On the Practical Strength of Two-Row Tableau Cuts

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Abstract

Following the flurry of recent theoretical work on cutting planes from two-row mixed integer group relaxations of an LP tableau, we report on computational tests to evaluate the strength of two-row cuts based on lattice-free triangles having more than one integer point on one side. A heuristic procedure to generate such triangles (referred to in the literature as "type 2" triangles) is presented, and then the coefficients of the integer variables are tightened by lifting.

To test the effectiveness of triangle cuts, we compare the gap closed using Gomory mixed integer cuts for one round, the gap closed in one round using all the triangle cuts generated by our heuristic and the gap closed by a small number of two-row split cuts. Our tests are carried out on randomly generated instances designed to represent different problem features by varying the number of integer non-basic variables, bounds, non-negativity constraints and density, as well as on the classical MIPLIB instances.

The outcome of this computational analysis is some insight into key characteristics of MIP instances whose presence makes two-row triangle cuts computationally effective. In particular it appears to be necessary that the tableau row pairs are dense, and more subjectively that the non-basic continuous variables are "important". Unfortunately these characteristics seem rarely to be present among real life instances, and more specifically the tableau rows of the MIPLIB instances are far from dense. A preliminary version of this work has been published in [18].

1 Introduction

Cutting planes have proven to be an indispensable tool in solving mixed integer programs (MIPs). For general MIPs, Gomory mixed integer (GMI) cuts [24] and mixed integer rounding (MIR) inequalities [28] have been found to be computationally important general-purpose cutting planes. Construction of many classes of general-purpose cutting planes can be viewed as deriving valid inequalities for some relaxation of MIPs. Seen from this perspective, GMI and MIR cuts can be classified as valid inequalities for single constraint relaxations of MIPs. A natural extension is to consider inequalities that are valid for multiple constraint relaxations of MIPs. This approach has been an active area of research recently.

A two constraint relaxation of a simplex tableau is the set:

$$\{(z,y,x) \in \mathbb{Z}^2 \times \mathbb{R}^p_+ \times \mathbb{Z}^{n-p}_+ \,|\, z = f + \sum_{j=1}^{p} r^j y_j + \sum_{j=p+1}^{n} r^j x_j\}, \tag{1}$$

where $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2, r^j \in \mathbb{Q}^2\ \forall j \in \{1,...,n\}$. This relaxation is obtained by first selecting two rows of a simplex tableau corresponding to integer basic variables that are not both currently at integral value(s) and then relaxing the non-negativity of the integer basic variables.

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A further relaxation, introduced in [4], is

$$\{(z, y) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ \mid z = f + \sum_{j=1}^{n} r^j y_j\}, \tag{2}$$

in which the integrality of the non-basic integer variables is also relaxed. Several variants of (2) have been studied; see for example [2, 3, 10, 11, 14, 15, 17, 20, 21, 29] and [19] for a recent survey on the topic. This relaxation (2) has at least two appealing features : i) it is possible to obtain a complete characterization of all facet-defining inequalities using intersections cuts derived from the so-called maximal lattice-free convex sets and ii) the cutting planes derived from these facet-defining inequalities have the strongest possible coefficients for the continuous variables. The emphasis on obtaining cutting planes in which the continuous variables have strong coefficients has been discussed in previous theoretical and computational studies; see for example [1, 23].

Though the continuous variables now have strong coefficients, the inequalities are valid for the set (2) which is a significant relaxation of the original two-row set. What would be preferable would be valid inequalities for (1) taking into account the integrality of the integer non-basic variables, even if these inequalities cut off parts of the set (2). One way to keep the strong coefficients of the continuous variables and to improve further the coefficients of the integer non-basic variables is to use lifting, based on the use of so-called 'fill-in functions' [25, 26] or the 'monoidal strengthening' technique [7].

1.1 Paper Overview and Main Contribution

From a computational standpoint, our initial goal was to show that a small subset of two-row cutting planes (that are valid inequalities for the two-row relaxations discussed above) could, together with one-row cuts, be useful in closing a significant amount of gap for general MIPs. However such a goal was too ambitious and we have had to considerably restrict it. More precisely, in this paper:

- We work with a specific subclass of two-row cuts, namely the type 2 triangle cuts (that is cuts based on lattice-free triangles having more than one integer point on one side), rather than with all such cuts. We apply 'monoidal strengthening [7]' to these inequalities;
- We work with a large number of type 2 triangle cuts simultaneously, specifically all those generated by our heuristic, and we do not select among the cuts generated [1].

Thus, the main goal of this paper is to try to answer the following (simpler?) question:

> What are the characteristics of a MIP in order for type 2 triangle cuts (obtained from the tableau of the first LP relaxation by our heuristic) to be useful in comparison to or together with GMI cuts?

To answer the above question we randomly generated knapsack-type instances (by suitably modifying a generator due to Atamtürk [5]) so as to have instances whose structure we understand and control. On this class of instances we conducted an extensive computational investigation in which type 2 triangle cuts were compared against and used in conjunction with GMI cuts. The following characteristics of the instances:

- number of rows,
- density,
- presence and influence of integer variables, and
- presence of bounds on the variables

[1] Cut selection is a fundamental issue, but it is definitely not restricted to the context of two-row cutting planes and it is a crucial research topic by itself.

have been isolated and the strength of the classes of cuts has been empirically evaluated for each characteristic.[2].

The results of the above investigation, i.e., our observations on the relation between strength of the cuts and characteristics of an instance, have been compared and corroborated by another extensive set of experiments, this time on MIPLIB instances both in their original form and with some modifications (to be discussed later).

Based on these experiments, we have been able to gain some insights regarding the question raised above. In particular, our comprehensive experiments suggest that two criteria that are significant for the good performance of type 2 triangle cutting planes are

1. the presence of "important" continuous non-basic variables[3], and

2. a high density of the constraint matrix of the instance.

In our experiments, we have also used a second family of two-row cuts, namely two-row split cuts, for the purposes of comparison. These cuts have been especially useful in our experiments because they are closely related to GMI cuts (both classes are associated to split disjunctions), and they are also two-row cutting planes, derived from a pair of tableau rows.

1.2 Literature Survey

The first computational investigation to test the practical impact of multi-row cuts was conducted in [22]. Multi-row cuts generated from 2 to 15 constraints were considered, and two classes of maximal lattice-free bounded convex sets were used for generating cuts. The first class consisted of simplices, while the second consisted of cross polytopes. The computational experiments compared CPLEX branch-and-cut enforced with multi-row cuts versus CPLEX default on a test bed of 87 MIPLIB 3.0 and MIPLIB 2003 instances. Despite some negative examples in which CPLEX default yielded a better dual bound at the root node and solved the problem faster, the results reported were judged to be encouraging, suggesting that the use of multi-row cutting planes may hold some potential. Those experiments differ significantly from the experiment conducted here in three important aspects: (1) In our paper only cuts based on two rows are considered. (2) The lattice-free triangles we use to generate cuts are carefully constructed, whereas in [22] the lattice-free bodies considered only have generic shapes. (3) Understanding the characteristics of MIPs for which multi-constraint cuts lead to improved performance is our primary goal, while it is not a goal in [22].

In [8] the authors tested two-row cuts that are derived specifically from pairs of rows in which one of the basic variables is integral and the other is fractional. The reason for selecting such pairs of rows is based on the theoretical insights gained in the paper [9]. The test bed consisted of instances from the MIPLIB library. In these experiments, many rounds of cuts were applied and the difference in gap closed was compared. One key insight obtained is that one must be careful in interpreting results in experiments involving GMI cuts as different implementations can lead to widely different results. The experiments in [8] differ significantly from the experiment conducted here in three important aspects: (1) Most of our comparisons are based on a single round of cuts. (2) The number of two-row cuts we add is significantly larger and the selection of the row pairs from which to generate two-row cuts is also different. (3) As stated before, understanding the characteristics of MIPs for which multi-constraint cuts lead to improved performance is our primary goal, while it is not a goal in [8].

1.3 Outline of Paper

The outline of this paper is the following. In Section 2, we summarize the relevant results regarding valid inequalities of (2) and the strengthened version of these inequalities valid for (1). We also discuss reasons for the selection of cutting planes using type 2 triangles for our experiments. In Section 3, we describe the heuristic used to generate the lifted two-row inequalities. In Section 4, we describe results of experiments conducted on random and MIPLIB instances. We make some final remarks in Section 5.

[2]Some of these experiments were first reported in [18].

[3]We consider a continuous variable to be important if it changes its value from the continuous relaxation to the optimal solution.

2 Review of Basic Results

2.1 Maximal Lattice-free Convex Sets

We begin with a definition of maximal lattice-free convex sets that are crucial in describing facet-defining inequalities for (2).

Definition 2.1 ([27]) *A set $M \subseteq \mathbb{R}^m$ is called lattice-free if $\text{int}(M) \cap \mathbb{Z}^m = \emptyset$. A lattice-free convex set M is maximal if there exists no lattice-free convex set $M' \neq M$ such that $M \subsetneq M'$.*

Proposition 2.1 ([20]) *Let M be a full-dimensional maximal lattice-free convex set in $\mathbb{R}^2$. Then M is one of the following:*

1. *A split set $\{(x_1, x_2) \,|\, b \leq a_1x_1 + a_2x_2 \leq b + 1\}$ where a_1 and a_2 are co-prime integers and b is an integer,*
2. *A triangle which is one of the following:*
   - (a) *A type 1 triangle: triangle with integral vertices and exactly one integral point in the relative interior of each edge,*
   - (b) *A type 2 triangle: triangle with at least one fractional vertex v, exactly one integral point in the relative interior of the two edges incident to v and at least two integral points on the third edge,*
   - (c) *A type 3 triangle: triangle with exactly three integral points on the boundary, one in the relative interior of each edge.*
3. *A quadrilateral containing exactly one integral point in the relative interior of each of its edges.*

2.2 Convex Hull of (2) via Intersection Cuts

Using the equality constraints $z = f + \sum_{j=1}^{n} r^j y_j$ present in (2), every valid inequality can be written in the form $\sum_{j=1}^{n} \alpha_j y_j \geq \alpha_0$. It is straight forward to show that in such an inequality $\alpha_j \geq 0$ for all $j \in \{1, ..., n\}$. An inequality is called nontrivial if $\alpha_0 > 0$.

Recall that f in (2) is a two-dimensional vector with at least one component fractional. A lattice-free convex set M containing f in its interior can be used to derive the intersection cut $\sum_{j=1}^{n} \pi(r^j) y_j \geq 1$ ([6]) for (2) where the coefficients are obtained as:

$$\pi(r^j) = \begin{cases} \lambda & \text{if } \exists \lambda > 0 \text{ s.t. } f + \frac{1}{\lambda} r^j \in \text{boundary}(M) \\ 0 & \text{if } r^j \text{ belongs to the recession cone of } M. \end{cases} \tag{3}$$

One example of intersection cuts is the split cut [16]. A split cut is derived from the split set described in Proposition 2.1 by using (3).

All non-trivial valid inequalities for (2) are intersection cuts and can be derived from lattice-free convex sets using (3). This can be verified by the construction of the set $L_\alpha = \{z \in \mathbb{R}^2 \,|\, z = f + \sum_{j=1}^{n} r^j y_j, \sum_{j=1}^{n} \alpha_j y_j \leq 1\}$ corresponding to the inequality $\sum_{j=1}^{n} \alpha_j y_j \geq 1$ and then verifying that: (a) L_α is a lattice-free convex set. (b) The intersection cut generated using L_α via (3) dominates the inequality $\sum_{j=1}^{n} \alpha_j y_j \geq \alpha_0$; see [4, 29] for details.

It is not difficult to prove that given two lattice-free convex sets M^1 and M^2 both containing f in the interior, if $M^1 \supseteq M^2$, then the inequality generated using M^1 dominates the inequality generated by M^2. Therefore it is sufficient to consider all maximal lattice-free convex sets described in Section 2.1 to obtain the convex hull of (2). In particular, type 2 triangles can lead to facet-defining inequalities for (2). We refer the reader to [4, 17] for a complete characterization of the facet-defining inequalities of the convex hull of (2).

2.3 Deriving all the facets for (2)

If the exact rational representation of the data in (2) is known, then it is possible to efficiently enumerate all the vertices of (2) by the use of the Euclidean algorithm (see [4]). Also the extreme rays of the convex hull of (2) are well understood. Once all the vertices are known, by setting up the polar, all facet-defining inequalities can be obtained. Since the exact rational representation may not typically be known, we will outline a different strategy to construct maximal type 2 lattice-free triangles.

2.4 Strength of Type 2 Triangle Cutting Plane Closure

Cutting plane closure is a theoretical measure of the strength of a class of inequalities. Given the linear programming relaxation of a mixed integer program and a class of cutting planes, the corresponding closure is defined as the set obtained by the addition of all possible inequalities of this class to the linear programming relaxation. Let the split closure $\mathcal{S}$, the triangle closure $\mathcal{T}$, and the quadrilateral closure $\mathcal{Q}$ be the sets obtained by adding to the continuous relaxation of (2) cuts obtained using (3) where M is all possible split sets, maximal lattice-free triangles and maximal lattice-free quadrilaterals, respectively. Let $\mathcal{P}$ be the convex hull of (2). Given a polyhedron $Q \subseteq \mathbb{R}^n_+ \setminus \{0\}$ of the form $Q = \{x \,|\, a_i x \geq b_i, i \in \{1, \ldots, m\}\}$ where $a_i \in \mathbb{R}^n_+$ and $b_i \geq 0$ for $i \in \{1, \ldots, m\}$ and given $\alpha > 0$ a scalar, αQ is defined to be the polyhedron $\alpha Q = \{x \,|\, \alpha a_i x \geq b_i, i \in \{1, \ldots, m\}\}$ [9].

Basu et al.proved the following result regarding the strength of different families of two-row inequalities.

