Facets of Two-Dimensional Infinite Group Problems

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January 6, 2006

Abstract

In this paper, we lay the foundation for the study of the two-dimensional mixed integer infinite group problem (2DMIIGP). We introduce tools to determine if a given continuous and piecewise linear function over the two-dimensional infinite group is subadditive and to determine whether it defines a facet of 2DMIIGP. We then present two different constructions that yield the first known families of facet-defining inequalities for 2DMIIGP. The first construction uses valid inequalities of the one-dimensional integer infinite group problem (1DIIGP) as building blocks for creating inequalities for the two-dimensional integer infinite group problem (2DIIGP). We prove that this construction yields all continuous piecewise linear facets of the two-dimensional group problem that have exactly two gradients. The second construction we present has three gradients and yields facet-defining inequalities of 2DMIIGP whose continuous coefficients are not dominated by those of facets of the one-dimensional mixed integer infinite group problem (1DMIIGP).

1 Introduction.

One of the most effective methods to solve unstructured mixed integer programs (MIPs) is branch-and-cut; see Marchand, Martin, Weismantel and Wolsey [16] and Johnson, Nemhauser and Savelsbergh [15] for recent expositions of this technique. Over the years many families of cutting planes have been introduced that can be added to the LP relaxations of MIPs to strengthen them. Interestingly, many of the cutting planes that are currently used in commercial software are derived from single-row relaxations of the problem. Because the interactions between multiple constraints cannot completely be captured by single constraints, cuts generated from multiple constraints of the problem should help in the solution of MIPs. Therefore, it is interesting to study how to derive strong inequalities from multiple row relaxations of MIPs.

Gomory and Johnson [12] recently put into light the possibility of using group-theoretic approaches to generate cuts from multiple rows of simplex tableaux. A strong appeal of this method is that it is an elegant approach to generate closed-form descriptions of cuts. Next we discuss its advantages and limitations further. On the negative side, Gomory [8] proved that starting with integer data for a pure integer program, simplex pivots may lead to tableaux in which any cut generated from a two-constraint group relaxation can also be generated using a single constraint. This is because the rows of fractional parts of the tableau form elements of a cyclic group. We mention however that this result does not hold for all pure integer problems. On the positive side, Gomory and Johnson [12] note that using the group approach with multiple rows allows a more accurate representation of the continuous variables as compared to using one-dimensional group approaches.

This should therefore lead to stronger coefficients for continuous variables in the cuts. This observation is important as recent computational studies by Dash and Günlük [5] on group cuts based on single row relaxations suggest that continuous coefficients in mixed integer cuts are relatively weak. Therefore, we believe that there is a significant advantage in using two-dimensional group cuts for the case of MIPs. In this paper we show that the use of two-dimensional cuts is indeed fruitful by deriving two large families of facet-defining inequalities for the two-dimensional mixed integer infinite group problem (2DMIIGP) whose continuous variables coefficients are not dominated by those of the one-dimensional group cuts.

In §2 we give a short introduction to integer and mixed integer infinite group problems. In §3 we introduce the concept of valid subadditive functions for group problem. In §4, we develop in Propositions 10 and 12, tools to verify that continuous and piecewise linear functions defined over the two-dimensional infinite group are subadditive. This method requires the verification of subadditivity of the function at a finite number of carefully selected points. In §5 we introduce a method to prove that given valid inequalities are facet-defining for the two-dimensional integer infinite group problem (2DIIGP). We then present the automorphism theorem of Johnson [14] in Theorem 28 and generalize the homomorphism theorem of Gomory and Johnson [12] in Theorem 30. These results can be used to create new facets for 2DIIGP from known facets. We then present in the following two sections the first proven families of facets for 2DIIGP. In §6 we present a family of subadditive functions for 2DIIGP that are obtained from subadditive functions of the one-dimensional integer infinite group problem (1DIIGP) through a simple constructive procedure. Necessary and sufficient conditions for these functions to be facet-defining for 2DIIGP are derived in Theorem 39. We also show in Theorem 42 that all continuous and piecewise linear facets of the two-dimensional group problem that have exactly two gradients can be generated using this construction. In §7 we present in Theorem 48 a second family of facets for 2DIIGP that cannot be derived using the procedure described in §6. In §8 we extend the facet-defining inequalities obtained in §6 and §7 for 2DIIGP to strong inequalities for the mixed integer extension of 2DIIGP that we call 2DMIIGP. We show that the coefficients of continuous variables in two-dimensional group cuts are not dominated by those of the Gomory mixed integer cuts from the individual constraints. We conclude the paper in §9 with remarks and directions for future research.

2   The Group Problem.

In this section we give a brief introduction to the integer and mixed integer group problems. A more detailed analysis of both the integer and mixed integer case can be found in Gomory [9], Gomory and Johnson [10, 11, 12] and Johnson [14]. We begin with the pure integer case. Consider the integer programming problem

$$
\begin{aligned}
\text{minimize} \quad & c^T t \\
s.t. \quad & At = b \\
& t \in \mathbb{Z}_+^n,
\end{aligned}
\tag{1}
$$

where we assume that $c \in \mathbb{R}^{n \times 1}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m \times 1}$. We let A_B be a basis matrix of the linear programming relaxation of (1) and let t_B be the corresponding basic variables. We also let A_N represent the columns of A that are not part of A_B and let t_N be the nonbasic variables associated with A_N. The feasible set of (1) may be written as

$$
A_B t_B + A_N t_N = b \quad \text{where} \quad t_B \in \mathbb{Z}_+^m \quad t_N \in \mathbb{Z}_+^{n-m}.
\tag{2}
$$

If the nonnegativity constraints of the basic variables in (8) are relaxed, we obtain

$$
A_N t_N \equiv b (mod \ A_B) \quad t_N \in \mathbb{Z}_+^{n-m},
\tag{3}
$$

where $c \equiv b (mod \ A_B)$ if $\exists z \in \mathbb{Z}^m$, such that $c = b + A_B z$. The convex hull of feasible solutions to (3) is the *corner polyhedron* introduced and studied by Gomory in [9]. Next if the inverse of the basis matrix A_B is left

multiplied to (8), the feasible set of (3) can be represented as

$$A_B^{-1} A_N t_N \equiv A_B^{-1} b (mod \ \bar{I}) \quad t_N \in \mathbb{Z}_+^{n-m}, \tag{4}$$

where $\bar{I}$ is the $m \times m$ identity matrix. Consider now a relaxation of (4) by removing all but one of its constraints. Multiplying this single row by $D = \det(A_B)$, we obtain

$$\sum_{i \in N} D \bar{a}_{ji} t_i \equiv D \bar{b}_j (mod \ D), \quad t_i \in \mathbb{Z}_+ \quad \forall i \in N, \tag{5}$$

where $\bar{a}_{ji}, \bar{b}_j$ are the coefficients of the jth row of $A_B^{-1} A_i$ and $A_B^{-1} b$ respectively and A_i is the column of A corresponding to the variable t_i. Clearly, $D \bar{a}_{ji}, D \bar{b}_j \in \mathbb{Z}$ if $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^{m \times 1}$.

Next we consider a relaxation of (5) in which there is a variable associated with each possible coefficient of the cyclic group of order D. More precisely, we consider

$$\begin{aligned} \sum_{u \in \mathbb{G}} u t_u &\equiv r \\ t_u \in \mathbb{Z}_+ &\quad \forall u \in \mathbb{G}, \end{aligned} \tag{6}$$

where the coefficients $D \bar{a}_{ji}$ of (5) are represented as the members of the finite cyclic group $\mathbb{G}$ and $D \bar{b}_j$ is denoted as r where $r \in \mathbb{G}$. The group problem (6) is a relaxation of (5) since every solution of (5) can be used to create a solution of (6) by setting the additional variables to zero. The convex hull of integer solutions of (6) is known as the *master cyclic group problem* $P(C_{D,r})$.

The master cyclic group problem is a useful relaxation of (1) since there exists an implicit representation of its facet-defining inequalities as extreme rays of a particular polyhedron; see Gomory [9]. Furthermore, Gomory [9] shows that the facet-defining inequalities of the convex hull of solutions of (5) may be obtained from a subset of the facet-defining inequalities of $P(C_{D,r})$. Over the years, many explicit families of facets for these finite group problems were obtained from Gomory's implicit characterization; see Gomory [9], Gomory, Johnson and Evans [13], Aráoz, Evans, Gomory and Johnson [3], Richard, Li and Miller [20] and Miller, Li and Richard [18]. Because these inequalities are facet-defining for $P(C_{D,r})$, they are valid for (1) and so can be used as cutting planes for general integer programs.

There is however a computational difficulty with the approach described above when the matrix A is rational. In such a case a sufficiently large integer D has to be multiplied to the system to make the coefficients in (5) integers. We note that this integer D may be difficult to determine and may give rise to cyclic group relaxations of large orders. To circumvent these difficulties, Gomory and Johnson [12] suggested the following variation of the approach. Instead of considering only the columns corresponding to $A_B^{-1} A_N$ in (4) they proposed to include all the columns of $\mathbb{R}^m$, considered as elements of a group where addition is done modulo 1 componentwise. The relaxation obtained in this fashion is the *integer infinite group problem*.

Definition 1 *The integer infinite group problem with right-hand-side r, where $r \in I^m \setminus \{o\}$, is defined as the set of functions $t : I^m \rightarrow \mathbb{Z}_+$ that satisfy*

1. *$t(u) > 0$ for some finite set $U \subset I^m$*

2. *$\sum_{u \in I^m} u t(u) = r$,*

where I^m is the infinite commutative group corresponding to the set $\{u \in \mathbb{R}^m \mid 0 \leq u_i < 1 \quad \forall i \in \{1, ..., m\}\}$ with addition modulo 1 componentwise and where we denote the element of I^m whose components are all equal to zero as $o \in I^m$.

