Optimization with Multiple Objectives

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Problems faced by Decision Makers

- Seeking solutions to problems with
  - Multiple competing criteria/objectives
  - Complex set of constraints
- Portfolio investment: maximize revenue, minimize risk; constraints on amount to invest and correlations between different investment options
- Scheduling of truck routes: minimize deadhead miles (driving empty truck), equalize work load among drivers, DOT restrictions
- Radiation therapy planning optimization: minimize normal tissue dose, maximize dose to tumor regions, “optimizing” coverage, conformity and other clinical factors
Job of the OR Specialist

- Incorporate important criteria into a mathematical model and find a satisfactory solution (one that meets or exceeds the decision maker’s goal for each criteria)
- “Optimize” multiple objective functions
- What does optimal mean in this case?
  - Provide a solution such that every criterion is optimized? Ideal, but not usually possible.
  - The idea of *dominated vs. non dominated*
  - Seek *non dominated* solutions
Single Objective Optimization

Single objective:
Maximize  \( z = x_1 - 3x_2 \)
Subject to \((x_1, x_2) \in X\)

Every point here is feasible and achieves objective value of 0

Optimal solution achieves objective value of 4
Multiple-Objective Optimization

- Given: $k$ objective functions involving $n$ decision variables satisfying a complex set of constraints.
- A feasible solution to a multiple objective problem is **efficient** (nondominated, Pareto optimal) if no other feasible solution is at least as good for every objective and strictly better in one.
- E.g. Given: two objectives, and the discrete set of points in “objective space” $\{[1,3],[1,2],[2,1]\}$. Assume more of each objective is better. Then the point $[1,2]$ is dominated by $[1,3]$, so it is not desirable. The efficient (non dominated) points are $[1,3]$ and $[2,1]$.
Multiple-Objective Optimization

- The set of all efficient points to a multiple-objective optimization problem is known as the *efficient frontier*.
- Regardless of how we prioritize the importance of each objective function, the best solution should be selected from the efficient frontier.
Example 1

Two Objectives; both maximize:

\[(z_1, z_2) = (x_1 - 3x_2, -4x_1 + x_2)\]
Solution Strategies

1. Multiobjective linear programming (linear constraints and linear objectives)

- Important in economics
- Algorithms exist to identify the entire efficient frontier, but computationally difficult for large problems
- Once efficient frontier is found, still need some method to select a final solution from among the (infinite) set of efficient points
Solution Strategies

2. Preemptive Optimization

- Perform the optimization by considering one objective at a time, based on priorities.
- Optimize one objective, obtain a bound (optimal objective value), put this objective as a constraint with this optimized bound and optimize using a second objective. Continue until all objectives are considered.
- If optimal objective value is obtained at each stage, the final solution is an efficient point of the original multiple-objective model.
Solution Strategies

3. Weighted Sum

- Convert multiple objectives into one single objective using weights and summation
- Determine the importance of each objective function by putting in appropriate weights. Add up all functions:

\[
Obj = \min (\omega_1 \; obj_1 + \omega_2 \; obj_2 + \ldots + \omega_n \; obj_n)
\]

\(\omega_i > 0\) for min obj, \(\omega_i < 0\) for max obj

- An optimal solution to this problem is an efficient point to the original multiple-objective model
Solution Strategies

4. Goal Programming

- Achieve target levels of each objective rather than maximized or minimized levels
- Easier to implement
- Suppose goal for \textit{obj}_1 is \textit{g}_1
  \[
  \textit{obj}_1 > \textit{g}_1, \ \textit{obj}_2 > \textit{g}_2, \ \ldots \ \textit{obj}_n > \textit{g}_n
  \]
- These goals are treated as soft constraints; i.e., they can be violated by the feasible solutions to the multiple objective model.
Solution Strategies

4. Goal Programming (cont…)

- Measure the *deviation* of each objective from targeted goal
- In essence, optimizing the objective:
  \[
  Obj = \min (\omega_1 |obj_1 - g_1| + \omega_2 |obj_2 - g_2| + \ldots + \omega_n |obj_n - g_n|)
  \]
- An optimal solution to this problem is not necessarily an efficient point to the original multiple-objective model
Example 2

Two Objectives; both maximize:

\[(z_1, z_2) = (x_1, -4x_1 + x_2)\]

Efficient Solutions in Decision Space

Efficient Frontier of Objective Set
Example of Preemptive Method

Two Objectives; both maximize:
\[ z_1 = x_1 \]
\[ z_2 = -4x_1 + x_2 \]
Assume \( z_1 \) is first priority.

- Maximize \( z_1 = x_1 \)
  subject to \( x \in X \)
- Obtain optimal objective value \( z_1 = 4 \).
  Every point along indicated segment is optimal.
- Maximize
  \[ z_2 = -4x_1 + x_2 \]
  subject to \( x \in X \) and \( x_1 = 4 \)
- Obtain solution \((4,1)\) with \( z_2 = -15 \)
Example of Weighted Sum Method

Two Objectives; both maximize:

\[ z_1 = x_1 \]
\[ z_2 = -4x_1 + x_2 \]

Assume objective \( z_1 \) is 3 times as important as objective \( z_2 \)

- Maximize \( 3z_1 + z_2 \)
  subject to \( x \in X \)
- Maximize \( -x_1 + x_2 \)
  subject to \( x \in X \)
- Obtain optimal objective value 3.5.
  Every point along line segment indicated is optimal

Optimal solutions of weighted sum objective
\( 3z_1 + z_2 \)

X

(0,0) (0, 3.5) (1, 4.5) (3.5, 2) (4, 1) (4, 0)
Multiple Objective Optimization for Radiation Therapy Treatment

- Employ some of these techniques along with techniques for handling discrete variables and/or techniques for managing nonlinear constraints and objectives, coupled with heuristic procedures such as simulated annealing, local search, genetic algorithms.

- Tractability (i.e., ability to return optimal solutions in a timely manner) can be an issue
Multiple Objective Optimization for Radiation Therapy Treatment

- Use of surrogate objectives – definitive purpose of the objectives may not be clear
- Model with only a selected subset of objectives and constraints in return for a quick solution
- Elements of uncertainty in constraints (ranges)