Theorem 2.1 ([9]) *1. Split versus triangle and quadrilateral closures: (i) $\mathcal{T} \subseteq \mathcal{S}$, (ii) $\mathcal{Q} \subseteq \mathcal{S}$.*

2. Triangle and quadrilateral closures versus $\mathcal{P}$: (i) $\mathcal{P} \subseteq \mathcal{T} \subseteq 2\mathcal{P}$, (ii) $\mathcal{P} \subseteq \mathcal{Q} \subseteq 2\mathcal{P}$.

3. Split closure versus $\mathcal{P}$: For all $\alpha \geq 1$, there is an instance of (2) such that $\mathcal{S} \nsubseteq \alpha\mathcal{P}$.

Based on the proof of Theorem 2.1 in [9], the results presented in parts (1.) and (2.) are true even when $\mathcal{T}$ is replaced by the closure obtained by using cutting planes from only maximal lattice-free triangles of type 2. Therefore, at least theoretically, we expect triangle cuts based on type 2 triangles to perform significantly better than split (and therefore GMI) cuts, whose closure $\mathcal{S}$, given the result in part (3.) can be arbitrarily far away from the convex hull of (2), i.e., $\mathcal{S} \nsubseteq \alpha\mathcal{P}$ for any $\alpha \geq 1$.

2.5 Coefficients of Integer Variables in (1)

As discussed in Section 1, valid inequalities for (2) are valid for (1). However, it is possible to strengthen such valid inequalities by taking into consideration the integrality of the variables x_j for $j \in \{p+1, \ldots, n\}$: It is possible to obtain the valid inequality ([25, 26],[7])

$$\sum_{j=1}^{p} \pi(r^j) y_j + \sum_{j=p+1}^{n} \phi(r^j) x_j \geq 1$$

for (1) where

$$\phi(r^j) = \inf_{u \in \mathbb{Z}^2} \{\pi(r^j + u)\}.$$

The function ϕ is called the 'fill-in function' [25, 26] or alternatively the above procedure is called the 'monoidal strengthening' technique [7]. The validity of the fill-in function rests on the fact that integer variables corresponding to columns r and $r + u$ where $u \in \mathbb{Z}^2$ must obtain the same coefficient in any facet-defining inequality for (1).

For an infinite version of the relaxation (1), it has been shown [20] that this strengthening yields extreme inequalities if π was obtained using (3) with M being a type 1 or type 2 triangle. Unfortunately however the valid inequalities for (2) strengthened in the way described above do not provide the complete list of facet-defining inequalities of the convex hull of (1).

In the case where M is a type 1 or type 2 triangle [20], the function ϕ can be evaluated as

$$\phi(r^j) = \min_{u \in \mathbb{Z}^2} \{\pi(r^j + u) \,|\, r^j + u + f \in M\}. \tag{4}$$

The paper [8] improves upon this result by presenting a closed-form formula to evaluate the function ϕ in the case in which π is generated using a type 1 or type 2 triangle. We have used this closed-form formula in our implementation.

2.6 Reasons for Using Type 2 Triangles

Our interest in type 2 triangles is motivated by both their theoretical properties and their relative ease of implementation as discussed in the previous section. We summarize these properties here:

1. Type 2 triangle cuts can yield facet-defining inequalities for (2).
2. The type 2 triangle closure of (2) dominates the split closure of (2) ([9]).
3. For type 2 triangles, the function $\phi(r^j)$ leads to extreme inequalities for the infinite relaxation of (2) ([20]).
4. The function $\phi(r^j)$ can be evaluated by a closed-form formula ([8]). This result is very useful as otherwise the computation of the function ϕ directly from its definition or using (4) would be a computational bottleneck.

Any type 2 maximal lattice-free triangle containing f in its interior leads to a valid inequality via (3) that cuts off the current fractional solution. The difficulty lies in identifying the correct subset of type 2 triangles.

In the next section we describe a heuristic to construct maximal lattice-free type 2 triangles.

3 A Heuristic to Generate Type 2 Triangles

Here we develop a heuristic to generate type 2 triangles. Potentially the number of such triangles is large, and even if we were able to restrict ourselves to triangles generating facet-defining inequalities for (2), the number would still be very large. Our starting point is the following observation:

A facet-defining inequality of the convex hull of (2) where $n = 3$ that corresponds to a maximal lattice-free triangle of type 2 can be obtained in the following way (see, [2, 17]):

1. Construct a facet-defining inequality of the form $\alpha_1 y_1 + \alpha_2 y_2 \geq 1$ for the set $Y^2 := \{(z, y) \in \mathbb{Z}^2 \times \mathbb{R}^2_+ \,|\, z = f + r^1 y_1 + r^2 y_2\}$ where $\alpha_1, \alpha_2 > 0$. The set $\text{conv}\{f, f + \frac{r^1}{\alpha_1}, f + \frac{r^2}{\alpha_2}\} \subseteq \mathbb{R}^2$ is lattice-free and the line segment between $f + \frac{r^1}{\alpha_1}$ and $f + \frac{r^2}{\alpha_2}$ contains at least two integer points.
2. Lift a third continuous variable y_3 to obtain the inequality $\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \geq 1$. The set $\text{conv}\{f + \frac{r^1}{\alpha_1}, f + \frac{r^2}{\alpha_2}, f + \frac{r^3}{\alpha_3}\} \subseteq \mathbb{R}^2$ is lattice-free and at least one of the sides of the triangle (other than the side between $f + \frac{r^1}{\alpha_1}$ and $f + \frac{r^2}{\alpha_2}$) contains one integer point.

Based on the above, the heuristic we now describe selects a relatively small subset of triangles, the choice motivated by the wish to obtain inequalities complementing the basic GMIC inequalities based on one-dimensional split cuts. We now present the details of our implementation for generating type 2 triangles.

1. Given three variables y_{j_1}, y_{j_2} and y_{j_3} such that $cone(r^{j_1}, r^{j_2}, r^{j_3}) = \mathbb{R}^2$, we first attempt to generate one facet $\alpha_{j_1} y_{j_1} + \alpha_{j_2} y_{j_2} \geq 1$ of $\text{conv}(Y^2)$ that is 'most different' from a GMI cut. Since the line segment between $f + \frac{r^{j_1}}{\alpha_{j_1}}$ and $f + \frac{r^{j_2}}{\alpha_{j_2}}$ is required to contain at least two integer points, we divide this step into two: one for each of the integer points to be generated. At the end of these two steps, either the integer points have been found, or no such pair exists. For simplicity let $j_i := i$ for $i = 1, 2, 3$.
   (a) Finding the first integer point: In this step we optimize an arbitrary objective function and use the optimal integer solution as the first integer point. The details are presented next. Let $c := (c_1, c_2)$ be a strict convex combination of r^1 and r^2. Solve $\min\{c^T z \,|\, \lambda_1 r^1 + \lambda_2 r^2 = -f + z,\ \lambda_1, \lambda_2 \geq 0, z \in \mathbb{Z}^2\}$. Let the optimal objective function be w^0 and the optimal solution be $(\bar{\lambda}^0_1, \bar{\lambda}^0_2, \bar{z}^0)$. The point $\bar{z}^0$ is one of the integer points belonging to the side of the triangle containing multiple integer points. (See, Figure 1(a).) In our implementation, c is selected in such a way that it bisects the angle between r^1 and r^2. This choice is motivated by the

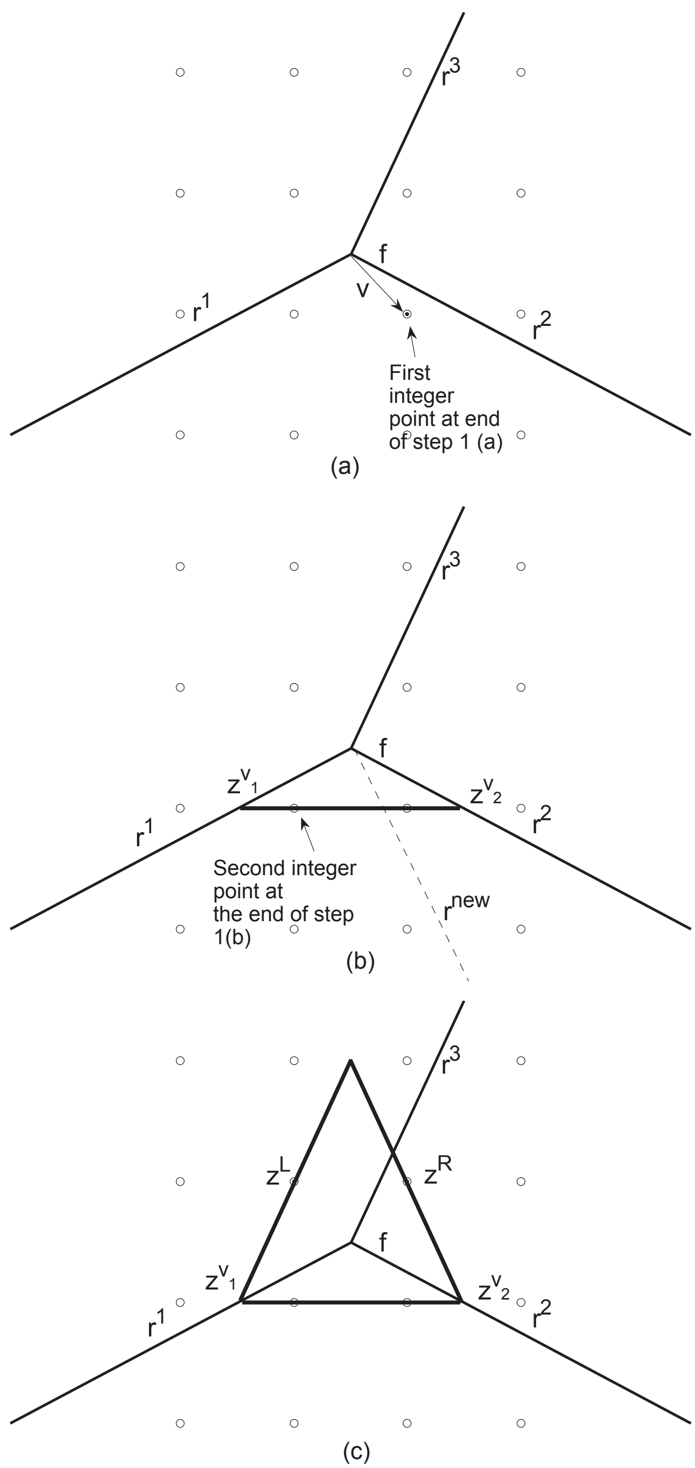

Figure 1: Illustration of heuristic method to generate type 2 triangles.

attempt to obtain an integer point which is not tight for one of the GMI cuts. Since the triangle inequality we generate will be satisfied at equality by the integer point we find in this step, in this way we expect to find an inequality different from the GMI cuts. However, we note here that the choice of c is entirely based on the above described geometric intuition and we do not have any formal theoretical justification for this choice.

(b) Finding the second integer point: Given an integer point, suppose we assume that the inequality defined by this integer point and the integer point found in the Step 1 (a) corresponds to a facet-defining inequality for the convex hull of Y^2 (and thus corresponds to one side of the triangle). To test this hypothesis, we optimize in the direction of the left-hand-side of this inequality over Y^2 . If our assumption is wrong, then we obtain a different integer that shows that the inequality is invalid. We now iterate using this new integer point. The details are presented next. Let $v = \bar{\lambda}_1^0 r^1 + \bar{\lambda}_2^0 r^2$ (i.e., $v = \bar{z}^0 - f$. See, Figure 1(a).). First an integer point belonging to the set $\{u \,|\, u = f + \mu_1 r^1 + \mu_2 r^2, \mu_1, \mu_2 \geq 0\}$ different from $\bar{z}^0$ is found to initiate the above iterative search. Let $r^{\text{new}} = v + \theta r^1$ for a suitable $\theta > 0$. In our implementation we do the following: Let α be the angle between v and r^1 and β be the angle between v and r^2. Assume without loss of generality that $\alpha > \beta$. Then in order to reduce numerical difficulties we look in the cone (v, r^1). We scale v and r^1 by integers (to avoid numerical erros) to have a l_2 norm close to 10. Finally, we set $r^{\text{new}} := v + 0.1r^1$. Thus r^{new} is a vector very close to v. For some c such that $c^T r^1 \geq 0$, $c^T r^{\text{new}} \geq 0$, solve $\min\{c^T z \,|\, \lambda_1 r^1 + \lambda_2 r^{\text{new}} = -f + z, \lambda_1, \lambda_2 \geq 0, z \in \mathbb{Z}^2\}$. In our implementation, c is selected in such a way that it bisects the angle between r^1 and r^{new}. Let the optimal solution be $(\bar{\lambda}_1^1, \bar{\lambda}_2^1, \bar{z}^1)$. Now we verify if the integer point $\bar{z}^1$ is suitable to be the second integer point on the side of the triangle. If not, we iteratively update this point. Let $e^1 \in \mathbb{R}^2$ be such that $(e^1)^T \bar{z}^0 = {e^1}^T \bar{z}^1 = 1$ and ${e^1}^T f < 1$. This is equivalent to verifying if ${e^1}^T x \geq 1$ is a valid inequality for Y^2. Repeat the following step: Solve

$$\min\{{e^i}^T z \,|\, \lambda_1 r^1 + \lambda_2 r^2 = -f + z, \lambda_1, \lambda_2 \geq 0, z \in \mathbb{Z}^2\}.$$

Let the optimal objective function be w^{i+1} and the optimal solution be $(\bar{\lambda}_1^{i+1}, \bar{\lambda}_2^{i+1}, \bar{z}^{i+1})$. Let $e^{i+1} \in \mathbb{R}^2$ be such that ${e^{i+1}}^T \bar{z}^0 = {e^{i+1}}^T \bar{z}^{i+1} = 1$ and ${e^{i+1}}^T f < 1$. If $w^{i+1} = 1$, then denote the point $\bar{z}^{i+1}$ by $\bar{z}^{i_0}$ and stop. If $w^{i+1} < 1$, then set $i \leftarrow i + 1$ and repeat this step. (See, Figure 1(b).)