Note that the summation in 2 is well-defined as t is considered to be a feasible solution to the group problem only if it has a finite support, i.e., $t(u) > 0$ for some finite set $U \subset I^m$. This assumption is legitimate since most integer programs encountered in applications have finitely many variables. We refer to the integer

infinite group problem as one-dimensional group problem ($1DIIGP$) in the case where $m = 1$. Gomory and Johnson analyze $1DIIGP$ in [12]. In this paper we analyze the case when $m = 2$ which we refer as 2DIIGP.

Next we describe the infinite group relaxation for mixed integer problems. Consider the mixed integer programming problem

$$\begin{aligned}
\text{minimize} \quad & c_I^T t + c_C^T s \\
s.t. \quad & A_I t + A_C s = b \\
& t \in \mathbb{Z}_+^n, s \in \mathbb{R}_+^n
\end{aligned} \tag{7}$$

where we assume that $c_I \in \mathbb{R}^{n_I \times 1}, c_C \in \mathbb{R}^{n_C \times 1}, A_I \in \mathbb{R}^{m \times n_I}, A_C \in \mathbb{R}^{m \times n_C}, b \in \mathbb{R}^{m \times 1}$. We let t_B represent the basic variables that are integer in some basic feasible solution of the linear programming relaxation of (7). The tableau rows corresponding to the variables t_B are

$$t_B + \bar{A}_I t_N + \bar{A}_C s_N = b \tag{8}$$

where $\bar{A}_I$ and $\bar{A}_C$ are the columns corresponding to integer and continuous non-basic variables. By following the same sequence of relaxation that produced (3), we obtain

$$\bar{A}_I t_N + \mathbb{P}(\bar{A}_C s_N) \equiv b(mod\ \bar{I}) \tag{9}$$

where $\bar{I}$ is the identity matrix and where the projection function $\mathbb{P} : \mathbb{R}^m \to I^m$ is defined as $\mathbb{P}(x_1, ..., x_m) = (x_1(mod1), ..., x_m(mod1))$.

Next we relax the problem by introducing more variables to (9). Since continuous variables can always be scaled, we only consider coefficients for continuous variables from the set $J^m = \{(w_1, ..., w_m) \mid max_{1 \leq i \leq m}|w_i| = 1\}$. It is clear that as the dimension increases the continuous variables are better represented in the group relaxation.

Definition 2 *The mixed integer infinite group problem with right-hand-side r, where $r \in I^m \setminus \{o\}$, is defined as the set of pairs of functions $t : I^m \to \mathbb{Z}_+$ and $s : J^m \to \mathbb{R}_+$ that satisfy*

1. *$t(u) > 0$ for some finite set $U \subset I^m$,*

2. *$s(w) > 0$ for some finite set $W \subset J^m$.*

3. *$\sum_{u \in I^m} ut(u) + \mathbb{P}(\sum_{w \in J^m} ws(w)) = r$.*

Johnson [14] analyzes general properties of the mixed integer infinite group problem and shows that strong valid inequalities of the mixed integer infinite group problem are related to those of the integer infinite group problem. In particular, strong inequalities for the two-dimensional mixed integer infinite group problem (2DMIIGP) can be obtained from strong inequalities of the 2DIIGP. For this reason, it is natural to focus first on purely integer infinite group problems as a way to obtain cuts for mixed integer infinite group problem. We note however that the full power of the two-dimensional group approach is expected to be realized for mixed integer problems.

3   Valid inequalities.

In this section, we first define the notion of valid functions over the two-dimensional group I^2. We then define subadditive functions and discuss the reason that it is sufficient to consider valid subadditive functions when studying group problems. We conclude this section by describing the subset of valid subadditive functions we study in this paper.

Definition 3 *A function $\phi : I^2 \to \mathbb{R}$ is defined to be a valid inequality for the two-dimensional group problem with respect to right-hand-side $r \in I^2$ if it satisfies the following conditions:*

1. $\sum_{u \in I^2} \phi(u)t(u) \geq 1 \ \forall t \in 2DIIGP$,

2. $\phi(o) = 0$,

3. $\phi(r) = 1$,

4. $\phi(u) \geq 0 \ \forall u \in I^2$.

Since valid inequalities are functions over I^2 we will use the terms valid function and valid inequality interchangeably. In the rest of this paper, we only consider valid inequalities. Whenever the word right-hand-side is used, it refers to r as presented in Definition 1. Next we introduce subadditive functions and describe their relations to valid inequalities for 2DIIGP and consequently for (1).

Definition 4 (Subadditive) *Let $\phi : I^2 \to \mathbb{R}$. We say that ϕ is subadditive if $\phi(a) + \phi(b) \geq \phi(a + b) \ \forall a, b \in I^2$.*

Subadditive functions play an important role in the group problem as they yield valid inequalities; see Wolsey and Nemhauser [19] for a more detailed presentation of subadditive functions. Although it is possible to construct valid functions that are not subadditive, these functions are always dominated by valid subadditive functions; see Gomory and Johnson [10]. Therefore it is sufficient to consider valid subadditive functions when studying the group problem.

The class of valid subadditive functions for I^2 is large. In this paper we will narrow down our study to a subset of these functions. In particular, we will study valid subadditive functions that have the following two properties.

Property 1 *ϕ is continuous.*

The reason for restricting our study to continuous functions is that it can be proven that the extreme inequalities of the mixed integer infinite group problem (MIIGP) are continuous; see Dey, Richard, Li and Miller [7]. However, we do not include continuity in the definition of valid functions since it is possible to construct discontinuous functions that define extreme inequalities for integer infinite group problem [7]. With this respect, our approach is different from that of Gomory and Johnson [12] who included continuity as part of the definition of valid functions.

We now introduce the second property.

Property 2 *ϕ is piecewise linear, i.e. I^2 can be decomposed into finitely many polytopes with non-empty interiors $P_1, ..., P_k$, such that $\phi(u) = \alpha_t^T u + \beta_t, \ \forall u \in P_t$, where $\alpha_t \in \mathbb{R}^2, \beta_t \in \mathbb{R} \ \forall t \in \{1, 2, ...k\}$.*

Note that in Property 2 we say that S is a polytope in I^2 if $\exists S'$, a polytope in $\mathbb{R}^2$, such that $S = \mathbb{P}(S')$. The reason we consider only functions that satisfy Property 2 is that Gomory and Johnson conjectured in [12] that all facets of MIIGP are piecewise linear.

4 Verifying subadditivity.

In this section we develop a method to verify that a given continuous and piecewise linear function over I^2 is subadditive by checking that it is subadditive at a finite number of points. Unless specified otherwise, if $u \in \mathbb{R}^2$ and it is said or implied that $u \in I^2$, then u should be considered to be $\mathbb{P}(u)$. For example, if it is said that $u = (1.5, -0.6) \in I^2$, then it should be understood that $u = (0.5, 0.4)$. Also, the term "slope" along a particular direction d will be often used to signify the directional derivative of the function in the direction d.

Definition 5 (Boundary) *For a continuous and piecewise linear function ϕ, we say that a point l belongs to the boundary of ϕ, denoted $\mathbb{B}(\phi)$, if l belongs to the intersection of two polytopes P_i and P_j where $i \neq j$ and the gradient of ϕ in P_i is not equal to the gradient of ϕ in P_j.*

Observe that for a piecewise linear and continuous function ϕ, ϕ is differentiable at a point u if $u \notin \mathbb{B}(\phi)$. In the next proposition, we show that it is sufficient to consider the points on the boundary of a piecewise linear and continuous function to prove that it is subadditive.

Proposition 6 *Let ϕ be a continuous, piecewise linear and nonnegative function over I^2. The function ϕ is subadditive iff*

$$\phi(l_1) + \phi(l_2) \geq \phi(l_1 + l_2) \tag{10}$$
$$\phi(l_1) + \phi(l_2 - l_1) \geq \phi(l_2) \tag{11}$$

$\forall l_1, l_2 \in \mathbb{B}(\phi)$.

Proof: The direct implication is straightforward. We therefore only prove the reverse implication. First consider the case where $\mathbb{B}(\phi) = \emptyset$. We have that $\phi(u) = \alpha_1 u + \beta_1 \quad \forall u \in I^2$. Let $\alpha_1 = (\alpha_1^x, \alpha_1^y)$. Because $\phi(u) \geq 0$ we obtain that $\alpha_1^x + \alpha_1^y + \beta_1 \geq 0$ by considering a sequence of points converging to $(1,1)$. Similarly, we also obtain $\alpha_1^x + \beta_1 \geq 0$, $\alpha_1^y + \beta_1 \geq 0$ and $\beta_1 \geq 0$. Therefore $\phi(u) + \phi(v) = \alpha_1^x(u^x + v^x) + \alpha_1^y(u^y + v^y) + 2\beta_1 \geq \phi(u + v)$ in all cases. Next we consider the case where $\mathbb{B}(\phi) \neq \emptyset$. Assume by contradiction that ϕ is not subadditive. Then there exists $u, v \in I^2$ such that

$$\phi(u) + \phi(v) < \phi(u + v). \tag{12}$$

We prove that (12) implies that (10) or (11) is violated, which gives the desired contradiction. The proof is in two steps. In the first step we show that there exists some $l \in \mathbb{B}(\phi)$, such that

$$\phi(l) + \phi(v') < \phi(l + v'). \tag{13}$$

In the second step we show that either (10) or (11) is violated.

