]\) \([-26,-20]\) \([-30,-20]\) \([-58,-42]\)

No distribution for \( \tilde{b} \) assumed
Robust repositioning framework

Nominal repositioning problem
Robust repositioning framework

An optimal solution for the nominal problem
Robust repositioning framework

Problem with uncertainty intervals
Robust repositioning framework

Problem with uncertainty intervals
Risk of a stock out
Robust repositioning framework

Alternative feasible solution for the nominal problem
Higher nominal cost, but no stockout risk
Robust repositioning framework

Given realization, nominal solution *not feasible*

Allow variable *adjustments* to recover feasibility
Robust repositioning framework

\[
\text{NP} \quad \min_{x \in \mathbb{Z}^n} \{ cx : Ax = b, \ x \geq 0 \}
\]

Net supply realization: \( b + \delta \)

\[ \delta \in \Gamma = \left\{ \delta \in \mathbb{Z}^m : -\bar{b} \leq \delta \leq \bar{b} \right\} \]

Adjustable decisions: \( y(\delta) \in W \)

\[ A(x + y(\delta)) = b + \delta \]

\[ x + y(\delta) \geq 0 \]
Allowable adjustments

\[ y(\delta) \in W \]

- Inventory flow adjustments
  - Assume no capacity limitations
  - Homogeneous inventory carrying cost
- Local repositioning flow adjustments
  - Allow sharing between neighbors
  - Only allow flow increases
Robust repositioning framework

Transformable robust optimization problem

\[
\text{RRP}(W, \Gamma) : \\
\min_{x \in \mathbb{Z}^n} \quad cx \\
A x = b \\
x \geq 0 \\
A(x + y(\delta)) = b + \delta \\
x + y(\delta) \geq 0 \\
y(\delta) \in W \\
\forall \delta \in \Gamma
\]
Robust repositioning framework

Transformable robust optimization problem

\[ \text{RRP}(W, \Gamma) : \]
\[
\min_{x \in \mathbb{Z}^n} c x \\
A x = b \\
x \geq 0 \\
-----------------------------------------------
\]

\[ x \in H(W, \delta) \quad \forall \delta \in \Gamma \]
Solving $\mathcal{RRP}(W, \varphi_k)$

• Develop sufficient constraint sets that guarantee $x \in H(W, \delta)$ for all $\delta \in \varphi_k$

• Constraint sets vary given feasible adjustments: $W$
Parametric conservatism

\[ \Gamma = \{ \delta \in \mathbb{Z}^{\lvert N \rvert} : -\bar{b} \leq \delta \leq \bar{b} \} \]

A robust solution with respect to \( \Gamma \) may be too conservative

\[ \varphi_k = \left\{ \delta \in \Gamma : \delta_v = z_v \bar{b}_v, \mid z_v \mid \leq 1, \sum_{v \in N} \mid z_v \mid \leq k \right\} \]

At most \( k \) time-space demands can simultaneously realize worst case value
Inventory-only problem

\[ \text{RRP}(W_1, \varphi_k) \]

where \( W_1 \) restricts adjustable variables s.t.

\[ y_a = 0 \quad \forall \ a \in R \]

Nominal flows on repositioning arcs cannot be adjusted, therefore each depot must hedge against uncertainty with only its own inventory
Inventory-only problem

**Depot A**

Analyzing vulnerability

<table>
<thead>
<tr>
<th>w=0</th>
<th>w=1</th>
<th>w=2</th>
<th>w=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>([-1,+1])</td>
<td>([-2,+2])</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{Maximum vulnerability } \vartheta(\alpha) \\
N\left(\alpha = (v_t^j, v_{t+1}^j)\right) &= \{v_{\ell}^j : \ell \leq t\} \\
\vartheta(\alpha) &= \sum_{\nu \in N_\alpha} b_\nu
\end{align*}
\]
Inventory-only problem

**Bounded vulnerability** \( \nu(N, k) \)

**Simple knapsack problem**

\[
\nu(N, k) = \sum (k \text{ largest } \bar{b}_v, \; v \in N)
\]
Inventory-only problem

\[
\begin{align*}
\nu(\cdot, 1) &= 1 & \nu(\cdot, 1) &= 2 \\
\nu(\cdot, 2) &= 1 & \nu(\cdot, 2) &= 3
\end{align*}
\]
Inventory-only problem

**Theorem**
A feasible solution $\mathbf{x}$ of the nominal problem is $k$-robust inventory feasible if and only if for all inventory arcs $a$

$$x_a \geq \nu(N_a, k)$$

Solvable in polynomial time by adding pre-computed lower-bounds on inventory arcs to the nominal problem.
Inventory-only problem

Depot A

Depot B

Maximum vulnerabilities $\vartheta(\nu^j_t)$
$k$-robust inventory example

The solution is 1-robust

Bounded vulnerabilities \( \nu(N_a, 1) \)
Bounded vulnerabilities $\nu(N_{a}, 2)$