One final check is needed to verify that $\bar{z}^{i_0}$ is indeed a vertex of $\text{conv}(Y^2)$. This check becomes relevant when $\text{conv}(Y^2)$ has only one vertex. Verify that $(\bar{z}^{i_0} - \bar{z}^0)$ and $(-\bar{z}^{i_0} + \bar{z}^0)$ do not belong to the cone formed by r^1 and r^2. If both belong to the cone, then we discard this pair of rays and do not construct any triangle.

2. Lifting the third continuous variable: From the previous step we have two integer points $\bar{z}^0$ and $\bar{z}^{i_0}$ that are tight at the inequality. In order to lift the third continuous variable, we need to find a maximal lattice-free triangle containing f in its interior. Thus, we need to find the other two integer points that belong to the relative interior of the two other sides of the triangle. These points are neighboring integer points on a lattice plane parallel to the one passing through $\bar{z}^0$ and $\bar{z}^{i_0}$. Let z^{v_1} and z^{v_2} be the two points obtained by extending the line segment passing through $\bar{z}^0$ and $\bar{z}^{i_0}$ and intersecting the two half lines $f + \mu_j r^j$, $\mu_j \geq 0$ $j \in \{1, 2\}$.

   Let $(a, b) = \bar{z}^0 - \bar{z}^{i_0}$. Update a and b by dividing them by their greatest common divisor and let $p, q \in \mathbb{Z}$ be such that $pa + qb = \pm 1$ and $(q, -p)^T r^3 > 0$. Then the two other integer points are of the form $\bar{z}^0 + (q, -p) + k(a, b)$ and $\bar{z}^0 + (q, -p) + (k + 1)(a, b)$ for some integer k, since as mentioned before these points are neighboring integer points lying on a lattice plane parallel to the line passing through z^{v_1} and z^{v_2}. The integer k can be calculated as follows. There are two cases:

   (a) f lies in the set $\{(w_1, w_2) \,|\, \bar{z}_1^0(-b) + \bar{z}_2^0 a \leq w_1(-b) + w_2(a) \leq \bar{z}_1^0(-b) + \bar{z}_2^0 a + 1\}$. In this case solve $\bar{z}^0 + (q, -p) + \lambda(a, b) = f + r^3\mu, \mu \geq 0$ for λ and μ and set $k = \lfloor \lambda \rfloor$.

   (b) f lies in the set $\{(w_1, w_2) \,|\, w_1(-b) + w_2(a) \geq \bar{z}_1^0(-b) + \bar{z}_2^0 a + 1\}$. Then solve $(q, -p) + \lambda(a, b) = \mu(f - \bar{z}^0)$, $\mu \geq 0$ for λ and μ and set $k = \lfloor \lambda \rfloor$.

Denote $\bar{z}^0 + (q, -p) + k(a, b)$ by z^L and $\bar{z}^0 + (q, -p) + (k+1)(a, b)$ by z^R. Construct the triangle by joining the two vertices z^{v_1} and z^{v_2} obtained in the previous step to z^L and z^R, and extending these line segments until they intersect at the third vertex. (See, Figure 1(c).)

3.1 Completeness and Strength

Note that if $R = (r^{j_1}, r^{j_2})$, the set Y^2 can be written as the two variable two constraint integer set: $\{z \in Z^2 | R^{-1}z \geq R^{-1}f\}$. The convex hulls of such sets typically have a large number of facets, and our heuristic only selects one of these facets, chosen so that the resulting triangle differs as much as possible from the split inequalities based on r^{j_1} and r^{j_2}. We also observe that for each choice of r^{j_1}, r^{j_2}, as one varies r^{j_3}, only a small number of distinct triangles can be obtained. We select only one of these triangles (see discussion in Section 4) to generate a cutting plane.

It is also natural to ask whether the triangles generated lead to facet-defining inequalities for (2). If the set (2) has only three rays, then the inequality generated by the heuristic is facet-defining as there are three tight integer points. However when more rays are present, the triangle provides valid lifting coefficients, but the resulting inequality is not facet-defining (unless, for example, if r^{j_3} goes through the third vertex of the triangle). General conditions for the resulting inequality to be facet-defining are non-trivial, see [17] and [12].

4 Computational Experiments

The aim of this computational section is to analyze the effectiveness of different families of cuts associated with a two-row relaxation of a simplex tableau. In particular, we would like to answer some specific questions:

1. How much can be gained from using cuts based on the two-row relaxation rather than the classical cuts generated from a one-row relaxation?

   GMI cuts from the simplex tableau can be derived (almost for free) by considering one row of the simplex tableau at a time. What can we gain in practice by considering two rows simultaneously?

2. What is the relative importance of different families of two-row cuts?

   As discussed in Section 2, all the facet-defining inequalities for the two-row relaxation (2) are intersection cuts arising from maximal two-dimensional lattice-free convex sets. In this work, we are mostly concerned with two-row cuts arising from type 2 triangles (i.e., type 2 triangle cuts) for reasons discussed in Section 2.6. We also consider two-row cuts from split sets (two-row split cuts). Split cuts from two rows are the most natural extension of classical GMI cuts, as GMI cuts from the simplex tableau are split cuts from a one-row relaxation of the problem at hand.

3. How is the effectiveness of different families of cuts based on the two-row relaxation affected by the structure of the MIP at hand?

   We aim at understanding *a priori* how the structure of the problem affects the importance of different families of two-row cuts.

The most natural approach often used when evaluating a new family of cuts is to study classical performance indicators on a suitable set of real-world instances by applying a cutting plane approach with and without the new family of cuts for a fixed number of iterations/rounds. The performance indicators typically gathered are the separation time, overall computing time, percentage gap closed, density of cuts, etc.

However, in this particular context there are a number of issues that prevent such an approach being satisfactory as is. Hence, to begin to answer the questions raised above we decided to test our cuts on MIPLIB 3.0 [13] instances, as well as on different sets of randomly generated instances for which we could control the size and (partially) the structure. This allows us to measure how the characteristics of a MIP may affect the quality of the different cuts we are considering. In addition, we decided to evaluate the impact of one round of cuts only, those generated from the first simplex tableau. In this way, we can add all the cuts we generate, without applying any (sophisticated) selection rule that might prevent us from understanding the real strength of the cuts. (Clearly, cut selection is a fundamental issue, but it is definitely not restricted to the context of two-row cutting planes and it is a crucial research topic by itself.)

More precisely, given the simplex tableau of the problem at hand, we generate one round of:

- GMI cuts (G cuts in the tables): one cut for each row associated with a basic fractional variable required to be integer.
- two-row cuts: cuts from any pair of rows whose associated basic variables are integer-constrained and with fractional values in the current basic solution. For each pair of rows, we consider the corresponding two-row relaxation (2) and we generate a set of:
  - Type 2 triangle cuts (T cuts in the tables): Given a set, say R, of suitable rays, for each pair of rays $r^1, r^2 \in R$ we select a third ray $r^3 \in R$ such that $cone(r^1, r^2, r^3) = \mathbb{R}^2$ and try to construct a type 2 triangle as described in Section 3. More precisely, given two rays r^1 and r^2, we select r^3 as the closest one to the bisector of the angle spanned by $-r^1$ and $-r^2$. If there is no r^3 such that the cone spanned by (r^1, r^2, r^3) is $\mathbb{R}^2$, then we skip the pair r^1, r^2. Of course, this is just a heuristic, as we never tried to separate more than one triangle for a fixed pair of rays. The set R contains all the rays r^j associated with continuous variables, together with rays $\hat{r}^j$ associated with integer variables obtained as follows:

    $$\hat{r}_i^j = \begin{cases} r_i^j - \lfloor r_i^j \rfloor & \text{if } f_i + r_i^j - \lfloor r_i^j \rfloor \leq 1 \\ r_i^j - \lfloor r_i^j \rfloor - 1 & \text{otherwise} \end{cases} \qquad i = 1, 2.$$

    The column r^j read from the tableau for an integer variable can be translated by integer vectors without affecting the coefficient the variable x_j receives in a cut; see discussion in Section 2. The motivation for the above choice of $\hat{r}^j$ is guided by the fact that these translations correspond to the translations needed to derive a GMI cut via monoidal strengthening of a split cut.

    Note that two-row cuts can be generated even from a row-pair in which one of the basic variables is integer at the current vertex. This is, for instance, the approach used in [8]. In this work, we decided to generate two-row cuts only from rows that can be used also for deriving one-row cuts, i.e., rows associated with fractional basic variables.
  - Non-elementary split cuts (S cuts in the tables): all the split cuts arising from non-elementary splits $\lfloor \pi f \rfloor \leq \pi_1 z_1 + \pi_2 z_2 \leq \lceil \pi f \rceil$, with $(\pi_1, \pi_2) \in [-3, 3]^2$, π_1, π_2 coprime integers. (We generate at most 14 non-elementary split cuts for any row pair by avoiding the generation of the same split twice.)

    The GMI cuts generated from one row of a simplex tableau are not always facet-defining for the set (2). However they are extremely efficient to generate. In the same spirit, the main motivation for the split cuts considered here is the ease with which they can be obtained.

The code has been implemented in C++, using IBM ILOG Cplex 10.0 as LP solver. In order to limit possible numerical issues in the cut generation, we adopted several tolerances and safeguards. In particular, we generated cuts only from tableau rows whose associated basic variables have a fractional part greater (resp. smaller) than 0.001 (resp. 0.999). In addition, we generated two-row cuts from a split or a triangle only if the fractional point f is safely in its interior. More precisely, splits and triangles are discarded if the Euclidean distance of f from the boundary of the corresponding convex body is smaller than 0.001.

4.1 From the Perfect Two-row Problem to the MIPLIB Instances

Let Opt(LP) and Opt(MIP) be the optimal objective function value of the linear programming relaxation and the mixed integer program respectively. Let $X_1, X_2, \ldots, X_k$ be families of cutting planes. Let Opt($X_1 X_2 ... X_k$) be the optimal objective function value of the linear programming relaxation after adding cutting planes from the families X_1, X_2, ..., X_k. We use the following notation for percentage gap closed:

$$\%\text{gap}(X_1 X_2 ... X_k) = \frac{\text{Opt}(X_1 X_2 ... X_k) - \text{Opt}(LP)}{\text{Opt}(MIP) - \text{Opt}(LP)} \times 100.$$

Table 1: Results on the "perfect" two-row relaxation.

%gap(G)	%gap(S)	%gap(T)	%gap(GS)	%gap(GT)	%gap(GST)
75.10	76.39	97.99	92.69	98.01	98.94

In addition, we also typically report the "relative importance" of each class of cuts, i.e., the marginal improvement in the gap closed by the cuts from a specific family. Specifically the importance of cuts from family $X \subseteq \{G, S, T\}$ (denoted as Imp(X)) is computed as

$$\text{Imp}(X) = \frac{\%\text{gap}(GST) - \%\text{gap}(GST \setminus X)}{\%\text{gap}(GST)} \times 100.$$

As a first experiment, we test our cuts on a set of randomly generated two-row instances with a fully dense simplex tableau of the form

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j y_j \\ & z = f + \sum_{j=1}^{n} r^j y_j, \\ & z \in \mathbb{Z}^2, \\ & y \geq 0, \end{aligned} \tag{5}$$

where $c_j \geq 0$, $f_i \notin \mathbb{Z}$ and $r_i^j \neq 0$ for all $j \in \{1, \ldots, n\}$ and $i \in \{1, 2\}$ (see Section 4.2 for a detailed description of the instances generated). In this very particular situation, the optimal basic solution associated with the LP relaxation of (5) is the fractional vertex $(z^*, y^*) = (f, 0)$ and, more importantly, the two-row relaxation (2) is not a relaxation, but is actually the feasible region of the problem.

On these instances, since the type 2 triangle cut closure is contained in the split closure, type 2 triangle cuts may be stronger than split cuts. The experiments are aimed at understanding if our heuristic procedure is able to capture a subset of type 2 triangles effective in practice and providing a good approximation of the convex hull of (2).

The average results obtained on 90 randomly generated instances of the form (5) are given in Table 1.

The first three columns report the average percentage gap closed by adding separately GMI cuts (G), two-row split cuts (S), and type 2 triangle cuts (T). The last three columns report the average percentage gap closed by combining different families of cuts: GMI cuts and two-row split cuts (GS), GMI cuts and type 2 triangle cuts (GT), and all the three families of cuts together (GST).

Table 1 shows that two-row cuts are useful on top of GMI cuts and stronger than GMI cuts. In particular, triangles appear superior to splits, as type 2 triangle cuts alone close more gap than one-row and two-row split cuts together. Although the triangle cuts we are generating do not completely dominate split cuts (split cuts added on top of triangle cuts move the integrality gap closed from 97.99% to 98.94%), our procedure seems effective in generating strong type 2 triangle cuts for the two-row relaxation (2).

As a second experiment, we tested our cuts on a subset of MIPLIB 3.0 [13] instances, in order to compare two-row cuts versus one-row cuts on real-world instances that are typically used as a testbed for evaluating general purpose cutting planes. Note that we did not apply any sophisticated criterion to select MIPLIB instances. We just selected manageable instances, for which we can separate a full round of triangle cuts (for all row pairs, for all pairs of non basic variables) in a reasonable computing time, even in the case of more complex experiments reported in Section 4.5.

The results are reported in Table 2. For each problem, we report again the percentage gap closed by each family of cuts, the gap closed by adding together different families and, in addition, we report also the relative importance of each class of cuts. The last row of the table gives the average results on all the instances tested.

Table 2: Results on MIPLIB instances.