Step 1: First note that if either of u or v belongs to $\mathbb{B}(\phi)$, (12) reduces to (13). So assume $u \notin \mathbb{B}(\phi)$ and $v \notin \mathbb{B}(\phi)$. Then consider at u any direction d with the following property: $\exists n^+, n^- \in \mathbb{R}_+$, such that $u + n^+ d \in \mathbb{B}(\phi)$ and $u - n^- d \in \mathbb{B}(\phi)$. Such a direction exists since $\mathbb{B}(\phi) \neq \emptyset$. Let the directional derivative of ϕ in the direction d and $-d$ for u and v be σ_1 and σ_2 respectively. There are two cases: $\sigma_1 + \sigma_2 \leq 0$ or $\sigma_1 + \sigma_2 > 0$. In the first case, let $\epsilon^1 = \min\{\epsilon | u + \epsilon d \in \mathbb{B}(\phi)\}$, let $\epsilon^2 = \min\{\epsilon | v - \epsilon d \in \mathbb{B}(\phi)\}$ where ϵ^2 is possibly ∞ and let $\epsilon^* = min(\epsilon^1, \epsilon^2)$. We obtain

$$\phi(u + \epsilon^* d) + \phi(v - \epsilon^* d) = \phi(u) + \phi(v) + \epsilon^*(\sigma_1 + \sigma_2) < \phi(u + v).$$

Furthermore either $u + \epsilon^* d \in \mathbb{B}(\phi)$ or $v - \epsilon^* d \in \mathbb{B}(\phi)$. Thus there exists a point $l \in \mathbb{B}(\phi)$ such that $\phi(l) + \phi(v') < \phi(l + v')$. If $\sigma_1 + \sigma_2 > 0$, we obtain the result similarly by considering the points $u - \epsilon d$ and $v + \epsilon d$.

Step 2: Now consider (13). We may assume that neither v' nor $(l + v')$ belong to $\mathbb{B}(\phi)$ since otherwise, we have a contradiction to either (10) or (11). Then consider at v' any direction d with directional derivative σ_1 such that $\exists n^+, n^- \in \mathbb{R}_+$, with $v' + n^+ d \in \mathbb{B}(\phi)$ and $v' - n^- d \in \mathbb{B}(\phi)$. Let the value of the directional derivative along d for $l + v'$ be σ_2. There are two cases: $\sigma_1 \leq \sigma_2$ or $\sigma_1 > \sigma_2$. In the first case, let $\epsilon^1 = \min\{\epsilon | v' + \epsilon d \in \mathbb{B}(\phi)\}$, let $\epsilon^2 = \min\{\epsilon | v' + l + \epsilon d \in \mathbb{B}(\phi)\}$ where ϵ^2 is possibly ∞ and let $\epsilon^* = min(\epsilon^1, \epsilon^2)$. We obtain,

$$\phi(l) + \phi(v' + \epsilon^* d) - \phi(l + v' + \epsilon^* d) = \phi(l) + \phi(v') + \epsilon^* \sigma_1 - \epsilon^* \sigma_2 - \phi(l + v') < 0.$$

Because either $v' + \epsilon^* d \in \mathbb{B}(\phi)$ or $l + v' + \epsilon^* d \in \mathbb{B}(\phi)$ we obtain a violation of either (10) or (11). For the other case $\sigma_1 > \sigma_2$, we obtain the result similarly by considering the points $v' - \epsilon d$ and $l + v' - \epsilon d$. $\qquad\square$

Observe that the result of Proposition 6 also holds if the dimension of the group studied is larger than two. Proposition 10 however is specific to the two-dimensional group and can be used to reduce the set of points at which subadditivity is verified. Before proving Proposition 10, we give a few definitions specific to I^2.

Definition 7 (Edges and Vertices of ϕ) *Let ϕ be a continuous and piecewise linear function over I^2 defined by the polytopes P_1, P_2,...,P_k. We define an edge Q of ϕ to be the one-dimensional intersection of two polytopes on which the gradient of ϕ is different. We let $\mathbb{Q}(\phi)$ be the set of all edges of the function ϕ. A vertex v of ϕ is defined to be the zero-dimensional intersection of at least three different polytopes on which the gradient of ϕ is different. Finally, we let $\mathbb{V}(\phi)$ be the set of all vertices of the function ϕ.*

Observe that both the sets $\mathbb{V}(\phi)$ and $\mathbb{Q}(\phi)$ are finite because there is a finite number of polytopes in the description of ϕ. Next we define parallel lines on I^2.

Definition 8 (Parallel lines) *The lines l_1 and l_2 are parallel (non-parallel) lines in I^2, if $\exists$ two lines $l_1^{'}$ and $l_2^{'}$ parallel (non-parallel) in $\mathbb{R}^2$ such that $\mathbb{P}(l_1^{'}) = l_1$ and $\mathbb{P}(l_2^{'}) = l_2$.*

Before we state the next proposition, we observe that there might be continuous and piecewise linear functions defined on I^2 that have edges without vertices. These edges cause difficulties when verifying subadditivity. To circumvent these difficulties, we introduce the notion of a supplemental vertex.

Definition 9 (Supplemental vertex) *A supplemental vertex is an arbitrarily selected point on an edge that has no vertex. We denote the set of supplemental vertices of ϕ as $\mathbb{V}'(\phi)$.*

We next make an observation about edges of ϕ with supplemental vertices that will be used in the proof of Proposition 10. Let Q be an edge without a vertex. The edge Q can be of two types. In the first case, Q wraps around itself. Therefore, if we move along Q we will eventually reach its supplemental vertex. In the other case, Q does not return to any point it passes over. This case however is not possible since edges are the intersection of a finite number of polytopes.

Proposition 10 *Let ϕ be a continuous, piecewise linear and nonnegative function on I^2. Then ϕ is subadditive iff*

$$\phi(v_1) + \phi(v_2) \geq \phi(v_1 + v_2) \qquad \forall v_1, v_2 \in \mathbb{V}(\phi) \cup \mathbb{V}'(\phi) \qquad (14)$$

$$\phi(v_1) + \phi(v_3 - v_1) \geq \phi(v_3) \qquad \forall v_1, v_3 \in \mathbb{V}(\phi) \cup \mathbb{V}'(\phi) \qquad (15)$$

$$\phi(v_1) + \phi(e_2) \geq \phi(e_3) \text{ where } e_2 \in q_2, e_3 \in q_3, v_1 + e_2 = e_3, \quad \forall v_1 \in \mathbb{V}(\phi) \cup \mathbb{V}'(\phi), \forall q_2, q_3 \in \mathbb{Q}(\phi) \qquad (16)$$

$$\phi(e_1) + \phi(e_2) \geq \phi(v_3) \text{ where } e_1 \in q_1, e_2 \in q_2, e_1 + e_2 = v_3, \quad \forall v_3 \in \mathbb{V}(\phi) \cup \mathbb{V}'(\phi), \forall q_1, q_2 \in \mathbb{Q}(\phi). \qquad (17)$$

Furthermore, if e_2 and e_3 (resp. e_1 and e_2) belong to identical or parallel edges, then (16) (resp. (17)) is redundant.

Proof: The direct implication is straightforward. For the reverse implication we use the result of Proposition 6. The boundary $\mathbb{B}(\phi)$ of piecewise linear functions on I^2 only contains edges and vertices. Therefore in (10) and (11), l_1 and l_2 can either be vertices or points on edges, while $l_1 + l_2$ and $l_1 - l_2$ can be vertices, points on edges or points in the interior of the polytopes P_1,...,P_k. We conclude that it is sufficient to verify

the following set of conditions to prove that ϕ is subadditive.

$$
\begin{align}
\phi(e_1) + \phi(e_2) &\geq \phi(n_3) \tag{18} \\
\phi(v_1) + \phi(e_2) &\geq \phi(n_3) \tag{19} \\
\phi(e_1) + \phi(n_2) &\geq \phi(e_3) \tag{20} \\
\phi(v_1) + \phi(n_2) &\geq \phi(e_3) \tag{21} \\
\phi(e_1) + \phi(e_2) &\geq \phi(e_3) \tag{22} \\
\phi(v_1) + \phi(v_2) &\geq \phi(n_3) \tag{23} \\
\phi(v_1) + \phi(n_2) &\geq \phi(v_3) \tag{24} \\
\phi(e_1) + \phi(e_2) &\geq \phi(v_3) \tag{25} \\
\phi(v_1) + \phi(e_2) &\geq \phi(e_3) \tag{26} \\
\phi(v_1) + \phi(e_2) &\geq \phi(v_3) \tag{27} \\
\phi(v_1) + \phi(v_2) &\geq \phi(e_3) \tag{28} \\
\phi(v_1) + \phi(v_2) &\geq \phi(v_3) \tag{29} \\
\phi(e_1) + \phi(n_2) &\geq \phi(v_3). \tag{30}
\end{align}
$$

In these inequalities n_i represents points that do not belong to $\mathbb{B}(\phi)$, e_i represents points on edges and v_i represents vertices. Furthermore, when writing $\phi(a_1) + \phi(a_2) \geq \phi(a_3)$ we always assume $a_1 + a_2 = a_3$. Inequalities (23) - (29) are directly implied by (14) - (17). We now have to prove that inequalities (18) - (22) and (30) also hold because of (14) - (17). We will prove the contrapositive. In particular, we will consider each of the conditions (18) - (22) and (30) iteratively and show that this condition is not satisfied only if some of the other remaining condition is not satisfied. At the end of the iteration, this process will imply that the conditions (18) - (22) and (30) are not satisfied if conditions (23) - (29) are not satisfied. Since inequalities (23) - (29) are directly implied by (14) - (17) the result will be proved.

For $u \in Q$ where Q is a edge of ϕ, we denote $\epsilon_d(u) = min\{\epsilon > 0 \mid u + \epsilon d \in V(\phi) \cup V'(\phi)\}$. If $u \notin \mathbb{B}(\phi)$, we denote $\epsilon_d(u) = min\{\epsilon > 0 \mid u + \epsilon d \in \mathbb{B}(\phi)\}$.

1. Assume that there is e_1, n_2 and v_3 that do not satisfy (30). We show that at least one of the inequalities (18) - (29) is not satisfied. Let d be a direction of non-decreasing slope for the edge to which e_1 belongs. Let σ_1, σ_2 be the slopes at e_1 and n_2 along d and $-d$ respectively. If $\sigma_1 + \sigma_2 \leq 0$ then for sufficiently small $\epsilon > 0$ we have that $\phi(e_1 + d\epsilon) + \phi(n_2 - d\epsilon) < \phi(v_3)$. If $\epsilon = min\{\epsilon_d(e_1), \epsilon_{-d}(n_2)\}$, then we obtain one of the following situations

$$
\begin{align*}
\phi(v_1') + \phi(n_2') < \phi(v_3) &\qquad \text{which violates (24)} \\
\phi(e_1') + \phi(e_2') < \phi(v_3) &\qquad \text{which violates (25)} \\
\phi(e_1') + \phi(v_2') < \phi(v_3) &\qquad \text{which violates (27)} \\
\phi(v_1') + \phi(e_2') < \phi(v_3) &\qquad \text{which violates (27)} \\
\phi(v_1') + \phi(v_2') < \phi(v_3) &\qquad \text{which violates (29).}
\end{align*}
$$

Note that if $\sigma_1 + \sigma_2 > 0$, then it suffices to consider $-d$ instead of d to obtain the same result.