The solution is not 2-robust
Inventory pooling

If we allow depot A and B to reactively reposition containers between them, the solution is 2-robust
ROP for reactive repositioning

$$\text{ROP}(W_2, \varphi_k) :$$

$$\min_x \{ cx : Ax = b, \ x \in H(W_2, \delta) \ \forall \ \delta \in \varphi_k \}$$

[Non-negative changes to repositioning arcs]

$$W_2 : [\text{Any integer change to inventory arcs}]$$

[No changes to first period repo arcs]
Feasibility for reactive repositioning

Depots supply/receive

\[ \begin{array}{cccc}
\text{Depot A} & 1 & 2 & 2 & 4 \\
\text{Depot B} & 1 & 1 & 2 & 3 \\
\text{Depot C} & 7 & 1 & 3 & 5 \\
\end{array} \]
Feasibility for reactive repositioning

Competing arcs: no reactive flow path
Feasibility for reactive repositioning

Not competing

w=0  w=1  w=2  w=3

Depot A

Depot B

Depot C
Feasibility for reactive repositioning

**Inbound-closed nodes:** no inbound reactive arcs

```
w=0    w=1    w=2    w=3
```

![Graph with nodes and arcs for feasible repositioning](attachment:image.png)
Feasibility for reactive repositioning

Not inbound-closed nodes

w=0  w=1  w=2  w=3

Dept A

Dept B

Dept C
Feasibility for reactive repositioning

Theorem

A solution $x$ to the nominal problem is feasible for the reactive repositioning robust repositioning problem if and only if for every set of competing arcs $K$ defining an inbound closed node set $U$:

$$\sum_{a \in K} x_a \geq \nu(U, k)$$

Arc set satisfies conditions for all $k \leq 3$
Feasibility for reactive repositioning

Theorem

A solution $x$ to the nominal problem is feasible for the reactive repositioning robust repositioning problem if and only if for every set of competing arcs $K$ defining an inbound closed node set $U$:

$$\sum_{a \in K} x_a \geq \nu(U, k)$$

- Potentially large number of constraints
- Resulting formulation requires IP (not LP)
- Constraint set size independent of uncertain outcome space size!!
Return…
Robust repositioning results

Table 2: Origin-destination distribution information for loaded demands between regions for computational test.

<table>
<thead>
<tr>
<th>Region</th>
<th>Probability of Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.15 0.00 0.10 0.10 0.20 0.15 0.20 0.20 0.05</td>
</tr>
<tr>
<td>2</td>
<td>0.35 0.05 0.00 0.05 0.15 0.10 0.30 0.30 0.05</td>
</tr>
<tr>
<td>3</td>
<td>0.10 0.05 0.05 0.00 0.15 0.10 0.30 0.30 0.05</td>
</tr>
<tr>
<td>4</td>
<td>0.10 0.10 0.20 0.10 0.00 0.10 0.20 0.25 0.05</td>
</tr>
<tr>
<td>5</td>
<td>0.05 0.10 0.25 0.15 0.05 0.00 0.15 0.25 0.05</td>
</tr>
<tr>
<td>6</td>
<td>0.05 0.10 0.30 0.15 0.20 0.15 0.00 0.00 0.10</td>
</tr>
<tr>
<td>7</td>
<td>0.10 0.10 0.30 0.15 0.20 0.15 0.00 0.00 0.10</td>
</tr>
<tr>
<td>8</td>
<td>0.10 0.05 0.05 0.10 0.15 0.15 0.20 0.30 0.00</td>
</tr>
</tbody>
</table>

To avoid beginning and ending effects created by this approach for generating time-space net supplies, we truncated the problem horizon. The first 9 weeks and the final 8 weeks were eliminated from the initial 57 weeks of data, resulting in an instance with a 40 week planning horizon. The size of the container fleet was set at 600. Initial inventories of containers at each depot were determined proportional to the probability of a demand originating in its corresponding region.

Figure 5: Total plan cost by fleet size and value of control parameter $k$ for TRP1 and TRP2.

The instance was solved using TRP1 and TRP2, where for the latter, reactive repositioning was only allowed between depots in the same region. Control parameter $k$ was varied from 0 (i.e., solution to the nominal problem) to 9. Figure 5 summarizes cost results.
reactive repositioning is allowed between depots in the same region, the plan is recoverable with respect to the same level of uncertainty, defined by parameter $k$, with far fewer containers of inventory per region.

Inventory builds to hedge against uncertainty.
What to remember

1. Stochastic and robust optimization are for *dynamic* decision planning problems
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1. Stochastic and robust optimization are for *dynamic* decision planning problems
2. Many ways to effectively incorporate parameter uncertainty in logistics optimization
3. Modeling and treatment of *recourse* especially critical
4. Ensure that your model is useful (and interesting), then solve
References