Pb	%gap G or S or T			%gap G + S + T			Cut contribution			
	G	S	T	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
bell3a	45.10	63.83	42.15	63.83	59.87	63.91	29.43	6.32	0.13	34.05
bell5	14.53	15.74	16.10	15.74	16.27	16.27	10.69	0.00	3.26	1.04
blend2	16.36	9.18	14.92	17.39	17.07	17.67	7.41	3.40	1.58	15.56
danoint	0.26	0.54	0.50	0.54	0.50	0.55	52.73	9.09	1.82	9.09
dcmulti	47.27	39.73	47.53	47.54	49.26	49.46	4.43	0.40	3.88	3.90
fiber	72.56	66.70	44.10	78.87	75.58	80.98	10.40	6.67	2.61	45.54
flugpl	11.74	12.66	14.47	12.74	14.47	14.54	19.26	0.48	12.38	0.48
gen	59.82	63.57	42.45	63.81	63.05	64.12	6.71	1.67	0.48	33.80
gesa2	27.12	59.17	22.83	61.17	36.42	63.38	57.21	42.54	3.49	63.98
gesa2_o	30.86	62.62	24.87	64.53	37.51	66.55	53.63	43.64	3.04	62.63
gesa3	43.95	68.37	25.61	75.28	62.32	75.41	41.72	17.36	0.17	66.04
gesa3_o	49.02	75.78	42.72	77.30	65.95	77.49	36.74	14.89	0.25	44.87
gt2	91.87	77.94	62.51	91.87	91.87	91.87	0.00	0.00	0.00	31.96
lseu	46.35	44.49	13.36	46.69	47.23	47.37	2.15	0.30	1.44	71.80
mas74	6.67	8.94	8.59	8.94	8.59	9.26	27.97	7.24	3.46	7.24
mas76	6.42	10.74	8.57	10.74	8.57	10.74	40.22	20.20	0.00	20.20
misc06	30.39	18.27	12.15	33.43	32.28	33.72	9.88	4.27	0.86	63.97
mod008	20.10	9.07	20.59	21.23	20.91	21.25	5.41	1.60	0.09	3.11
modglob	15.62	26.62	15.01	27.34	17.48	27.34	42.87	36.06	0.00	45.10
p0033	56.82	12.69	0.13	57.03	57.03	57.03	0.37	0.00	0.00	99.77
p0282	3.70	5.95	6.41	5.97	6.48	7.10	47.89	8.73	15.92	9.72
p0548	41.79	13.54	0.54	41.87	41.82	41.87	0.19	0.12	0.00	98.71
pp08a	51.44	59.89	15.72	71.31	52.81	71.48	28.04	26.12	0.24	78.01
pp08aCUTS	31.53	37.31	33.38	41.08	38.75	42.01	24.95	7.76	2.21	20.54
qnet1_o	39.48	34.04	28.44	43.35	43.14	43.64	9.53	1.15	0.66	34.83
rout	1.35	1.18	1.34	1.35	1.49	1.49	9.40	0.00	9.40	10.07
vpm1	10.00	3.82	0.00	10.18	10.00	10.18	1.77	1.77	0.00	100.00
vpm2	13.10	25.73	9.86	26.96	17.70	27.04	51.55	34.54	0.30	63.54
avg.	31.62	33.15	20.53	39.93	35.52	40.49	22.59	10.58	2.42	40.70

Besides a few instances, such as `flugpl`, where type 2 triangles appear slightly superior to splits, the results are contrary to those reported in Table 1 for the "perfect two-row" instances. Namely two-row cuts are quite useful as they move the average gap closed from 31.62% up to 40.49%. However such an improvement is mostly due to the two-row split cuts. Type 2 triangles alone appear much weaker than GMI cuts, and the improvement in the gap that can be obtained by adding triangles on top of split cuts is almost negligible, as the average gap closed rises from 39.93% (generating GMI and two-row split cuts) to 40.49% (generating also type 2 triangle cuts).

Overall Tables 1 and 2 present completely different pictures, so this necessarily calls for an explanation. Even if MIPLIB instances are quite heterogeneous, each one with its own particular structure, there are several relevant differences between all the MIPLIB instances and the "perfect two-row" instances of the form (5), and each difference raises a different question:

1. Integrality of the non-basic variables.

   All the non-basic variables in the random instances of the form (5) are continuous variables, while in MIPLIB instances most of the non-basic variables are integer constrained. The strength of the triangle closure w.r.t. the split closure has been theoretically proven for a model where all the non-basic variables are relaxed to be continuous. Therefore, the theory developed so far considers generating facets of the "continuous" two-row relaxation (2) and then strengthening the coefficient of integer variables in the cut. In other words, the focus is mostly on the continuous component of the problem because all integer non-basic variables are considered afterwards.

   What is the impact of non-basic integer variables on the strength of the inequalities generated?

2. Bounds on the variables.

All the inequalities considered are valid for a relaxation that assumes no bounds on the integer basic variables and only non-negativity of the non-basic variables as in the random instances of type (5). However in the MIPLIB instances (most of) the variables are bounded.

What is the role played by bounds on the variables?

3. Number of rows.

   The random instances of the form (5) are two-row instances, while MIPLIB instances have more constraints. For a two-row problem, it seems quite reasonable that a cut generation approach that considers both tableau rows simultaneously should result in stronger inequalities. On the other hand the strength of cuts based on the two-row relaxation might deteriorate when moving from a two-row to an m-row problem (with $m > 2$).

   What is the impact of the number of tableau rows on the strength of cuts based on the two-row relaxation?

4. Row correlation (tableau density).

   On the one hand the random instances of type (5) have a fully dense tableau, i.e., the two tableau rows are 'completely related' to each other. In such a situation since two-row cuts use information from two rows simultaneously, they may be important on top of GMI cuts that are generated by considering one row at a time. On the other hand the first LP tableau of MIPLIB instances is generally sparse, and therefore the tableau rows might only be 'weakly related' to each other.

   What is the effect of sparsity on the strength of the inequalities generated?

   Note that in the perfect two-row case sparsity does not matter too much. Table 3 reports the same results as Table 1 on instances of the form (5) but with different "levels of sparsity", obtained by randomly fixing to 0 the tableau coefficients of the non-basic variables with different probabilities. In addition, the relative standard deviation of the %gap closed, computed as the standard deviation of the gap closed divided by the average gap closed (RSD) is also reported. As expected, by reducing

Table 3: Results on the "perfect" two-row relaxation on sparse instances.

Tableau density	%gap(G) (RSD)	%gap(S) (RSD)	%gap(T) (RSD)	%gap(GS) (RSD)	%gap(GT) (RSD)	%gap(GST) (RSD)
100%	75.10 (0.16)	76.39 (0.17)	97.99 (0.05)	92.69 (0.07)	98.01 (0.05)	98.94 (0.03)
80%	74.79 (0.18)	66.82 (0.23)	94.02 (0.09)	92.32 (0.08)	96.66 (0.07)	98.43 (0.04)
60%	80.06 (0.16)	56.79 (0.27)	91.38 (0.14)	90.29 (0.10)	97.19 (0.06)	97.70 (0.05)

   the row correlation we can observe that the difference between the percentage gap closed by GMI cuts and triangle cuts decreases, mainly because GMI cuts are much more effective on sparse problems. However, even for the *extreme case*[4] of instances with an average tableau density almost equal to 60%, triangle cuts still appear to be more important than GMI cuts and 2-row split cuts together. Observe that in Table 3, when the average %gap closed in high, the RSD value is low and when the %gap closed is lower, the RSD value is sometimes a little higher. This is true also for the standard deviations in the Table 3 (for example the smallest standard deviation is 2.9% for the case of Tableau density 60% and column %gap(GST) and the largest standard deviation is 15.3% for the case Tableau density 60% and column %gap(S)). This shows that when the cutting planes performed well, this good performance was consistent across almost all the instances. Overall, the RSD values appear to be sufficiently small for all the conclusions we derive.

   The results in Table 3 show also that our separation heuristic continues to be effective for the sparse case too.

   A more detailed analysis of the row correlation impact is given in Section 4.4, where we consider row correlation and tableau density simultaneously, because of their strong relationship.

[4] Instances with $m = 2$ and density equal to 60% are almost completely uncorrelated and GMI cuts are known to be very effective.

It is clear that each of the factors highlighted above may influence the effectiveness of GMI, two-row split and type 2 triangle cuts in a different way. In order to measure, from a computational standpoint, the relation between the characteristics of the instance and the strength of the different classes of cuts, we have designed a computational investigation on randomly generated instances, in which each specific characteristic of the instances, namely

- presence and influence of integer variables,
- presence of bounds on the variables,
- number of rows, and
- row correlation

has been isolated and the strength of the classes of cuts has been empirically evaluated[5]. The results of these experiments are reported and discussed in Sections 4.2–4.4.

The conclusions we have derived from this investigation, on the relation between strength of the cuts and characteristics of an instance, are then compared and brought into question by another set of experiments on MIPLIB instances, both in their original version and with some modifications. The results of this investigation are reported in Section 4.5.

4.2 Integrality Experiments

This section is aimed at understanding the effect of the integrality of certain non-basic variables on the strength of the different classes of cuts. To this end, we have considered random instances of the form

$$
\begin{array}{lrcl}
\min & \multicolumn{3}{l}{\displaystyle\sum_{j=1}^{p} c_j y_j + \sum_{j=p+1}^{n} c_j x_j} \\
 & z & = & \displaystyle f + \sum_{j=1}^{p} r^j y_j + \sum_{j=p+1}^{n} r^j x_j, \\
 & z & \in & \mathbb{Z}^m, \\
 & y & \geq & 0, \\
 & x & \geq & 0, \text{ integer},
\end{array}
\tag{6}
$$

with $c \geq 0$, $f \notin \mathbb{Z}^m$ and a "small" number of rows (i.e., we have tested the cases $m = 2$ and $m = 5$).

These instances are generated by using the multidimensional knapsack generator of Atamturk [5]. First, we obtained multidimensional knapsack instances having m rows and ℓ non-negative integer variables. These instances are of the form

$$
\begin{array}{ll}
\max & \displaystyle\sum_{j=1}^{\ell} q_j x_j \\
 & \displaystyle\sum_{j=1}^{\ell} w_{ij} x_j \leq b_i, \quad i = 1, \ldots, m, \\
 & x_j \geq 0, \text{ integer}, \quad j = 1, \ldots, \ell,
\end{array}
\tag{7}
$$

where all the data entries are strictly positive. After solving the continuous relaxation of these instances, we modified them slightly so as to obtain four sets having characteristics suitable for our tests. The modifications are as follows:

A. We relax the m basic variables to be free variables, we change 10% of the nonbasic variables to be non-negative continuous variables and we remove the other nonbasic variables.
In this way, we obtain instances with only continuous variables (besides the basic ones) but without too many variables so that the number of type 2 triangles is relatively small. The instances of the

[5] Some of these experiments were first reported in [18].

form (5) addressed in Section 4.1 have been obtained exactly in this way. In particular, the results reported in Table 1 are average values over 90 instances with $m = 2$ (30 with $\ell = 50$, 30 with $\ell = 100$, 30 with $\ell = 150$).

B. As for set A, but the remaining 90% are kept as non-negative integer variables.
This is now a mixed integer program in which the continuous variables (approximately 10%) need to interact with the integer variables.

C. As for set B, but the objective coefficients of the continuous variables are divided by 100.
In this way, the "importance" of the integer variables is significantly increased, in the sense that the non-basic integer variables are more likely to become appealing because of their small(er) reduced costs.

D. As for set B, but all the non-basic variables are continuous.
Essentially the instances in set D are relaxations of instances in set B. These instances also allow one to evaluate the effect of the presence of continuous variable versus the presence of integer variables on the performance of triangle cuts, split cuts, and GMI cuts.

All the instances generated of the form (6) have a completely dense tableau. The optimal basic solution associated with the LP relaxation of (6) is the fractional vertex $(z^*, y^*, x^*) = (f, 0, 0)$, where all the basic variables are integer-constrained free variables and all the non-basic variables are non-negative. Therefore the only differences between these instances and the "perfect" two-row instances (5) considered in Section 4.1 lie in the number of rows and in the presence of non-basic integer variables. In addition, it is important to note that the way used to derive them accomplishes a very important goal. The cuts generated for the three sets A, B and C are exactly the same (except for the fact that coefficients for the missing integer variables are not present in case of cuts generated for instances in set A). What changes is the impact of the cuts on the solution process even though (i) the number of continuous variables is constant from A to C, (ii) the number of integer variables is the same from B to C, and (iii) the columns of all variables stay the same from A to C. The impact changes are instead due to the fact that (a) no integer variables are present in A and (b) the objective function coefficients of the continuous variables have been decreased from B to C. On the other hand the cuts generated for instances in set D are different from those generated for instances in sets B and C, since the strengthening of coefficients of integer variables for instances in set B and C is not applicable for D. In particular, the impact of presence of integer variables and the corresponding cut strengthening technique can be judged by comparing the result for instances in set B and D which have the same objective function coefficients.

For $m = 2$, the results obtained on each set of instances A, B, C and D are reported in Table 4, where all entries are average values obtained on 60 instances (20 with $\ell = 50$, 20 with $\ell = 100$, 20 with $\ell = 150$). Table 5 contains the corresponding results for the case $m = 5$.

Table 4: 2-row dense instances: effect of non-basic integer variables.

	Characteristics					G			S			T		
Set	ny	nx	nz.s	nz.y	nz.x	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
A	9.8	—	0.0	2.0	—	75.10 (0.16)	2.0	1.6	76.39 (0.17)	14.0	2.1	97.99 (0.05)	68.1	2.1
B	9.8	90.2	0.0	2.0	0.5	65.33 (0.22)	2.0	1.5	78.88 (0.15)	14.0	2.3	91.13 (0.09)	2740.8	2.5
C	9.8	90.2	0.3	0.1	2.8	16.27 (0.36)	2.0	1.3	36.09 (0.29)	14.0	2.7	37.17 (0.30)	2740.8	3.0
D	100.0	—	0.0	2.0	—	70.11 (0.18)	2.0	1.6	67.12 (0.21)	13.4	2.3	97.56 (0.06)	1675.4	2.7

	%gap G or S or T			%gap G + S + T			Cut contribution			
Set	G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
A	75.10	76.39	97.99	92.69 (0.07)	98.16 (0.05)	98.94 (0.03)	24.55	0.92	6.45	1.09
B	65.33	78.88	91.13	88.58 (0.09)	91.13 (0.09)	93.65 (0.07)	31.60	2.77	5.76	2.78
C	16.27	36.09	37.17	37.38 (0.28)	37.19 (0.29)	41.93 (0.26)	61.35	11.76	10.82	11.88
D	70.11	67.12	97.56	88.71 (0.10)	97.56 (0.06)	98.13 (0.05)	28.54	0.59	9.82	0.59

Table 5: 5-row dense instances: effect of non-basic integer variables.