2. Assume that there is e_1, e_2 and n_3 that do not satisfy (18). We show that at least one of the inequalities (19) - (29) is not satisfied. Let d be a direction of nondecreasing slope for the edge to which e_1 belongs. Let σ_1, σ_3 be the slopes at e_1 and n_3 along d respectively. If $\sigma_1 \leq \sigma_3$, then for sufficiently small $\epsilon > 0$ we have that $\phi(e_1 + d\epsilon) + \phi(e_2) < \phi(n_3 + d\epsilon)$. If $\epsilon = min\{\epsilon_d(e_1), \epsilon_d(n_3)\}$, then we obtain one of the

following situations

$$\phi(v_1') + \phi(e_2) < \phi(n_3') \qquad \text{which violates (19)}$$
$$\phi(e_1') + \phi(e_2) < \phi(e_3') \qquad \text{which violates (22)}$$
$$\phi(e_1') + \phi(e_2) < \phi(v_3') \qquad \text{which violates (25)}$$
$$\phi(v_1') + \phi(e_2) < \phi(e_3') \qquad \text{which violates (26)}$$
$$\phi(v_1') + \phi(e_2) < \phi(v_3') \qquad \text{which violates (27).}$$

Note that if $\sigma_1 > \sigma_3$, then it suffices to consider $-d$ instead of d to obtain the same result.

3. Assume that there is v_1, e_2 and n_3 that do not satisfy (19). We show that at least one of inequalities (20) - (29) is not satisfied. Let d be a direction of nondecreasing slope for the edge to which e_2 belongs. Let σ_2, σ_3 be the slopes at e_2 and n_3 along d respectively. If $\sigma_2 \leq \sigma_3$, then for sufficiently small $\epsilon > 0$ we have that $\phi(v_1) + \phi(e_2 + d\epsilon) < \phi(n_3 + d\epsilon)$. If $\epsilon = min\{\epsilon_d(e_2), \epsilon_d(n_3)\}$, then we obtain one of the following situations

$$\phi(v_1) + \phi(v_2') < \phi(n_3') \qquad \text{which violates (23)}$$
$$\phi(v_1) + \phi(e_2') < \phi(e_3') \qquad \text{which violates (26)}$$
$$\phi(v_1) + \phi(e_2') < \phi(v_3') \qquad \text{which violates (27)}$$
$$\phi(v_1) + \phi(v_2') < \phi(e_3') \qquad \text{which violates (28)}$$
$$\phi(v_1) + \phi(v_2') < \phi(v_3') \qquad \text{which violates (29).}$$

Note that if $\sigma_2 > \sigma_3$, then it suffices to consider $-d$ instead of d to obtain the same result.

4. Assume that there is e_1, n_2 and e_3 that do not satisfy (20). We show that at least one of conditions (21) - (29) is not satisfied. Let d be a direction of nondecreasing slope for the edge to which e_1 belongs. Let σ_1, σ_2 be the slopes at e_1 and n_2 along d and $-d$ respectively. If $\sigma_1 + \sigma_2 \leq 0$ then for sufficiently small $\epsilon > 0$, we have that $\phi(e_1 + d\epsilon) + \phi(n_2 - d\epsilon) < \phi(e_3)$. If $\epsilon = min\{\epsilon_d(e_1), \epsilon_{-d}(n_2)\}$, then we obtain one of the following situations

$$\phi(v_1') + \phi(n_2') < \phi(e_3) \qquad \text{which violates (21)}$$
$$\phi(e_1') + \phi(e_2') < \phi(e_3) \qquad \text{which violates (22)}$$
$$\phi(e_1') + \phi(v_2') < \phi(e_3) \qquad \text{which violates (26)}$$
$$\phi(v_1') + \phi(e_2') < \phi(e_3) \qquad \text{which violates (26)}$$
$$\phi(v_1') + \phi(v_2') < \phi(e_3) \qquad \text{which violates (28).}$$

Note that if $\sigma_1 + \sigma_2 > 0$, then it suffices to consider $-d$ instead of d to obtain the same result.

5. Assume that there is v_1, n_2 and e_3 that do not satisfy (21). We show that at least one of inequalities (22) - (29) is not satisfied. Let d be a direction of nondecreasing slope for the edge to which e_3 belongs. Let σ_2, σ_3 be the slopes at n_2 and e_3 along d respectively. Then if $\sigma_2 \leq \sigma_3$, then for sufficiently small $\epsilon > 0$ we have that $\phi(v_1) + \phi(n_2 + d\epsilon) < \phi(e_3 + d\epsilon)$. If $\epsilon = min\{\epsilon_d(n_2), \epsilon_d(e_3)\}$, then we obtain one of the following situations

$$\phi(v_1) + \phi(e_2') < \phi(e_3') \qquad \text{which violates (26)}$$
$$\phi(v_1) + \phi(n_2') < \phi(v_3') \qquad \text{which violates (24)}$$
$$\phi(v_1) + \phi(e_2') < \phi(v_3') \qquad \text{which violates (27)}$$
$$\phi(v_1) + \phi(v_2') < \phi(e_3') \qquad \text{which violates (28)}$$
$$\phi(v_1) + \phi(v_2') < \phi(v_3') \qquad \text{which violates (29).}$$

Note that if $\sigma_2 > \sigma_3$, then it suffices to consider $-d$ instead of d to obtain the same result.

6. Assume that there is e_1, e_2 and e_3 that do not satisfy (22). We show that at least one of inequalities (23) - (29) is not satisfied. There are two subcases:

   (a) The edges to which e_1 and e_2 belong are parallel. Let d be the direction of nondecreasing slope for the edge to which e_1 belongs. Let σ_1, σ_2 be the slopes at e_1 and e_2 along d respectively. Then if $\sigma_1 \leq \sigma_2$, then for sufficiently small $\epsilon > 0$ we have that $\phi(e_1 + d\epsilon) + \phi(e_2 - d\epsilon) < \phi(e_3)$. If $\epsilon = min\{\epsilon_d(e_1), \epsilon_{-d}(e_2)\}$, then we obtain one of the following situations

$$\phi(v_1') + \phi(e_2') < \phi(e_3) \qquad \text{which violates (26)}$$
$$\phi(e_1') + \phi(v_2') < \phi(e_3) \qquad \text{which violates (26)}$$
$$\phi(v_1') + \phi(v_2') < \phi(e_3) \qquad \text{which violates (28).}$$

   Note that if $\sigma_1 > \sigma_2$, then it suffices to consider $-d$ instead of d to obtain the same result.

   (b) The edges to which e_1 and e_2 belong are not parallel. In this case, there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha u_1 + \beta u_2 = u_3$, where u_i is a unit vector in the direction of nondecreasing slope for each of the three edges to which e_1, e_2, e_3 belong. Let σ_1, σ_2 and σ_3 be the slopes at e_1, e_2 and e_3 along u_1, u_2 and u_3 respectively. Then if $\alpha\sigma_1 + \beta\sigma_2 \leq \sigma_3$, then for sufficiently small $\epsilon > 0$ we have that $\phi(e_1 + \alpha u_1\epsilon) + \phi(e_2 + \beta u_2\epsilon) < \phi(e_3 + u_3\epsilon)$. If $\epsilon = min\{\epsilon_{\alpha u_1}(e_1), \epsilon_{\beta u_2}(e_2), \epsilon_{u_3}(e_3)\}$, then we obtain one of the following situations

$$\phi(v_1') + \phi(e_2') < \phi(e_3') \qquad \text{which violates (26)}$$
$$\phi(e_1') + \phi(v_2') < \phi(e_3') \qquad \text{which violates (26)}$$
$$\phi(e_1') + \phi(e_2') < \phi(v_3') \qquad \text{which violates (25)}$$
$$\phi(v_1') + \phi(v_2') < \phi(e_3') \qquad \text{which violates (28)}$$
$$\phi(e_1') + \phi(v_2') < \phi(v_3') \qquad \text{which violates (27)}$$
$$\phi(v_1') + \phi(e_2') < \phi(v_3') \qquad \text{which violates (27)}$$
$$\phi(v_1') + \phi(v_2') < \phi(v_3') \qquad \text{which violates (29).}$$

   Note that if $\alpha\sigma_1 + \beta\sigma_2 > \sigma_3$, then it suffices to consider $-d$ instead of d to obtain the same result.

This concludes the proof of the first part of the proposition as we have shown in cases (i)-(vi) that if ϕ is not subadditive then one of the conditions (14) - (17) is violated.

Finally we consider the case where e_2 and e_3 belong to parallel or identical edges. We show next that in this case (16) is not necessary as it is implied by (14) or (15). Let d be a direction of nonnegative slope for the edge to which e_2 belongs. Let this slope be σ_2. Let the slope at e_3 along direction d be σ_3. If $\sigma_2 \leq \sigma_3$, we have $\phi(v_1) + \phi(e_2 + \epsilon d) < \phi(e_3 + \epsilon d)$. If $\epsilon = min\{\epsilon_d(e_2), \epsilon_d(e_3)\}$, then we obtain a contradiction to either (14) or (15). If $\sigma_2 > \sigma_3$, consider $-d$ instead of d to obtain a similar contradiction. Similarly it can be proven that if e_1 and e_2 belong to parallel or identical edges then (17) is not necessary as it is implied by (14) or (15). □

It follows from Proposition 6 and Proposition 10 that it is possible to verify that a continuous and piecewise linear function defined over I^2 is subadditive by verifying that a finite number of inequalities hold. We prove in Proposition 12 that the number of inequalities that need to be checked can be further reduced if ϕ satisfies a specific symmetry condition that is required for an inequality to be strong. This notion of symmetry is presented in Johnson [14] and is a generalization of similar condition for one-dimensional group problems; see Gomory and Johnson [12].