Set	Characteristics					G			S			T		
	ny	nx	nz.s	nz.y	nz.x	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
A	9.5	—	0.3	4.7	—	37.76 (0.26)	5.0	2.3	53.56 (0.18)	139.6	3.9	62.83 (0.16)	1260.0	4.1
B	9.5	90.5	0.1	5.0	0.9	32.18 (0.26)	5.0	2.1	52.20 (0.18)	139.6	4.9	56.79 (0.18)	30430.8	5.1
C	9.5	90.5	2.9	0.3	4.5	8.97 (0.34)	5.0	1.9	20.55 (0.23)	139.6	4.8	18.80 (0.25)	30430.8	4.4
D	100.0	—	0.0	5.0	—	35.28 (0.28)	5.0	2.0	46.42 (0.20)	139.6	3.9	60.09 (0.16)	26385.1	4.4

Set	%gap G or S or T			%gap G + S + T			Cut contribution			
	G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
A	37.76	53.56	62.83	57.58 (0.16)	62.83 (0.16)	63.72 (0.16)	41.48	1.38	9.44	1.38
B	32.18	52.20	56.79	55.62 (0.17)	56.81 (0.18)	60.26 (0.16)	46.83	6.33	7.39	6.45
C	8.97	20.55	18.80	21.71 (0.21)	18.99 (0.24)	22.52 (0.21)	60.89	17.05	3.64	18.32
D	35.28	46.42	60.09	54.32 (0.17)	60.09 (0.16)	61.28 (0.15)	46.40	3.76	11.51	3.76

The first part of the tables gives the characteristics of the instances in the set and the results obtained by generating each class of cuts separately. In particular, the first five columns report the number of y variables (ny), the number of x variables (nx), the number of non-basic slacks that are nonzero in the MIP optimal solution (nz.s), the number of non-basic y variables that are nonzero in the MIP optimal solution (nz.y) and the number of non-basic x variables that are nonzero in the MIP optimal solution (nz.x). Then, for each of the cut families, we report four columns: the percentage gap closed (%gap), the relative standard deviation of the gap closed, computed as the standard deviation of the gap closed divided by the average gap closed (RSD), the number of cuts generated (#cuts) and the number of cuts binding in the second optimal tableau, i.e., after reoptimization (#bind). The second parts of Tables 4 and 5 report the same information as in Table 2, i.e., gap closed by each family of cuts, gap closed by adding together different families of cuts and "relative importance" of each class.

The results for the case $m = 5$ are consistent with those obtained with $m = 2$. However, instances with 5 rows turn out to be harder to solve than instances with 2 rows, and each family of cuts is less effective in the case $m = 5$. For this reason, the differences we can observe on the each of the classes of problems A, B, C and D are more radical for 5-row models. Observe that the RSD values for the C instances are usually the highest within each column. This shows that not only are the C instances more difficult to solve, but there is also more variability in the performance of the cutting planes on these instances. Overall, the RSD values appear to be sufficiently small for all the conclusions we derive.

Below we give possible interpretations of the results in Tables 4 and 5.

- Two-row vs one-row cuts:
  - Two-row cuts appear to be important. On all the four sets of instances they close much more of the gap than GMI cuts.
  - If we consider adding all the cuts together, then we observe that the relative importance of two-row cuts increases from instances of type A to instances of type C (by comparing Imp(ST) values). This is interesting as from A to C the instances become harder to solve.
  - Even if the number of two-row cuts generated is much larger than the number of GMI cuts, the number of two-row cuts needed (i.e., binding) is comparable to the number of GMI cuts. This suggests that it may be of some importance to find the "right" two-row cuts.
- Type 2 triangle cuts vs split cuts:
  - On the set A, type 2 triangle cuts appear stronger than split cuts. In particular, type 2 triangle cuts close 62.83% gap and split cuts close 53.56% gap for $m = 5$ with corresponding values of 97.99% and 92.69% for m=2.
  - On the set B, the gap closed by split cuts (one-row and two-row splits together) is comparable with the gap closed by triangles.

  - On the set C, the situation is the opposite relative to the set A. On these instances, split cuts appear stronger than triangle cuts.
  - The average importance of two-row cuts (i.e. Imp(ST)) is almost equal for instances in sets B and D. However, this effect is produced by a significant increase in importance of triangle cuts and a corresponding decrease in the importance of split cuts as we go from set B to set D. The increase in importance of the triangle cuts in set D is consistent with the behavior observed in sets A,B and C. Thus it appears that as the importance (or number) of continuous variables increases, the marginal benefit obtained from triangle cuts increases for both $m = 2$ and $m = 5$.
  - Because MIPLIB instances are closer to instances of type B and C (most of the non-basic variables are integer constrained), this behavior can be seen as a first explanation of the different results obtained on the perfect two-row instances (see Table 1) and on MIPLIB instances (see Table 2).

- Need for study of new relaxations:

  The instances in which the integer variables are more important (set C) show that the performance of all the classes of cuts drastically deteriorate. This suggests that analysis of other relaxations based on integer non-basic variables should be pursued.

As already observed, the relative importance of the different classes of cuts is highly affected by the presence (and the importance) of integer non-basic variables. Indeed, as shown in Tables 4–5, as we go from set D (i.e., all the non-basic variables are continuous) to set B (i.e., 90% of the non-basic variables are integer), and from set B to set C (i.e., 90% of the non-basic variables are integer and their importance is increased), the relative importance of split cuts, namely Imp(S) and Imp(GS), improves, while the relative importance of triangle cuts, namely Imp(T), deteriorates. Thus, the larger the difference between the optimal solution of (6) and the optimal solution of its relaxation (5) (where all the non-basic integer variables are relaxed to be continuous), the smaller the importance of triangle cuts w.r.t. split cuts.

However, even if instances of sets B, C and D are described by the same tableau, it is worth noting that the cuts separated on instances of set D are different from those separated on instances of set B and C, in which cut coefficients of non-basic integer variables are lifted. To evaluate the impact of integer lifting on the different types of cuts, we considered separating the cuts on instances of type B without applying integer lifting. The results are reported in Table 6 for the case of $m = 2$ instances and in Table 7 for the case of $m = 5$ instances. (For RSD of individual cuts, the values that cannot be found in Tables 4-5 are those for "Non lifted cuts, Set B, 2 rows" 0.32%, 0.33% and 0.25% for G, S and T, respectively, and those for "Non lifted cuts, Set B, 5 rows" 0.34%, 0.23% and 0.23% for G, S and T, respectively.)

Table 6: 2-row dense instances: effect of integer lifting.

		%gap G or S or T			%gap G + S + T			Cut contribution			
	Set	G	S	T	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
Lifted cuts	B	65.33	78.88	91.13	88.58 (0.09)	91.13 (0.09)	93.65 (0.07)	31.60	2.77	5.76	2.78
Non lifted cuts	B	47.15	42.29	63.92	58.39 (0.27)	63.92 (0.25)	64.12 (0.25)	28.54	0.59	9.82	0.59
	D	70.11	67.12	97.56	88.71 (0.10)	97.56 (0.06)	98.13 (0.05)	28.54	0.59	9.82	0.59

Table 7: 5-row dense instances: effect of integer lifting.

		%gap G or S or T			%gap G + S + T			Cut contribution			
	Set	G	S	T	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
Lifted cuts	B	32.18	52.20	56.79	55.62 (0.17)	56.81 (0.18)	60.26 (0.16)	46.83	6.33	7.39	6.45
Non lifted cuts	B	21.81	27.00	36.10	32.37 (0.22)	36.11 (0.23)	36.76 (0.22)	46.40	3.76	11.51	3.76
	D	35.28	46.42	60.09	54.32 (0.17)	60.09 (0.16)	61.28 (0.15)	46.40	3.76	11.51	3.76

As expected, integer lifting is rather important for all the different classes of cuts, as the gap closed deteriorates on set B if we do not lift cut coefficients of non-basic integer variables. However, the results indicate that the lifting is more effective on split (one-row split or two-row split) than on triangle cuts. Indeed, on the 2-row models, we have Imp(S)=2.77, Imp(GS)=2.78, Imp(T)=5.76 for the “lifted case”, and Imp(S)=0.59, Imp(GS)=0.59, Imp(T)=9.82 for the “non-lifted case”, and a similar behavior can be observed on the 5-row models. Thus, integer lifting increases the relative importance of split cuts w.r.t. triangle cuts. It is also worth noting that, as we go from set D to set B without applying integer lifting, all the different classes of cuts close less gap, but their relative importance remains exactly the same.

4.3 Bound Experiments

As discussed in Section 4.1, all the inequalities considered (one-row split cuts, two-row split cuts and type 2 triangle cuts) are generated from a relaxation that assumes no bounds on the integer basic variables and only non-negativity on the non-basic variables.

In order to measure the impact of variable bounds on the strength of the different families of cuts, we have considered the instances of type B and C addressed in Section 4.2, whose simplex tableau has the form (6), and we have introduced bounds on basic and non-basic variables, separately. For the generic set Z (Z $\in \{B, C\}$), we have generated three different types of instances:

Z.1: bounds on the basic variables: z binary, $y \geq 0$, $x \geq 0$;

Z.2: bounds on the non-basic integer variables: z free, $y \geq 0$, x binary;

Z.3: bounds on the non-basic continuous variables: z free, $0 \leq y \leq 1$, $x \geq 0$.

The results are reported in Table 8 and Table 9 for the cases $m = 2$ and $m = 5$, respectively. All entries are average values over the same 60 instances with $m = 5$ (20 with $\ell = 50$, 20 with $\ell = 100$, 20 with $\ell = 150$) considered in Section 4.2 except for the addition of the bounds, and the tables have the same structure as Table 5.

Table 8: 2-row dense instances: effect of variable bounds.

	Characteristics					G			S			T		
Set	ny	nx	nz.s	nz.y	nz.x	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
B	9.8	90.2	0.0	2.0	0.5	65.33 (0.22)	2.0	1.5	78.88 (0.15)	14.0	2.3	91.13 (0.09)	2740.8	2.5
B.1			0.0	2.0	0.5	69.60 (0.19)		1.6	78.03 (0.17)		2.2	90.94 (0.10)		2.5
B.2			0.0	2.0	0.4	66.10 (0.20)		1.6	74.53 (0.18)		2.3	91.08 (0.09)		2.4
B.3			0.0	2.0	0.5	63.55 (0.22)		1.5	76.51 (0.16)		2.3	88.53 (0.10)		2.5
C	9.8	90.2	0.3	0.1	2.8	16.27 (0.36)	2.0	1.3	36.09 (0.29)	14.0	2.7	37.17 (0.30)	2740.8	3.0
C.1			0.5	0.2	3.3	17.36 (0.31)		1.4	31.38 (0.28)		3.1	34.18 (0.26)		3.5
C.2			0.4	0.3	3.2	14.68 (0.33)		1.5	27.74 (0.26)		3.1	31.57 (0.24)		3.6
C.3			0.3	0.1	2.9	16.20 (0.36)		1.3	35.97 (0.29)		2.7	37.02 (0.30)		3.0

	%gap G or S or T			%gap G + S + T			Cut contribution			
Set	G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
B	65.33	78.88	91.13	88.58 (0.09)	91.13 (0.09)	93.65 (0.07)	31.60	2.77	5.76	2.78
B.1	69.60	78.03	90.94	86.24 (0.12)	90.94 (0.10)	91.28 (0.10)	24.64	0.68	5.91	0.68
B.2	66.10	74.53	91.08	85.06 (0.12)	91.08 (0.09)	91.60 (0.08)	29.24	0.63	8.01	0.63
B.3	63.55	76.51	88.53	85.94 (0.10)	88.54 (0.10)	91.05 (0.08)	31.67	2.77	6.17	2.78
C	16.27	36.09	37.17	37.38 (0.28)	37.19 (0.29)	41.93 (0.26)	61.35	11.76	10.82	11.88
C.1	17.36	31.38	34.18	32.63 (0.27)	34.18 (0.26)	36.48 (0.25)	53.69	7.01	11.23	7.01
C.2	14.68	27.74	31.57	28.39 (0.26)	31.57 (0.24)	33.24 (0.24)	57.14	5.16	14.76	5.16
C.3	16.20	35.97	37.02	37.26 (0.28)	37.04 (0.29)	41.78 (0.26)	61.35	11.76	10.82	11.88

Below we give possible interpretations of the results in Tables 8-9.

- One-row split cuts (i.e., GMI cuts) seem to be less affected than two-row cuts by the presence of bounds on the variables.
- On both sets B and C, the relative importance of split cuts (two-row split cuts alone and also one-row and two-row split cuts together) substantially deteriorates on instances with bounds on the basic

Table 9: 5-row dense instances: effect of variable bounds.

Set	Characteristics					G			S			T		
	ny	nx	nz.s	nz.y	nz.x	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
B	9.5	90.5	0.1	5.0	0.9	32.18 (0.26)	5.0	2.1	52.20 (0.18)	139.6	4.9	56.79 (0.18)	30430.8	5.1
B.1			0.3	4.7	1.1	28.53 (0.31)		2.4	37.60 (0.27)		4.8	44.67 (0.24)		5.3
B.2			0.1	4.9	1.0	32.66 (0.24)		2.3	49.38 (0.21)		4.8	57.31 (0.19)		5.7
B.3			0.1	5.0	0.9	30.41 (0.27)		2.1	49.10 (0.20)		4.8	52.99 (0.19)		5.0
C	9.5	90.5	2.9	0.3	4.5	8.97 (0.34)	5.0	1.9	20.55 (0.23)	139.6	4.8	18.80 (0.25)	30430.8	4.4
C.1			3.6	0.6	4.2	9.18 (0.25)		2.2	16.05 (0.18)		6.0	16.13 (0.20)		6.4
C.2			3.5	0.5	4.1	8.34 (0.28)		2.1	16.66 (0.24)		5.6	16.35 (0.23)		5.9
C.3			2.9	0.3	4.5	8.97 (0.34)		1.9	20.55 (0.23)		4.8	18.80 (0.25)		4.4

Set	%gap G or S or T			%gap G + S + T			Cut contribution			
	G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
B	32.18	52.20	56.79	55.62 (0.17)	56.81 (0.18)	60.26 (0.16)	46.83	6.33	7.39	6.45
B.1	28.53	37.60	44.67	40.39 (0.25)	44.67 (0.24)	45.08 (0.24)	39.12	1.19	11.34	1.20
B.2	32.66	49.38	57.31	52.54 (0.19)	57.31 (0.19)	58.36 (0.19)	44.07	1.87	10.24	1.87
B.3	30.41	49.10	52.99	52.28 (0.19)	53.01 (0.19)	56.38 (0.18)	46.72	6.31	7.26	6.44
C	8.97	20.55	18.80	21.71 (0.21)	18.99 (0.24)	22.52 (0.21)	60.89	17.05	3.64	18.32
C.1	9.18	16.05	16.13	16.55 (0.19)	16.13 (0.20)	17.26 (0.19)	47.45	6.96	4.28	6.96
C.2	8.34	16.66	16.35	17.15 (0.24)	16.35 (0.23)	17.97 (0.24)	53.44	8.68	4.71	8.68
C.3	8.97	20.55	18.80	21.71 (0.21)	18.99 (0.24)	22.52 (0.21)	60.89	17.05	3.64	18.32

variables and on instances with upper bounds on the non-basic integer variables (i.e., on sets B.1, B.2, C.1, C.2). On the same sets of instances, the cut contribution of type 2 triangle cuts increases.

Even if the presence of variable bounds may affect the performance of each class of cuts differently, the variations we observe in Table 9 seem less important than those observed in the experiments conducted in Section 4.2. In summary, the introduction of bounds does not appear to significantly affect the relative strength of two-row versus one-row cuts, or of type 2 triangle cuts versus split cuts.

4.4 Density and Multi-row Experiments

In this section we report on experiments that are aimed at understanding the impact of the number of rows and of the correlation among the tableau rows on the strength of the different classes of cuts we are concerned with, namely, GMI cuts, two-row split cuts and type 2 triangle cuts, in isolation and/or together. We consider density and number of rows in the same section because density cannot be changed in an interesting way without increasing m.

To this end, we have considered random instances of type D, generated as discussed in Section 4.2. Recall that for these instances, the simplex tableau associated with the optimal basic solution of the LP relaxation has the form (6), but all non-basic variables are continuous.

In order to obtain instances with different tableau density, we enforced some random sparsity in the original knapsack model (7), by fixing to 0 the weights w_{ij} of the items corresponding to non-basic variables with a certain probability. By varying the number and the density of the knapsack constraints, we considered 6 different sets of instances, namely

D.r05.d20 (5 rows, density of almost 20%),

D.r10.d10 (10 rows, density of almost 10%),

D.r10.d20 (10 rows, density of almost 20%),

D.r20.d05 (20 rows, density of almost 5%),

D.r20.d10 (20 rows, density of almost 10%),

D.r20.d20 (20 rows, density of almost 20%), and for each set we generated 50 random instances with 50 non-basic variables (i.e., $p = 50$).

The correlation among the tableau rows of those instances is measured as follows. For any row-pair (i, k) of the tableau (6) we computed the number of null directions r^j (i.e., the number of columns r^j with $r^j_i = 0$ and $r^j_k = 0$), the number of elementary directions r^j (i.e., the number of columns r^j with either $r^j_i = 0$ or $r^j_k = 0$), and the number of non-elementary directions r^j (i.e., the number of columns r^j with $r^j_i \neq 0$ and $r^j_k \neq 0$). Then we denote by p00, p01 and p11 the percentage of null, elementary, and non-elementary directions over all the row pairs respectively. In this way, the ratio p01/p11 gives a measure of the *entanglement* of the tableau. In particular, for a completely dense tableau we have p01/p11=0, while p01/p11 tends to infinity if the tableau rows are completely unrelated to each other. Note that the numbers are computed as follows: First, for each instance, we take p01, p11 and p01/p11. Then, we report the average of these values over the set of instances. Therefore average(p01/p11) is different from average(p01)/average(p11). Note that in the completely unrelated case two-row cuts are useless on top of split cuts, as split cuts always will close 100% of the integrality gap. This is because such instances can be decomposed row-wise and solved independently. Since each row has only one integer variable, the split closure is the convex hull; see [16].

As a first experiment, we have tested the impact of the correlation among the tableau rows on instances with the same number of constraints. Table 10 reports the results obtained on instances with $m = 20$ and an increasing tableau density, namely instances of the sets D.r20.d05, D.r20.d10 and D.r20.d20. All

Table 10: 20-row instances of type D: effect of sparsity.

	Characteristics					G			S			T		
Set	dens	p00	p01	p11	p01/p11	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
D.r20.d05	15.53	75.62	17.71	6.67	4.44	67.31 (0.16)	8.6	5.5	69.71 (0.17)	383.8	10.1	69.29 (0.18)	116.7	9.0
D.r20.d10	35.83	52.17	23.99	23.84	1.64	36.89 (0.25)	11.1	4.4	43.97 (0.18)	637.1	8.2	47.80 (0.19)	2314.2	7.9
D.r20.d20	89.24	8.28	4.97	86.76	0.09	9.07 (0.30)	16.9	3.0	15.57 (0.23)	1736.9	6.1	17.24 (0.23)	113548.8	6.6

		%gap G or S or T			%gap G + S + T			Cut contribution			
Set	p01/p11	G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
D.r20.d05	4.44	67.31	69.71	69.29	85.20 (0.08)	85.64 (0.09)	87.82 (0.07)	25.08	2.78	3.13	20.35
D.r20.d10	1.64	36.89	43.97	47.80	52.03 (0.16)	52.74 (0.17)	55.82 (0.15)	34.99	5.78	6.70	13.68
D.r20.d20	0.09	9.07	15.57	17.24	16.32 (0.22)	17.43 (0.23)	18.01 (0.22)	50.21	3.69	8.74	4.60

entries in the table are average values over 50 instances, and the table has the same structure as Table 5. The first part of the table gives the characteristics of the instances in each set and the results obtained by generating each class of cuts separately. In particular, the first 5 columns report the tableau density (dens) of (6), the percentage of null, elementary, non-elementary directions over all tableau row-pairs (p00, p01, p11, respectively), and the entanglement of the tableau (p01/p11). The second part of the table reports the tableau entanglement, the gap closed by each family of cuts, the gap closed by adding together different families of cuts, and the "relative importance" of each class.

Below we give possible interpretations of the results in Table 10.

- Two-row vs one-row cuts:
  - Two-row cuts appear to be important on all the three sets of instances. On the sparsest instances (i.e., set D.r20.d05), the gap closed by GMI cuts is comparable with that closed by two-row split cuts and type 2 triangle cuts. However, even in this case the improvement achieved by adding two-row cuts on top of GMI cuts is relevant, as the gap closed rises from 67.31% to 87.82%.
  - The relative importance of two-row cuts increases with the entanglement of the tableau. The strength of cuts based on the two-row relaxation w.r.t. those based on the one-row relaxation increases if the tableau rows are more related to each other.
- Type 2 triangle cuts vs split cuts:

- Type 2 triangle cuts appear stronger than split cuts (one-row and two-row split cuts together) only when the tableau is almost completely dense, i.e., on the set D.r20.d20.
- Even in the case in which all the non-basic variables are continuous (all instances, here), split cuts (GMI and two-row split cuts together) appear definitely stronger than type 2 triangle cuts on sparse instances, i.e., on the sets D.r20.d05 and D.r20.d10.
- The relative importance of triangle cuts increases with the entanglement of the tableau, while the contribution of split cuts increases as the tableau becomes sparser.
- Type 2 triangle cuts are generated by exploiting the structure of the tableau rows, i.e., by using tableau columns as generating directions for building the triangles. Therefore, the number of triangle cuts generated explodes if the tableau is almost completely dense. However, on all the three sets of instances the number of binding type 2 triangle cuts is similar to the number of binding split cuts.

• From two-row to m-row relaxations:

By increasing the correlation among the tableau rows, the importance of the two-row relaxation w.r.t. the one-row relaxation increases. However, the performance of all the classes of cuts deteriorates drastically. This suggests that on dense instances it might be important to consider more than two rows at a time i.e., consider m-row cuts with $m > 2$.

Very similar observations apply by considering instances with a density of almost 20% (in the original constraints) and an increasing number of rows, namely instances of the sets D.r05.d20, D.r10.d20 and D.r20.d20. Indeed, as it can be observed in the first part of Table 11 (which has the same structure as Table 10), the resulting instances have an average entanglement very similar to those in Table 10. More precisely, by keeping the same average number of non-zero entries in the original constraints and by increasing the number of rows, the simplex tableau tends to become much denser and the tableau rows much more related to each other. Hence, from D.r05.d20 to D.r20.d20, the contribution of two-row cuts

Table 11: Instances of type D with similar density in the original constraints: effect of the number of constraints.

	Characteristics					G			S			T		
Set	dens	p00	p01	p11	p01/p11	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
D.r05.d20	36.05	41.18	45.54	13.28	4.85	76.95 (0.18)	4.9	3.7	64.54 (0.12)	107.3	6.0	65.01 (0.13)	98.5	4.7
D.r10.d20	52.74	30.33	33.86	35.81	1.30	40.66 (0.24)	9.4	4.1	45.87 (0.17)	456.3	7.3	48.52 (0.18)	2645.8	6.9
D.r20.d20	89.24	8.28	4.97	86.76	0.09	9.07 (0.30)	16.9	3.0	15.57 (0.23)	1736.9	6.1	17.24 (0.23)	113548.8	6.6

		%gap G or S or T			%gap G + S + T			Cut contribution			
Set	p01/p11	G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
D.r05.d20	4.85	76.95	64.54	65.01	91.27 (0.08)	90.44 (0.09)	93.87 (0.08)	18.76	3.93	2.83	30.47
D.r10.d20	1.30	40.66	45.87	48.52	56.05 (0.16)	57.11 (0.17)	59.80 (0.17)	35.30	5.17	6.36	18.43
D.r20.d20	0.09	9.07	15.57	17.24	16.32 (0.22)	17.43 (0.23)	18.01 (0.22)	50.21	3.69	8.74	4.60

and in particular of type 2 triangle cuts becomes more important.

As a further experiment, we have evaluated the impact of the number of rows on instances with a similar entanglement in the tableau constraints (instead of in the original ones), i.e., on instances of the sets D.r05.d20, D.r10.d10 and D.r20.d05. The outcome of these experiments is given in Table 12, that again has the same structure as Tables 10–11.

Below we give possible interpretations of the results reported in Table 12.

• Two-row vs one-row cuts:

- GMI cuts are comparable with two-row cuts on all three sets of instances. In particular, on the instances D.r05.d20 and D.r10.d10, GMI cuts close more gap than two-row split and type

Table 12: Instances of type D with similar tableau entanglement: effect of the number of constraints.

Set	Characteristics					G			S			T		
	dens	p00	p01	p11	p01/p11	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind	%gap (RSD)	#cuts	#bind
D.r05.d20	36.05	41.18	45.54	13.28	4.85	76.95 (0.18)	4.9	3.7	64.54 (0.12)	107.3	6.0	65.01 (0.13)	98.5	4.7
D.r10.d10	23.22	61.47	30.63	7.90	5.36	77.58 (0.13)	7.5	5.6	69.67 (0.15)	281.3	9.1	61.20 (0.17)	109.7	7.2
D.r20.d05	15.53	75.62	17.71	6.67	4.44	67.31 (0.16)	8.6	5.5	69.71 (0.17)	383.8	10.1	69.29 (0.18)	116.7	9.0

Set	p01/p11	%gap G or S or T			%gap G + S + T			Cut contribution			
		G	S	T	GS (RSD)	GT (RSD)	GST (RSD)	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
D.r05.d20	4.85	76.95	64.54	65.01	91.27 (0.08)	90.44 (0.09)	93.87 (0.08)	18.76	3.93	2.83	30.47
D.r10.d10	5.36	77.58	69.67	61.20	89.63 (0.06)	88.32 (0.07)	91.95 (0.05)	16.15	4.02	2.63	33.55
D.r20.d05	4.44	67.31	69.71	69.29	85.20 (0.08)	85.64 (0.09)	87.82 (0.07)	25.08	2.78	3.13	20.35

2 triangle cuts. However, in all the three cases two-row cuts appear to be important, and the improvement achieved by adding two-row cuts on top of GMI cuts is always relevant.

- The relative importance of two-row splits and type 2 triangles does not seem to be affected by the number of rows. Interestingly, the effectiveness of cuts based on the two-row relaxation w.r.t. the classical one-row relaxation cuts does not deteriorate by increasing the number of constraints.

- Type 2 triangle cuts vs split cuts:
  - Split cuts (one-row and two-row split cuts together) appear stronger than type 2 triangle cuts on all the three sets of problem.
  - There is no clear winner among the three families of cuts (one-row splits, two-row splits and type 2 triangles, separately). On all the sets of instances considered, the three classes of cuts seem to be different from each other, their effect is somehow complementary and the contribution of each family is always non negligible.