Definition 11 (Symmetry) *A function $\phi : I^2 \to \mathbb{R}$ is said to be symmetric if $\phi(u) + \phi(r - u) = 1 \ \forall u \in I^2$, where r is the right-hand-side. Furthermore, for any point $u \in I^2$, the point $r - u$, is said to be the complementary point of u.*

If symmetry conditions are satisfied for a function ϕ, it can easily be seen that the complement of an edge of ϕ is also an edge of ϕ and the complement of a vertex is also a vertex. Therefore, if a supplemental vertex v

is defined for ϕ we also define its complementary point as a supplemental vertex. In Proposition 12, we show that the conditions needed to verify that a function ϕ is subadditive can be simplified when it is symmetric.

Proposition 12 *Let ϕ be a continuous and piecewise linear function on I^2 that is symmetric (with complementary supplemental vertices). Then ϕ is subadditive iff (14) and (17) hold.*

Proof: Assume for a contradiction that (15) is violated, i.e.,

$$\phi(v_1) + \phi(v_3 - v_1) < \phi(v_3).$$

We show that this implies that an inequality of the form (14) is also violated. Because ϕ is symmetric, (31) implies that

$$\phi(v_1) + \phi(r - v_3) < \phi(r - v_3 + v_1).$$

Since, v_3 is a vertex, $r - v_3$ is also a vertex, which yields the desired contradiction to (14). Similarly, we can prove that if (16) is violated, then $\phi(r - e_3) + \phi(e_2) < \phi(r - v_1)$, where $r - v_1$ is a vertex, yielding a contradiction to (17). □

5 Facets of 2DIIGP.

In this section, we develop conditions on the function ϕ under which the inequality

$$\sum_{u \in I^2} \phi(u)t(u) \geq 1 \tag{31}$$

is facet-defining for 2DIIGP. The following property is a necessary condition for a valid inequality (31) to be strong.

Definition 13 (Minimal) *Let $r \in I^2$ and let $\phi : I^2 \to \mathbb{R}$ be a valid function. We say that ϕ is minimal if there does not exist a valid function ϕ^* different from ϕ such that $\phi^*(u) \leq \phi(u) \ \forall u \in I^2$.*

Necessary and sufficient conditions for ϕ to be minimal were derived in Gomory and Johnson [11] and are given in the following theorem.

Theorem 14 *Let $r \in I^2$ and let $\phi : I^2 \to \mathbb{R}$ be a valid function. Then ϕ is minimal iff ϕ is subadditive and the symmetry condition $\phi(u) + \phi(r - u) = 1$ holds for every $u \in I^2$.* □

Since the minimality of a function ϕ is related to the right-hand-side r by the symmetry condition, we say that a function is minimal with respect to right-hand-side r. Because the infinite group has an uncountable number of columns the definition of "facet" is more technical than that used in finite group problems. The following definition is from Gomory and Johnson [12].

Definition 15 (Facet) *Let $P(\phi)$ be the set of points t that satisfy a given inequality ϕ at equality, i.e., $t \in P(\phi)$ if $t \in 2DIIGP$ and $\sum_{u \in I^2, t(u) > 0} \phi(u)t(u) = 1$. We say that an inequality ϕ is facet-defining for I^2 if there does not exists a valid function ϕ^* such that $P(\phi^*) \supsetneq P(\phi)$.*

A valid subadditive function that induces a facet for 2DIIGP is called a *facet-defining function*. Facet-defining functions are minimal; see [11]. However, not all minimal functions yield facet-defining inequalities for 2DIIGP. Proving that a subadditive function ϕ is facet-defining is typically difficult. Gomory and Johnson [12] introduced the following definition to help simplify these proofs.

Definition 16 (Equality Set) *For each point $u \in I^2$, define $g(u)$ to be the variable corresponding to the point u. We define the set of equalities of ϕ to be the system of equations $g(u) + g(v) = g(u + v)$ for all $u, v \in I^2$ such that $\phi(u) + \phi(v) = \phi(u + v)$. We denote it as $E(\phi)$.*

11

The following theorem gives a procedure to verify that ϕ is a facet-defining inequality for 2DIIGP. Although it is proven in Gomory and Johnson [12] in the context of the one-dimensional group problem and for continuous valid functions, the proof is general and works in our case.

Theorem 17 (Facet Theorem) *If ϕ is minimal and subadditive, and if the set $E(\phi)$ of all equalities of ϕ has no solution in the space of valid functions other than ϕ then ϕ is a facet.* □

We next present two propositions that will be used extensively to prove that given families of inequalities are facet-defining for 2DIIGP. Proposition 18 and its converse Proposition 19 are variations of the Facet Theorem with very similar proofs to that of the Facet Theorem; see Dey [6].

Proposition 18 *Let ϕ be a minimal function for 2DIIGP. If ϕ is not facet-defining then there exists a valid subadditive and minimal function $\phi' \neq \phi$ such that $E(\phi') \supsetneq E(\phi)$.* □

Proposition 19 *Let ϕ be a minimal function for 2DIIGP. If there exists a valid subadditive and minimal function $\phi' \neq \phi$ such that $E(\phi') \supsetneq E(\phi)$, then ϕ is not facet-defining.* □

To prove that $E(\phi)$ has a unique solution, Gomory and Johnson [12] used a result that they refer to as the Interval Lemma. However, this result does not apply here as it is specific to one-dimensional problems. In Proposition 24, we introduce a new result that can be used to verify that $E(\phi)$ has an unique solution for two-dimensional problems. The following proposition, given in Aczél [1], is used in the proof of Proposition 23.

Proposition 20 *Let K be the closed interval $[0, \epsilon] \subset \mathbb{R}$ for $\epsilon > 0$. If $g : K \to \mathbb{R}$ is such that $g(x) + g(y) = g(x + y) \; \forall x, y \in K$ and $g(x) \geq 0$ for arbitrarily small $x \in K$, then $g(x) = cx \; \forall x \in K$, where $c \in \mathbb{R}$.* □

We give a two-dimensional version of this result in Proposition 24. Before we prove this result, we introduce the notation $[a, b]$ to represent the line segment between a and b. This notation is slightly ambiguous in I^2 since there are multiple line segments with end points $a, b \in I^2$. If we denote $\tilde{a}, \tilde{b}$ to be elements of $\mathbb{R}^2$ that have the same numerical values as a and b then we will use $[a, b]$ to refer to the line segment $\mathbb{P}([\tilde{a}, \tilde{b}])$ unless otherwise specified.

Definition 21 (Star-shaped) *A set $S \subseteq \mathbb{R}^2$ is defined to be star-shaped with respect to a point $u \in S$, if the line segment $[u, v] \in S \; \forall v \in S$.*

We define a set $S \in I^2$ to be star-shaped with respect to a point u, if there is a star-shaped set $S' \in \mathbb{R}^2$ with respect to a point $u' \in \mathbb{R}^2$ such that $\mathbb{P}(S') = S$ and $\mathbb{P}(u') = u$.

Definition 22 (Path Connected) *A set S is path-connected if for any two points $x, y \in S$, there is a continuous function $f : [0, 1] \to S$ such that $f(0) = x$ and $f(1) = y$.*

Proposition 23 *Let U and V be closed sets in $\mathbb{R}^2$. Let g be a real-valued function defined over U, V and $U + V$. Assume that*

1. *U is star-shaped with respect to the origin and U has a non-empty interior.*

2. *V is path connected.*

3. *$g(u) + g(v) = g(u + v), \; \forall u \in U, \; \forall v \in V$.*

4. *$\sum_{i \in S} g(u_i) = g(\sum_{i \in S} u_i) \; \forall u_i \in U$ such that $\sum_{i \in S} u_i \in U$ and $\forall S$ with $|S| \leq 3$.*

5. *$g(u) \geq 0, \; \forall u \in U$.*

Then g is a linear function with the same gradient in U, V and $U + V$.

Proof: Let o represent the origin. It is easily verified that 1, 4 and 5 imply that $g(o) = 0$. Moreover, because of 4 and because $g(o) = 0$ we can easily show that $\mu_1 g(u_1) + \mu_2 g(u_2) = g(\mu_1 u_1 + \mu_2 u_2)$, where $\mu_1, \mu_2 \in \{1, -1\}$ and $u_1, u_2, (\mu_1 u_1 + \mu_2 u_2) \in U$. Consider now two points $p, q \in U$ such that p, q, o are affinely independent. Such points exist since the interior of the set U is non-empty. It follows from Proposition 20 and conditions 1, 4 and 5 that $g(\theta p) = c_1 \theta$ and $g(\theta q) = c_2 \theta$ for all $\theta \in [0, 1]$ and for some $c_1, c_2 \in \mathbb{R}_+$. For any point $t \in U$, $\exists \alpha_1, \alpha_2 \in \mathbb{R}$ such that $t = \alpha_1 p + \alpha_2 q$. Now, $\exists \delta \in \mathbb{Z}$, $\delta \geq 1$, such that $\alpha_1 = \delta \mu_1 \theta_1$ and $\alpha_2 = \delta \mu_2 \theta_2$, where $\theta_1, \theta_2 \in [0, 1]$, and $\mu_1, \mu_2 \in \{-1, 1\}$. Because U is star-shaped and because of assumption 4, we have $g(t) = g(\alpha_1 p + \alpha_2 q) = g(\delta \mu_1 \theta_1 p + \delta \mu_2 \theta_2 q) = \delta g(\mu_1 \theta_1 p + \mu_2 \theta_2 q) = \delta \mu_1 g(\theta_1 p) + \delta \mu_2 g(\theta_2 q) = \delta \mu_1 \theta_1 c_1 + \delta \mu_2 \theta_2 c_2 = \alpha_1 c_1 + \alpha_2 c_2$. Therefore, g is linear over U. We can hence write $g(x) = \alpha^T x$.