Figure 2 reports the contribution of each class of cuts w.r.t. the tableau entanglement p01/p11 on all the six sets of instances considered in this section. From the picture it is clear that the impact of the different cut families is closely related to the parameter p01/p11 we have chosen to represent each set of instances. On the one hand, the importance of two-row cuts (type 2 triangle and two-row split cuts together) consistently increases with the tableau entanglement, i.e., when p01/p11 decreases. On the other hand, the contribution of split cuts (two-row and one-row split cuts together) becomes more important when p01/p11 increases. In particular, type 2 triangles appear superior to splits only in the almost-fully-dense case, when p01/p11=0.09. This behavior can be seen as another explanation of the different results obtained on the perfect two-row instances and on MIPLIB instances (see Table 1 and Table 2, respectively). In fact, the average tableau entanglement on the MIPLIB instances considered in Section 4.1 is p01/p11=13.62.

4.5 Back to MIPLIB Instances

The experiments presented in the previous sections suggest that two-row cuts from the simplex tableau might be helpful in practice to strengthen the LP relaxation of MIPs, and that they can be profitably used in conjunction with one-row cuts, i.e., GMI cuts. However, the computational investigation conducted in Sections 4.2–4.4 on randomly generated instances has shown that the performance of different families of two-row cuts, namely type 2 triangle and two-row split cuts, are heavily affected by the structure of the problem. In particular,

- type 2 triangle cuts are important on instances in which one needs strong cut coefficients on non-basic continuous variables and on instances characterized by a high density of the tableau.

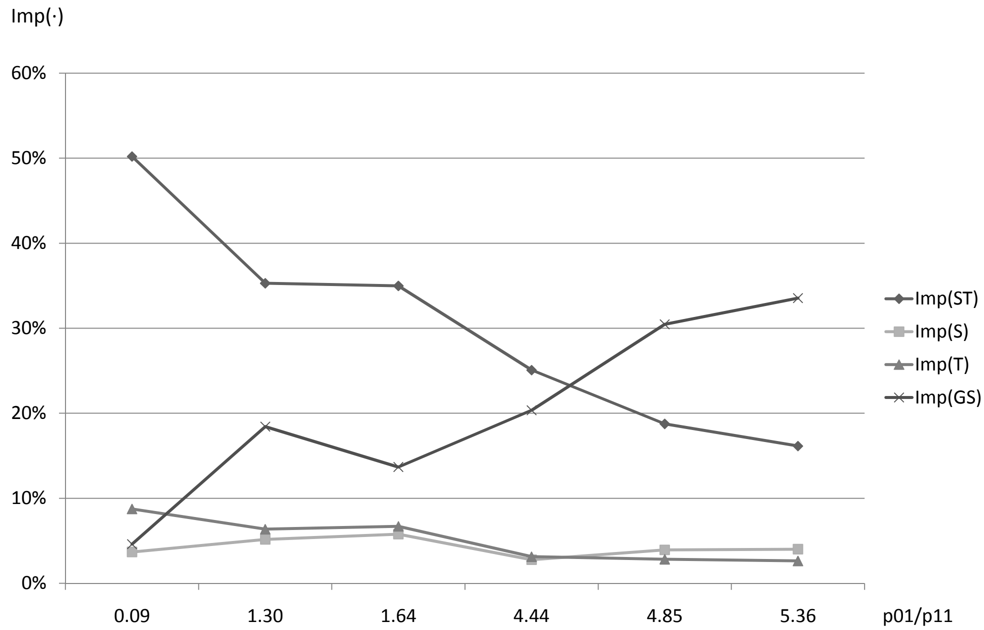

Figure 2: Cut contribution on sparse instances of type D.

- two-row split cuts provide stronger cut coefficients on non-basic integer variables and appear superior to type 2 triangles on sparse instances.

In order to understand if the insights gained on the experiments conducted on "artificial" instances are meaningful, we designed another set of experiments, this time on modified MIPLIB instances. In particular, given the MIPLIB instances considered in Section 4.1, we obtained the following sets of instances:

- MIPLIB.D. We generate 5 rounds of GMI cuts from the simplex tableau, we keep in the formulation all the GMI cuts that are binding at the last optimal basic solution and we discard all the other cuts.
  In this way, we do not change the set of feasible mixed-integer solutions, but we are likely to increase the density of the tableau and therefore the correlation among the tableau rows.

- MIPLIB.D.C. As for set MIPLIB.D, but we relax all the non-basic variables in the last basic solution to be continuous.
  In this way, we obtain a set of instances in which all the non-basic variables are continuous, but we keep the bounds on the variables in order to (partially) preserve the structure of the original problem.

The aggregated results on the three sets of MIPLIB instances (MIPLIB, MIPLIB.D, MIPLIB.D.C) are reported in Table 13. The table has the same structure as Tables 10–12. Detailed results on all the instances are given in the Appendix.

The results in Table 13 confirm the observations previously made on randomly generated instances. In particular:

- From MIPLIB to MIPLIB.D the tableau rows become much more related to each other, the performance of all classes of cuts drastically deteriorates, but the importance of two-row cuts (both type 2 triangle and split cuts) significantly increases.

- From MIPLIB.D to MIPLIB.D.C[6], the gap closed by all the classes of cuts improves (even if such

[6]The tableau characteristics of sets MIPLIB.D and MIPLIB.D.C are similar but not identical, because integral tableau coefficients associated with integer non-basic variables are treated as zero entries in MIPLIB.D. In fact, all the cuts generated are derived by relaxing all the bounds on basic variables and upper bounds on non-basic variables, and hence the tableau coefficients associated with integer non-basic variables can be replaced w.l.o.g. by their fractional part.

Table 13: Results on modified MIPLIB instances.

	Characteristics					G			S			T		
Set	dens	p00	p01	p11	p01/p11	%gap	#cuts	#bind	%gap	#cuts	#bind	%gap	#cuts	#bind
MIPLIB	27.67	64.18	16.31	19.51	13.62	31.62	31.9	16.4	33.15	9549.4	25.5	20.53	6660.9	13.0
MIPLIB.D	49.19	43.74	14.13	42.13	1.61	4.10	32.1	7.1	11.83	8666.2	10.8	8.12	43747.8	9.9
MIPLIB.D.C	49.31	43.86	13.66	42.48	1.59	5.16	33.1	7.3	12.84	8393.8	11.3	10.24	40490.6	10.6

	%gap G or S or T			%gap G + S + T			Cut contribution			
Set	G	S	T	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
MIPLIB	31.62	33.15	20.53	39.93	35.52	40.49	22.59	10.58	2.42	40.70
MIPLIB.D	4.10	11.83	8.12	11.91	8.40	12.25	57.47	17.35	4.06	19.54
MIPLIB.D.C	5.16	12.84	10.24	13.51	10.71	14.73	52.69	15.86	6.41	19.43

an improvement is not comparable with the deterioration we observe from MIPLIB to MIPLIB.D), the importance of type 2 triangles increases, while the importance of two-row splits decreases.

Besides this consistent behavior, we have to note that on all the three sets of instance (MIPLIB, MIPLIB.D, MIPLIB.D.C) two-row splits appear to be definitely superior to type 2 triangles. If we consider each family of cuts separately, two-row split cuts always close more gap than type 2 triangle cuts. If we consider generating the cuts all together, then the relative importance of two-row splits is again always superior to that of type 2 triangles. This can be explained by noting that, even for MIPLIB.D.C instances, the average tableau density and tableau entanglement are not comparable with those observed for the set of instances D.r20.d20 addressed in Section 4.4. For MIPLIB.D.C we have dens=49.31 and p01/p11=1.59, while for D.r20.d20 we have dens=89.24 and p01/p11=0.09.

5 Conclusions

In this paper we have carried out experiments with type 2 triangle cuts, split cuts and GMI cuts on different classes of randomly generated instances designed to study the effect of basic characteristics of mixed integer programming problems such as integer non-basic variables, presence of bounds on variables, the number of rows and the correlation between coefficients of different rows of an instance (the entanglement effect). These experiments were also carried out on MIPLIB instances in their original and modified forms. By comparing the gap closed by these inequalities, we were able to obtain some understanding of how the structure of the problem affects the absolute and relative performance of these families of cutting planes. We believe that the computational methodology introduced in this paper may prove to be useful in the future to analyze multi-row cuts whose inherent structure is rather different from that of the classical 1-row cuts.

We list some conclusions regarding the performance of various families of cutting planes. Most of these conclusions are based on observations that were consistently found across all the types of instances tested in this paper.

- The triangle cuts generated by our heuristic procedure provide a good approximation of the convex hull of the two-row relaxation (2) (see the experiments on the "perfect" two-row relaxation in Section 4.1).
- The improvement in the gap closed by using two-row cuts together with GMI cuts from the simplex tableau is relevant.
- Among the two-row cuts, the contribution of two-row splits is more significant than that of type 2 triangle cuts. In fact, on both randomly generated and MIPLIB instances, two-row split cuts based on "standard" shapes (thus not carefully discerning the actual tableau structure) produce a non-negligible improvement in the gap closed over the case in which only GMI cuts are used.
- The marginal contribution of type 2 triangle cuts becomes relevant only on instances characterized by

1. a high density of the tableau (in particular instances with low p00/p01 value),
2. the presence of important continuous variables (i.e., instances in which one needs strong cut coefficients on non-basic continuous variables).

In all the other situations they appear to be weaker than split cuts. Since many real life instances do not typically satisfy the above two criteria, the experiments in this paper suggest that type 2 triangle cuts are not competitive with split cuts.

The above observations lead to the following open questions and research directions.

- Other classes of facets of (1). Our experiments suggest that generating facets of the continuous two-row relaxation (2) and then strengthening them to obtain valid inequalities for (1) does not lead to useful cutting planes for real-life instances. Of course this investigation is focused only on one class of facets of (2) (i.e., type 2 triangles) and it is clearly not exhaustive. However, a study of facets of (1) that are not derived by lifting facets of (2) should be pursued.

- Cut selection. In all experiments very few type 2 triangles are binding. Being able to obtain a small number of two-row cuts that contain this set of binding cuts is a challenging but important question.

- Multi-row relaxations. The experiments conducted in Section 4.4 have shown that the marginal importance of the two-row relaxation w.r.t. the one-row relaxation increases with the increase in tableau entanglement. However, the absolute performance of all the classes of cuts drastically deteriorates on instances characterized by a high density of the tableau. This suggests that on dense instances it may be necessary to consider more than two rows simultaneously to generate cutting planes.

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Appendix: Detailed Results on MIPLIB Instances

Table 14: Detailed results on original MIPLIB instances.

	Characteristics					G			S			T			%gap G + S + T			Cut contribution			
Pb	dens	p00	p01	p11	p01/p11	%gap	#cuts	#bind	%gap	#cuts	#bind	%gap	#cuts	#bind	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
bell3a	11.80	79.28	17.86	2.87	6.22	45.10	32	10	63.83	5777	13	42.15	271	6	63.83	59.87	63.91	29.43	6.32	0.13	34.05
bell5	19.58	66.34	28.16	5.50	5.12	14.53	22	7	15.74	2347	11	16.10	84	4	15.74	16.27	16.27	10.69	0.00	3.26	1.04
blend2	27.84	64.70	14.92	20.38	0.73	16.36	6	3	9.18	204	6	14.92	1329	6	17.39	17.07	17.67	7.41	3.40	1.58	15.56
danoint	99.18	0.28	1.07	98.65	0.01	0.26	34	5	0.54	7833	7	0.50	55848	8	0.54	0.50	0.55	52.73	9.09	1.82	9.09
dcmulti	16.67	70.95	24.75	4.30	5.76	47.27	42	19	39.73	10551	42	47.53	2463	28	47.54	49.26	49.46	4.43	0.40	3.88	3.90
fiber	5.11	90.74	8.29	0.97	8.55	72.56	23	14	66.70	3407	28	44.10	2000	10	78.87	75.58	80.98	10.40	6.67	2.61	45.54
flugpl	65.00	20.93	28.15	50.93	0.55	11.74	9	5	12.66	269	7	14.47	107	8	12.74	14.47	14.54	19.26	0.48	12.38	0.48
gen	6.07	89.43	9.00	1.57	5.73	59.82	39	9	63.57	9642	19	42.45	4541	8	63.81	63.05	64.12	6.71	1.67	0.48	33.80
gesa2	1.72	96.60	3.36	0.04	84.00	27.12	58	43	59.17	22530	60	22.83	429	35	61.17	36.42	63.38	57.21	42.54	3.49	63.98
gesa2_o	1.97	96.13	3.81	0.07	54.43	30.86	73	47	62.62	35410	70	24.87	889	37	64.53	37.51	66.55	53.63	43.64	3.04	62.63
gesa3	4.51	92.42	6.15	1.44	4.27	43.95	85	32	68.37	41228	47	25.61	7334	15	75.28	62.32	75.41	41.72	17.36	0.17	66.04
gesa3_o	4.64	92.05	6.62	1.33	4.98	49.02	99	32	75.78	57311	58	42.72	15827	27	77.30	65.95	77.49	36.74	14.89	0.25	44.87
gt2	12.62	75.53	23.69	0.77	30.77	91.87	11	11	77.94	685	23	62.51	161	7	91.87	91.87	91.87	0.00	0.00	0.00	31.96
lseu	29.62	49.74	41.27	8.99	4.59	46.35	11	6	44.49	685	11	13.36	442	8	46.69	47.23	47.37	2.15	0.30	1.44	71.80
mas74	100.00	0.00	0.00	100.00	0.00	6.67	12	7	8.94	923	17	8.59	6600	11	8.94	8.59	9.26	27.97	7.24	3.46	7.24
mas76	99.94	0.00	0.12	99.88	0.00	6.42	11	6	10.74	768	4	8.57	5500	7	10.74	8.57	10.74	40.22	20.20	0.00	20.20
misc06	7.78	86.22	12.00	1.78	6.74	30.39	16	14	18.27	1622	12	12.15	254	9	33.43	32.28	33.72	9.88	4.27	0.86	63.97
mod008	92.73	0.00	14.55	85.45	0.17	20.10	5	1	9.07	140	4	20.59	1000	4	21.23	20.91	21.25	5.41	1.60	0.09	3.11
modglob	11.14	79.58	18.56	1.86	9.98	15.62	29	16	26.62	4682	31	15.01	812	25	27.34	17.48	27.34	42.87	36.06	0.00	45.10
p0033	19.70	65.25	30.10	4.65	6.47	56.82	6	4	12.69	179	6	0.13	14	1	57.03	57.03	57.03	0.37	0.00	0.00	99.77
p0282	8.95	84.07	13.96	1.97	7.09	3.70	26	6	5.95	4396	15	6.41	2186	8	5.97	6.48	7.10	47.89	8.73	15.92	9.72
p0548	7.32	85.84	13.68	0.48	28.50	41.79	49	44	13.54	14865	54	0.54	1187	14	41.87	41.82	41.87	0.19	0.12	0.00	98.71
pp08a	3.91	92.36	7.47	0.18	41.50	51.44	53	46	59.89	14863	69	15.72	212	14	71.31	52.81	71.48	28.04	26.12	0.24	78.01
pp08aCUTS	32.46	47.32	40.43	12.24	3.30	31.53	46	27	37.31	12956	39	33.38	17933	29	41.08	38.75	42.01	24.95	7.76	2.21	20.54
qnet1_o	17.60	68.28	28.25	3.48	8.12	39.48	11	9	34.04	741	15	28.44	975	13	43.35	43.14	43.64	9.53	1.15	0.66	34.83
rout	56.97	21.93	42.20	35.87	1.18	1.35	34	4	1.18	7274	3	1.34	57846	6	1.35	1.49	1.49	9.40	0.00	9.40	10.07
vpm1	3.59	93.06	6.71	0.24	27.96	10.00	20	15	3.82	1324	12	0.00	26	6	10.18	10.00	10.18	1.77	1.77	0.00	100.00
vpm2	6.24	88.00	11.53	0.47	24.53	13.10	31	16	25.73	4772	31	9.86	236	10	26.96	17.70	27.04	51.55	34.54	0.30	63.54
avg.	27.67	64.18	16.31	19.51	13.62	31.62	31.9	16.4	33.15	9549.4	25.5	20.53	6660.9	13.0	39.93	35.52	40.49	22.59	10.58	2.42	40.70