Next we prove that $\exists \epsilon > 0$ such that $\forall b_1, b_2 \in V$ with $|b_1 - b_2| < \epsilon$, $(b_1 + U) \cap (b_2 + U) \neq \emptyset$. Since U has a non empty interior, $\exists u_1 \in U$ and $\epsilon > 0$ such that the open ball of radius 2ϵ around u_1 is completely contained in U, i.e., $B(u_1, 2\epsilon) \subseteq U$. Therefore $u_1 + b_1 - b_2 \in U$, i.e., $\exists u_2 \in U$ such that $u_1 + b_1 = u_2 + b_2$. Thus $(b_1 + U) \cap (b_2 + U) \neq \emptyset$.

For any $b \in V$, g is linear with gradient α over the set $(b + U) \subset (U + V)$ since g is linear over U with gradient α and 3 holds. Next suppose that we have $b_1, b_2 \in V$ such that $(b_1 + U) \cap (b_2 + U) \neq \emptyset$. Then let g on $(b_1 + U)$ and $(b_2 + U)$ be $\alpha^T x + \beta_1$ and $\alpha^T x + \beta_2$ respectively. For any $x_0 \in (b_1 + U) \cap (b_2 + U)$ we have $\alpha^T x_0 + \beta_1 = \alpha^T x_0 + \beta_2$, or $\beta_1 = \beta_2$. Thus, if $(b_1 + U) \cap (b_2 + U) \neq \emptyset$, the function is linear and identical over the two sets. Finally we show that for $b_1, b_2 \in V$ such that $(b_1 + U) \cap (b_2 + U) = \emptyset$, g is linear and identical over both sets. Since V is path-connected, there exists a path connecting b_1 and b_2. Therefore, we can construct a finite sequence of points $c_1, ..., c_n$ on this path such that $|b_1 - c_1| \leq \epsilon/2$, $|c_i - c_{i+1}| \leq \epsilon/2$ and $|c_n - b_2| \leq \epsilon/2$. Thus if g over $b_1 + U$ is $\alpha^T x + \beta$, then it is the same over all $c_i + U$ and therefore also over $b_2 + U$. Thus g over $U + V$ is of the form $\alpha^T x + \beta$. Finally, we note that $V \subset U + V$, since $o \in U$. Thus g is linear over U, V and $U + V$ and has the same gradient α on these sets. $\qquad \square$

Note that the condition 4 in Proposition 23 does not follow from a condition of the form $g(u_1) + g(u_2) = g(u_1 + u_2)$, $\forall u_1, u_2 \in U$ since, it may be that $u_i + u_j \notin U$ even though $\sum_{i \in S} u_i \in U$. Note also that it may be that the function satisfying the conditions of Proposition 23 is of the form $g(u) = \alpha^T u$ $\forall u \in U$ and $g(u) = \alpha^T u + \beta$ $\forall u \in V, U + V$, where $\beta \in \mathbb{R}$.

Next we prove a variant of Proposition 23 that is valid for I^2 rather than $\mathbb{R}^2$.

Proposition 24 *Let U' and V' be closed sets in $\mathbb{R}^2$ which satisfy conditions 1 and 2 of Proposition 23. Let $U = \mathbb{P}(U')$ and $V = \mathbb{P}(V')$. Assume that $\forall u \in U$, $\exists \tilde{u} \in U'$ such that $\mathbb{P}(\tilde{u}) = u$ and for all directions d, there exists a sufficiently small positive ϵ such that if $u + \epsilon d \in U$, then $\tilde{u} + \epsilon d \in U'$. Also assume that the previous property holds for V and V', and for $U + V$ and $U' + V'$. Finally assume that there is a real-valued function g defined over U, V and $U + V$, that satisfies conditions 3, 4, 5 of Proposition 23. Then g is a linear function on U, V and $U + V$.*

Proof: Consider the real-valued function g' defined over U', V' and $U' + V'$ as, $g'(\tilde{u}) = g(\mathbb{P}(\tilde{u}))$. Since, $\mathbb{P}(x) + \mathbb{P}(y) = \mathbb{P}(x + y)$, g' satisfies 3, 4, 5 of Proposition 23. Therefore by Proposition 23, g' is a linear function over U', V' and $U' + V'$. Now consider any point $u \in U$ and any direction d such that $u + \epsilon d \in U$ for sufficiently small ϵ. By assumption, there exists $\tilde{u}$, such that $\mathbb{P}(\tilde{u}) = u$ and $\tilde{u} + \epsilon d \in U'$. Therefore,

$$
\begin{aligned}
g(u + \epsilon d) - g(u) &= g(\mathbb{P}(\tilde{u} + \epsilon d)) - g(\mathbb{P}(\tilde{u})) \\
&= g'(\tilde{u} + \epsilon d) - g'(\tilde{u}) \\
&= \sigma_d \epsilon, \tag{32}
\end{aligned}
$$

where σ_d is the slope of g' in the direction d. Thus,

$$
\lim_{\epsilon \to 0^+} \frac{g(u + \epsilon d) - g(u)}{\epsilon} = \sigma_d. \tag{33}
$$

We conclude that for all directions d and all points in U, the slope of g and g' are equal. Since g' is linear over U', it follows that g is linear over U. A similar argument can be made to prove that g is linear over V and $U + V$. $\qquad \square$

Note that Proposition 24 plays a role analogous to the Interval Lemma of Gomory and Johnson [12]. Proposition 24 is different from the Interval Lemma in that it does not assume continuity of g and is defined over a two dimensional group. Next we give results about automorphisms and homomorphisms over I^2. These results are presented because they allow the derivation of new valid inequalities for I^2 from known valid inequalities of I^2. Propositions 25 -27 and Theorem 28 are either from or adapted from Gomory and Johnson [10] and Johnson [14].

Proposition 25 *The rotation function $\rho : I^2 \to I^2$, $\rho(x, y) = (1 - y, x)$ is an automorphism.* $\qquad\square$

Proposition 26 *The reflection function $\varsigma : I^2 \to I^2$, $\varsigma(x, y) = (1 - x, y)$ is an automorphism.* $\qquad\square$

Figures 1 and 2 illustrate the rotation and reflection functions. We also note that all finite compositions
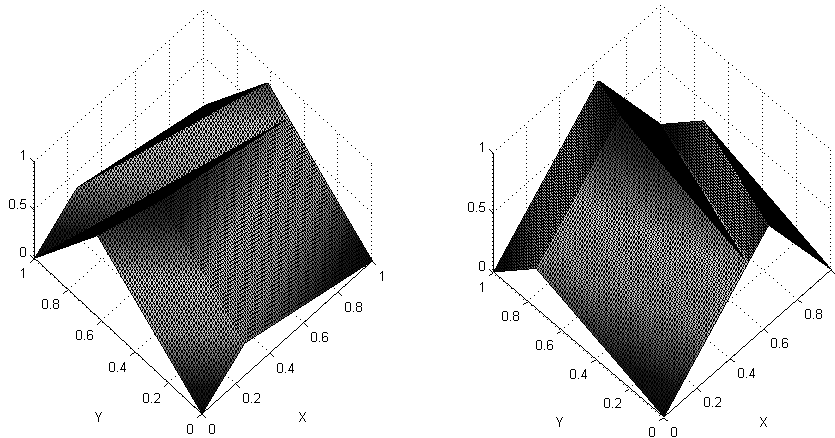

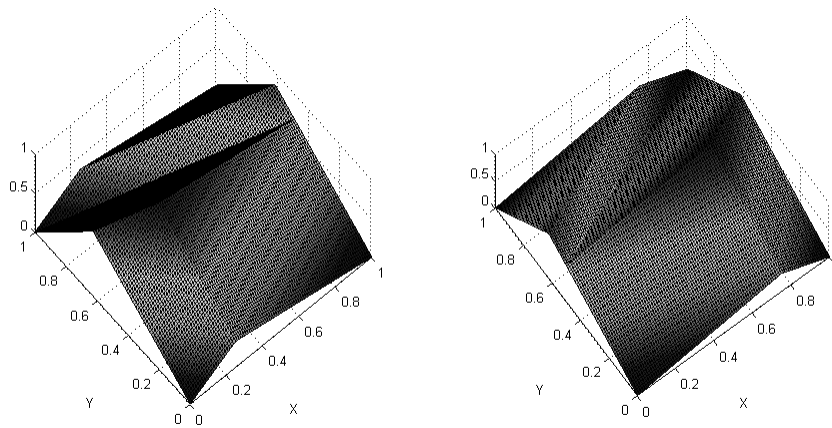Figure 1: Facet ϕ and its rotation $\phi \circ \rho$ where $\rho(x, y) = (1 - y, x)$.

Figure 2: Facet ϕ and its reflection $\phi \circ \varsigma$ where $\varsigma(x, y) = (1 - x, y)$.

of the above rotation and reflection functions are automorphisms. The next proposition is adapted from Gomory and Johnson [10].

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Proposition 27 *Let $\omega : I^2 \to I^2$ be a surjective map that is a homomorphism. The function ϕ with respect to the right-hand-side r is subadditive and minimal iff the function $\phi \circ \omega(x) = \phi(\omega(x))$ is subadditive and minimal with respect to right-hand-side v, for any v such that $\omega(v) = r$.*

Proof: First we assume that ϕ is subadditive and minimal. We prove first that $\phi \circ \omega$ is subadditive. For $u_1, u_2 \in I^2$, we have $\phi \circ \omega(u_1) + \phi \circ \omega(u_2) = \phi(\omega(u_1)) + \phi(\omega(u_2)) \geq \phi(\omega(u_1) + \omega(u_2)) = \phi(\omega(u_1 + u_2)) = \phi \circ \omega(u_1 + u_2)$. The first inequality holds because ϕ is subadditive while the last but one equality holds because ω is a homomorphism. We now prove that $\phi \circ \omega$ is minimal. We similarly write that $\phi \circ \omega(u_1) + \phi \circ \omega(v - u_1) = \phi(\omega(u_1)) + \phi(\omega(v - u_1)) = \phi(\omega(u_1)) + \phi(\omega(v) - \omega(u_1)) = 1$.

Now we assume that $\phi \circ \omega$ is subadditive and minimal. First we prove ϕ is subadditive. Since ω is surjective we can choose $u_1' \in \{z \mid \omega(z) = u_1\}$ and $u_2' \in \{z \mid \omega(z) = u_2\}$, for any $u_1, u_2 \in I^2$. Because ω is a homomorphism we have, $\omega(u_1' + u_2') = \omega(u_1') + \omega(u_2') = u_1 + u_2$. Thus, $\phi(u_1) + \phi(u_2) = \phi \circ \omega(u_1') + \phi \circ \omega(u_2') \geq \phi \circ \omega(u_1' + u_2') = \phi(\omega(u_1' + u_2')) = \phi(u_1 + u_2)$. The inequality holds because $\phi \circ \omega$ is subadditive. To prove ϕ is minimal, we note that $\omega(v - u_1') = \omega(v) - \omega(u_1') = r - u_1$. Thus, $\phi(u_1) + \phi(r - u_1) = \phi \circ \omega(u_1') + \phi \circ \omega(v - u_1') = \phi \circ \omega(v) = \phi(\omega(v)) = 1$. $\qquad\square$

Proposition 27 shows that the subadditivity and minimality of a function are preserved under surjective homomorphisms. For automorphisms, an even stronger result can be proven. This result is given in Proposition 28.

Theorem 28 *Let $\omega : I^2 \to I^2$ be an automorphism and let ϕ be facet-defining for 2DIIGP. Then $\phi \circ \omega$ is facet-defining for 2DIIGP.* $\qquad\square$

Because there exists an unique automorphism ω^{-1} where $\omega \circ \omega^{-1}$ is the identity function on I^2, Theorem 28 also implies that ϕ is facet-defining if $\phi \circ \omega$ is facet-defining.

Next we prove that the facet-defining property of a function is also preserved for the following multiplicative homomorphism $\lambda : I^2 \to I^2$ defined as $\lambda(x, y) = (\lambda_1 x (mod 1), \lambda_2 y (mod 1))$, where λ_1, λ_2 are positive integers. The following proposition is easily verified.

Proposition 29 *λ is a surjective homomorphism on I^2.* $\qquad\square$

The proof of the following theorem is similar to the proof for the one-dimensional case given by Gomory and Johnson [12].

Theorem 30 *ϕ is facet-defining with respect to right-hand-side r iff $\phi \circ \lambda$ is facet-defining with respect to right-hand-side v, where $\lambda(v) = r$.*

Proof: Assume first that $\phi \circ \lambda$ is facet-defining. It follows from Proposition 27 that ϕ is minimal and subadditive. Assume by contradiction that ϕ is not facet-defining. Then by Proposition 18 there exists a valid subadditive and minimal function $\phi^* \neq \phi$ such that $E(\phi^*) \supsetneq E(\phi)$. First note that, by Proposition 29, $\phi \neq \phi^*$ implies that $\phi \circ \lambda \neq \phi^* \circ \lambda$. Second, consider any equality that is satisfied by $\phi \circ \lambda$, $\phi \circ \lambda(u_1) + \phi \circ \lambda(u_2) = \phi \circ \lambda(u_1 + u_2)$. We also have $\phi^*(\lambda(u_1)) + \phi^*(\lambda(u_2)) = \phi^* \lambda(u_1 + u_2)$ or equivalently, $\phi^* \circ \lambda(u_1) + \phi^* \circ \lambda(u_2) = \phi^* \circ \lambda(u_1 + u_2)$. Thus, $\phi^* \circ \lambda$ satisfies all the equalities of $E(\phi \circ \lambda)$. Also, since $E(\phi^*) \supsetneq E(\phi)$, there exists $\overline{u}_1, \overline{u}_2$, such that $\phi^*(\overline{u}_1) + \phi^*(\overline{u}_2) = \phi^*(\overline{u}_1 + \overline{u}_2)$ and $\phi(\overline{u}_1) + \phi(\overline{u}_2) > \phi(\overline{u}_1 + \overline{u}_2)$. Let $\overline{v}_1, \overline{v}_2 \in I^2$ such that $\lambda(\overline{v}_1) = \overline{u}_1$ and $\lambda(\overline{v}_2) = \overline{u}_2$. Such point exists because λ is surjective. We obtain that $\phi^* \circ \lambda(\overline{v}_1) + \phi^* \circ \lambda(\overline{v}_2) = \phi^* \circ \lambda(\overline{v}_1 + \overline{v}_2)$ and $\phi \circ \lambda(\overline{v}_1) + \phi \circ \lambda(\overline{v}_2) > \phi \circ \lambda(\overline{v}_1 + \overline{v}_2)$. Thus $E(\phi^* \circ \lambda) \supsetneq E(\phi \circ \lambda)$ which is a contradiction to the fact that $\phi \circ \lambda$ is facet-defining, by Proposition 19.

Next we assume that ϕ is facet-defining. From Proposition 27, we have that $\phi \circ \lambda$ is subadditive and minimal. Assume by contradiction that $\phi \circ \lambda$ is not facet-defining. It follows from Proposition 18 that there exists a valid subadditive and minimal function η such that $E(\phi \circ \lambda) \subsetneq E(\eta)$. First consider the vector $(\hat{x}, \hat{y}) = \left(\frac{n_1}{\lambda_1}, \frac{n_2}{\lambda_2} \right)$ where n_1 and n_2 are nonnegative integers. Then, $\lambda(t(\hat{x}, \hat{y})) = (0, 0)$ for $t \in \mathbb{Z}$ and thus $\phi(\lambda(t(\hat{x}, \hat{y}))) = \phi(0, 0) = 0$. Further, if l is the least common multiple of λ_1 and λ_2, we have $(l)(\hat{x}, \hat{y}) \equiv (0, 0)$ and therefore $(l)\phi(\lambda(\hat{x}, \hat{y})) = 0 = \phi(0, 0) = \phi(\lambda((l(\hat{x}, \hat{y})))$. Since $E(\phi \circ \lambda) \subsetneq E(\eta)$, it is easy to verify that $(l)\eta \left(\frac{n_1}{\lambda_1}, \frac{n_2}{\lambda_2} \right) = \eta(l(\hat{x}, \hat{y})) = \eta(0, 0) = 0$ or equivalently $\eta \left(\frac{n_1}{\lambda_1}, \frac{n_2}{\lambda_2} \right) = 0$. Furthermore for

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any vector $(x, y) \in I^2$ we have that $\phi(\lambda(x + \frac{1}{\lambda_1}, y)) = \phi(\lambda(x, y)) = \phi(\lambda(x, y)) + \phi(\lambda(\frac{1}{\lambda_1}, 0))$ and also that $\phi(\lambda(x, y + \frac{1}{\lambda_2})) = \phi(\lambda(x, y)) = \phi(\lambda(x, y)) + \phi(\lambda(0, \frac{1}{\lambda_2}))$. Hence η must satisfy these equalities. This implies that η has the same repeating pattern as $\phi \circ \lambda$ and the value of η over I^2 is completely defined by the value of η over $C = \{(x, y) \mid 0 \leq x < 1/\lambda_1, 0 \leq y < 1/\lambda_2\}$.

Now we construct the function ϕ^* on I^2 as $\phi^*(x, y) = \eta(\frac{x}{\lambda_1}, \frac{y}{\lambda_2})$. Using the fact that η has the same repeating pattern as $\phi \circ \lambda$ it is easy to verify that ϕ^* is a valid subadditive inequality. Since by assumption $\eta \neq \phi \circ \lambda$ we have that $\phi^* \neq \phi$. Now for any equality $\phi(x_1, y_1) + \phi(x_2, y_2) = \phi(x_1 + x_2, y_1 + y_2)$ that ϕ satisfies, we have $\phi \circ \lambda(\frac{x_1}{\lambda_1}, \frac{y_1}{\lambda_2}) + \phi \circ \lambda(\frac{x_2}{\lambda_1}, \frac{y_2}{\lambda_2}) = \phi \circ \lambda(\frac{(x_1 + x_2)}{\lambda_1}, \frac{(y_1 + y_2)}{\lambda_2})$. Since η satisfies all the equalities of $\phi \circ \lambda$, we have $\eta(\frac{x_1}{\lambda_1}, \frac{y_1}{\lambda_2}) + \eta(\frac{x_2}{\lambda_1}, \frac{y_2}{\lambda_2}) = \eta(\frac{(x_1 + x_2)}{\lambda_1}, \frac{(y_1 + y_2)}{\lambda_2})$. If follows from the definition of ϕ^* that $\phi^*(x_1, y_1) + \phi^*(x_2, y_2) = \phi^*(x_1 + x_2, y_1 + y_2)$. Thus, ϕ^* satisfies all the equalities that ϕ satisfies. Moreover, $E(\phi \circ \lambda) \subsetneq E(\eta)$. Therefore, there exists $\overline{u}_1, \overline{u}_2 \in I^2$ such that $\eta(\overline{u}_1) + \eta(\overline{u}_2) = \eta(\overline{u}_1 + \overline{u}_2)$ and $\phi \circ \lambda(\overline{u}_1) + \phi \circ \lambda(\overline{u}_2) > \phi \circ \lambda(\overline{u}_1 + \overline{u}_2)$. This implies that $\phi^*(\lambda(\overline{u}_1)) + \phi^*(\lambda(\overline{u}_2)) = \phi^*(\lambda(\overline{u}_1 + \overline{u}_2))$ and $\phi(\lambda(\overline{u}_1)) + \phi(\lambda(\overline{u}_2)) > \phi(\lambda(\overline{u}_1 + \overline{u}_2))$. Thus $E(\phi^*) \supsetneq E(\phi)$. We conclude from Proposition 19 that this is a contradiction to the fact that ϕ is facet-defining. $\square$