Table 15: Detailed results on “denser” MIPLIB instances (MIPLIB.D instances).

	Characteristics					G			S			T			%gap G + S + T			Cut contribution			
Pb	dens	p00	p01	p11	p01/p11	%gap	#cuts	#bind	%gap	#cuts	#bind	%gap	#cuts	#bind	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
bell3a	43.86	41.05	30.18	28.77	1.05	1.06	6	2	1.35	163	6	1.28	592	4	1.35	1.40	1.40	24.29	0.00	3.57	8.57
bell5	13.46	75.16	22.77	2.08	10.95	7.25	17	10	11.20	1676	14	5.64	63	7	11.21	8.08	11.38	36.29	29.00	1.49	50.44
blend2	38.34	61.27	0.79	37.95	0.02	0.59	14	2	1.25	1261	4	1.22	8569	5	1.25	1.22	1.26	53.17	3.17	0.79	3.17
danoint	99.72	0.26	0.03	99.71	0.00	0.01	35	3	0.02	8308	5	0.02	59500	6	0.02	0.02	0.02	50.00	0.00	0.00	0.00
dcmulti	69.76	27.30	5.89	66.81	0.09	0.85	48	4	1.33	15330	9	1.20	133892	7	1.33	1.20	1.35	37.04	11.11	1.48	11.11
fiber	20.88	69.88	18.49	11.63	1.59	11.09	22	9	22.44	3055	9	20.49	41529	12	22.44	20.49	22.45	50.60	8.73	0.04	8.73
flugpl	75.00	25.00	0.00	75.00	0.00	0.33	8	4	0.94	177	2	0.98	418	4	0.96	0.98	0.99	66.67	1.01	3.03	1.01
gen	21.00	70.94	16.12	12.94	1.25	0.85	27	10	2.92	5297	9	3.43	24229	9	2.92	3.43	4.08	79.17	15.93	28.43	15.93
gesa2	8.04	87.09	9.76	3.16	3.09	2.38	46	12	41.43	14093	16	13.12	29012	23	41.44	13.12	41.66	94.29	68.51	0.53	68.51
gesa2_o	7.98	86.40	11.24	2.36	4.76	3.86	56	10	63.66	21456	11	16.83	49350	12	63.67	16.84	63.95	93.96	73.67	0.44	73.68
gesa3	6.74	90.09	6.35	3.56	1.78	14.93	59	5	47.31	16612	15	37.64	20150	12	47.31	37.64	47.98	68.88	21.55	1.40	21.55
gesa3_o	15.13	81.84	6.07	12.09	0.50	3.37	65	8	32.21	20094	10	24.64	140913	12	32.21	24.64	33.31	89.88	26.03	3.30	26.03
gt2	40.12	37.86	44.04	18.09	2.43	23.52	12	4	35.11	802	8	35.11	2217	5	35.11	35.11	35.11	33.01	0.00	0.00	0.00
lseu	93.05	5.55	2.80	91.65	0.03	0.21	15	3	0.55	1584	8	0.51	11902	7	0.55	0.51	0.58	63.79	12.07	5.17	12.07
mas74	99.93	0.00	0.13	99.87	0.00	0.01	20	3	0.04	2652	5	0.03	19000	3	0.04	0.03	0.04	75.00	25.00	0.00	25.00
mas76	99.96	0.00	0.07	99.93	0.00	0.03	18	2	0.06	2133	3	0.04	15300	3	0.06	0.04	0.06	50.00	33.33	0.00	33.33
misc06	16.22	80.12	7.31	12.56	0.58	7.61	13	6	9.48	1069	8	12.60	5024	7	9.48	12.60	12.62	39.70	0.16	24.88	0.16
mod008	99.41	0.35	0.49	99.16	0.00	0.19	9	3	0.82	502	2	0.72	3600	2	0.82	0.72	0.82	76.83	12.20	0.00	12.20
modglob	41.58	47.29	22.26	30.45	0.73	7.15	42	11	8.68	10896	17	9.57	72927	18	9.45	9.71	9.80	27.04	0.92	3.57	2.35
p0033	50.00	40.34	19.33	40.34	0.48	0.88	10	4	7.91	553	5	5.88	1694	5	7.91	5.88	7.95	88.93	26.04	0.50	26.04
p0282	35.83	43.93	40.47	15.59	2.60	0.66	27	5	1.79	5229	11	1.57	20566	7	1.81	1.58	1.86	64.52	15.05	2.69	15.59
p0548	16.36	70.95	25.37	3.67	6.91	15.16	75	26	16.82	37841	55	12.06	37924	32	17.41	17.04	17.49	13.32	2.57	0.46	31.05
pp08a	70.70	11.19	36.22	52.59	0.69	4.79	49	10	6.17	15561	12	8.20	94810	18	6.67	8.20	8.28	42.15	0.97	19.44	0.97
pp08aCUTS	77.61	7.44	29.91	62.66	0.48	5.01	47	14	8.71	14375	17	8.25	85164	21	8.86	8.35	9.39	46.65	11.08	5.64	12.14
qnet1_o	90.29	9.57	0.29	90.14	0.00	0.37	42	5	1.71	12172	5	1.52	173100	6	1.71	1.52	1.71	78.36	11.11	0.00	11.11
rout	85.13	14.86	0.02	85.12	0.00	0.67	43	6	0.71	12970	12	0.80	126414	10	0.82	0.80	0.82	18.29	2.44	0.00	2.44
vpm1	11.04	81.87	14.19	3.94	3.60	0.58	30	7	2.82	5374	11	2.31	5533	8	2.82	2.31	2.98	80.54	22.48	5.37	22.48
vpm2	30.26	57.25	24.99	17.77	1.41	1.25	43	10	3.68	11418	13	1.82	41546	12	3.71	1.82	3.76	66.76	51.60	1.33	51.60
avg.	49.19	43.74	14.13	42.13	1.61	4.10	32.1	7.1	11.83	8666.2	10.8	8.12	43747.8	9.9	11.91	8.40	12.25	57.47	17.35	4.06	19.54

Table 16: Detailed results on “denser–continuous” MIPLIB instances (MIPLIB.D.C instances).

	Characteristics					G			S			T			%gap G + S + T			Cut contribution			
Pb	dens	p00	p01	p11	p01/p11	%gap	#cuts	#bind	%gap	#cuts	#bind	%gap	#cuts	#bind	GS	GT	GST	Imp(ST)	Imp(S)	Imp(T)	Imp(GS)
bell3a	42.11	42.71	30.38	26.92	1.13	0.81	6	3	0.87	147	4	0.78	346	4	0.87	0.87	0.87	6.90	0.00	0.00	10.34
bell5	13.52	75.04	22.88	2.08	11.00	8.03	17	14	11.90	1496	17	5.76	54	11	11.97	8.71	12.04	33.31	27.66	0.58	52.16
blend2	39.48	60.13	0.79	39.08	0.02	1.47	14	2	3.08	1159	4	3.04	8409	4	3.08	3.04	3.12	52.88	2.56	1.28	2.56
danoint	99.72	0.26	0.03	99.70	0.00	0.01	34	2	0.02	7840	5	0.02	56100	6	0.02	0.02	0.02	50.00	0.00	0.00	0.00
dcmulti	70.74	26.39	5.75	67.87	0.08	0.20	49	3	0.29	15313	8	0.27	133389	7	0.30	0.27	0.30	33.33	10.00	0.00	10.00
fiber	20.03	69.96	20.03	10.01	2.00	14.01	36	4	26.74	7774	11	30.08	41375	10	27.91	30.08	40.64	65.53	25.98	31.32	25.98
flugpl	75.00	25.00	0.00	75.00	0.00	0.33	8	4	0.94	177	2	0.98	418	4	0.96	0.98	0.99	66.67	1.01	3.03	1.01
gen	18.61	74.69	13.39	11.92	1.12	0.84	23	9	2.84	3978	11	3.37	19919	7	2.84	3.37	3.80	77.89	11.32	25.26	11.32
gesa2	8.05	87.08	9.75	3.17	3.08	2.47	45	12	38.63	11465	20	13.99	26387	23	38.64	13.99	38.78	93.63	63.92	0.36	63.92
gesa2_o	7.52	87.30	10.35	2.34	4.42	3.77	57	10	57.64	18114	13	11.10	45978	14	57.64	11.11	57.73	93.47	80.76	0.16	80.77
gesa3	6.80	90.05	6.29	3.66	1.72	9.05	59	6	41.90	13643	16	35.49	11045	15	41.91	35.49	42.76	78.84	17.00	1.99	17.00
gesa3_o	15.19	81.81	6.02	12.18	0.49	3.58	65	9	34.18	18824	12	25.29	97642	12	34.18	25.29	35.41	89.89	28.58	3.47	28.58
gt2	51.91	26.85	42.48	30.67	1.39	16.91	12	7	2.03	771	8	15.49	2624	10	16.91	16.91	16.91	0.00	0.00	0.00	8.40
lseu	91.57	7.88	1.09	91.03	0.01	0.72	16	3	1.62	1565	6	1.50	11908	6	1.62	1.50	1.90	62.11	21.05	14.74	21.05
mas74	99.93	0.00	0.13	99.87	0.00	0.25	20	3	0.53	2647	5	0.47	19000	5	0.53	0.47	0.53	52.83	11.32	0.00	11.32
mas76	99.96	0.00	0.07	99.93	0.00	0.41	18	2	0.62	2108	6	0.50	15300	5	0.62	0.50	0.62	33.87	19.35	0.00	19.35
misc06	15.35	82.55	4.20	13.25	0.32	7.61	11	5	9.49	714	7	12.50	3451	8	9.49	12.50	12.53	39.27	0.24	24.26	0.24
mod008	99.37	0.40	0.45	99.15	0.00	12.81	9	3	37.75	504	5	35.88	3600	4	37.75	35.88	39.10	67.24	8.24	3.45	8.24
modglob	41.94	47.04	22.05	30.91	0.71	8.43	42	11	9.54	10913	17	8.14	73092	16	10.39	10.64	10.74	21.51	0.93	3.26	24.21
p0033	50.00	40.34	19.33	40.34	0.48	18.05	8	4	29.74	413	4	38.95	953	5	29.74	38.95	38.95	53.66	0.00	23.65	0.00
p0282	35.72	44.22	40.13	15.65	2.56	0.67	27	4	2.00	4098	10	1.74	16870	8	2.01	1.76	2.06	67.48	14.56	2.43	15.53
p0548	14.76	73.57	23.34	3.09	7.55	18.01	75	33	18.89	28869	54	13.96	26019	34	19.47	20.07	20.13	10.53	0.30	3.28	30.65
pp08a	70.44	11.62	35.89	52.50	0.68	5.34	49	10	6.58	15543	13	9.28	95061	20	7.34	9.28	9.36	42.95	0.85	21.58	0.85
pp08aCUTS	77.76	7.38	29.73	62.89	0.47	5.57	47	15	9.70	13884	18	8.97	84481	17	9.84	9.11	10.37	46.29	12.15	5.11	13.50
qnet1_o	90.07	9.88	0.09	90.02	0.00	0.35	59	5	1.64	23161	6	1.50	169512	6	1.64	1.50	1.66	78.92	9.64	1.20	9.64
rout	85.11	14.86	0.05	85.09	0.00	2.21	51	3	2.44	16044	9	2.67	126784	7	2.82	2.69	2.82	21.63	4.61	0.00	5.32
vpm1	9.78	83.85	12.73	3.42	3.72	0.95	27	8	2.73	3782	12	2.36	4371	15	2.74	2.36	2.98	68.12	20.81	8.05	20.81
vpm2	30.27	57.24	24.99	17.77	1.41	1.75	42	9	5.13	10080	14	2.55	39650	15	5.18	2.55	5.24	66.60	51.34	1.15	51.34
avg.	49.31	43.86	13.66	42.48	1.59	5.16	33.1	7.3	12.84	8393.8	11.3	10.24	40490.6	10.6	13.51	10.71	14.73	52.69	15.86	6.41	19.43