In Figure 3 we show an example of an application of the homomorphism result of Theorem 30 when applied to the three-gradient facet that is discussed in detail in §7. Both the functions displayed in Figure 3 are facet-defining for 2DIIGP.
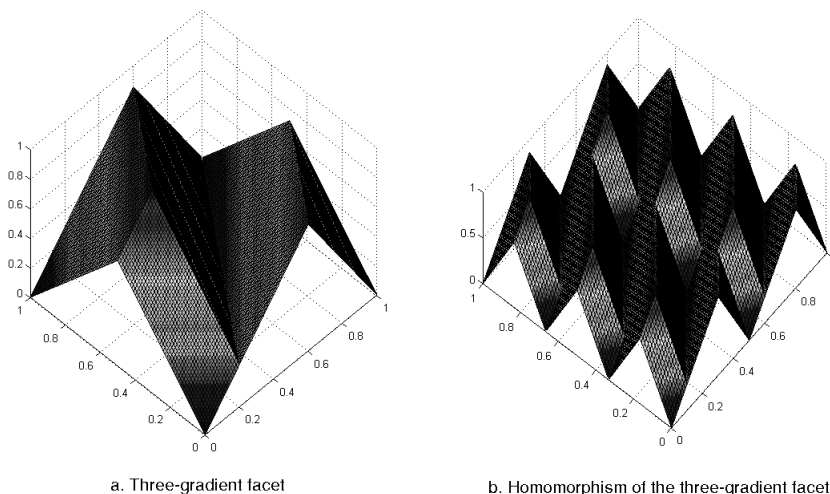

a. Three-gradient facet   b. Homomorphism of the three-gradient facet

Figure 3: Generating new facets using the Homomorphism Theorem ($\lambda_1 = 2$, $\lambda_2 = 3$)

6   A simple family of facets.

In this section we describe a method to construct facets of 2DIIGP from facets of 1DIIGP. We then show that all facets of 2DIIGP that have only two gradients can be derived using this construction.

Construction 31 *Given ζ a piecewise linear and continuous valid inequality for 1DIIGP with right-hand-side c, we construct the function κ for 2DIIGP with right-hand-side (f_1, f_2) where $f_1 + f_2 = c$ as $\kappa(x, y) = \zeta((x + y)(mod 1))$.*

An example of Construction 31 is presented in Figure 4 where the function ζ that was used to generate κ is a three-slope facet introduced in Gomory and Johnson [12]. The edges of κ are represented in Figure 5. Note in particular that κ has no vertex and that the edges of κ are parallel. It can easily be seen that the

intersection of these edges with the x-axis corresponds exactly to the points where ζ is non-differentiable. Also the number of edges of κ is equal to the number of points where ζ is non-differentiable.

Construction 31 has a natural interpretation. First, the defining constraints of 2DIIGP are added. The resulting constraint defines an instance of 1DIIGP that is a relaxation of the initial 2DIIGP. Then, a valid inequality is created for the relaxation. Clearly this inequality is valid for the initial 2DIIGP. We will show in Theorem 34 that if ζ is facet-defining for 1DIIGP then κ is facet-defining for 2DIIGP. Finally, we note that some functions that can be derived through Construction 31 are given in Johnson [14]. However, they are not proven to be facet-defining for 2DIIGP and they are obtained through a more involved procedure.

Although it is clear that the inequalities generated by Construction 31 are valid, it is not obvious that they are subadditive over I^2. We prove next in Proposition 32 that inequalities obtained from Construction 31 are indeed subadditive.
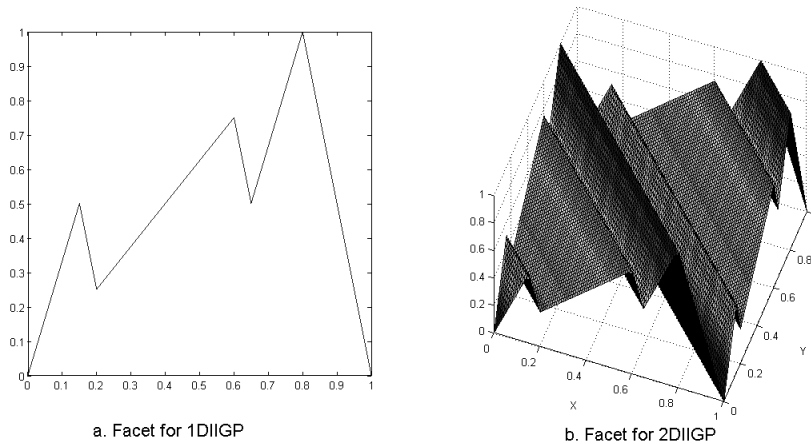

a. Facet for 1DIIGP

b. Facet for 2DIIGP

Figure 4: Construction 31 applied to a three-slope facet of 1DIIGP.

Proposition 32 κ *is subadditive iff ζ is subadditive.*

Proof: The direct implication is straightforward. To prove the reverse implication, assume that ζ is subadditive. We will use Proposition 10 to prove that κ is subadditive. It is easily seen that κ has no vertices. Therefore, we place a supplemental vertex on each of the edges of κ along the x-axis. We do not add complementary vertices since they are not required in Proposition 10. Since the edges of κ are parallel, it follows from Proposition 10 that it is sufficient to show that (14) and (15) hold to prove that κ is subadditive. Finally since all the vertices of κ are supplemental vertices that are located on the x-axis, (14) and (15) are satisfied because ζ is subadditive. □

Proposition 33 *The function κ is minimal iff ζ is minimal.*

Proof: We first prove the reverse implication. We have $\kappa(x,y) + \kappa(f_1 - x, f_2 - y) = \zeta((x + y)(mod1)) + \zeta((f_1 + f_2 - x - y)(mod1)) = \zeta((x + y)(mod1)) + \zeta((c - (x + y))(mod1)) = 1$. We now prove the direct implication. For any $x \in [0, 1)$, we have that $\kappa(x, 0) + \kappa(f_1 - x, f_2) = 1$. Since by construction $\kappa(x, 0) = \zeta(x)$ and $\kappa(f_1 - x, f_2) = \zeta(c - x)$, we obtain that $\zeta(x) + \zeta(c - x) = 1$. □ The proof of the next theorem requires us to identify specific points of I^2. We introduce these points next and illustrate them in Figure 5. Note that points described with the same letter are the same points in I^2. First let the line between $c1$ and $d1$ represent the first edge up to which κ has the same gradient, i.e., κ has the same gradient at all points in the interior of the triangle formed by $\{o1, c1, d1\}$. Let $a1$, $b1$ be the midpoints of $[c1, o1]$ and $[o1, d1]$. Observe that κ has no edge between $a1$ and $b1$ or $a2$ and $b2$. Note also that there may
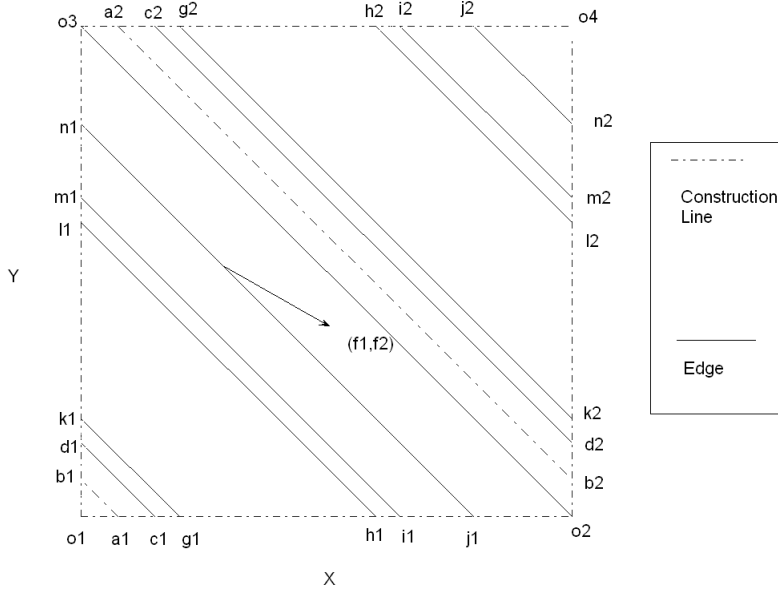
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Figure 5: Edges of κ

be more edges than shown in Figure 5 in the regions $\{o3, d1, c1, o2\}$ and $\{o4, c2, d2, o4\}$. However, we do not need these edges in the proof of Theorem 34.

Theorem 34 *κ is facet-defining for 2DIIGP iff ζ is facet-defining for 1DIIGP.*

Proof: Suppose first that κ is facet-defining for 2DIIGP. Assume by contradiction that ζ is not facet-defining. Then from Proposition 18 we know that there exists a valid subadditive and minimal inequality $\zeta^* \neq \zeta$, such that $E(\zeta^*) \supsetneq E(\zeta)$. Define $\kappa^*(x, y) = \zeta^*((x + y)(mod 1))$. Since $\zeta^* \neq \zeta$, we have that $\kappa^* \neq \kappa$. Now consider any equality satisfied by κ, $\kappa(x_1, y_1) + \kappa(x_2, y_2) = \kappa(x_1 + x_2, y_1 + y_2)$. By construction of κ, we have that $\zeta(x_1+y_1)+\zeta(x_2+y_2) = \zeta(x_1+x_2+y_1+y_2)$. Therefore $\zeta^*(x_1+y_1)+\zeta^*(x_2+y_2) = \zeta^*(x_1+x_2+y_1+y_2)$. But it follows from the construction of κ^* that $\kappa^*(x_1, y_1)+\kappa^*(x_2, y_2) = \kappa^*(x_1+y_1, x_2+y_2)$. Thus κ^* satisfies all the equalities of κ. Moreover, since $E(\zeta^*) \supsetneq E(\zeta)$, there exists $\overline{u}_1, \overline{u}_2 \in I^1$ such that $\zeta^*(\overline{u}_1)+\zeta^*(\overline{u}_2) = \zeta^*(\overline{u}_1+\overline{u}_2)$ and $\zeta(\overline{u}_1) + \zeta(\overline{u}_2) > \zeta(\overline{u}_1 + \overline{u}_2)$. Then by construction of κ and κ^* we have that $\kappa^*(\overline{u}_1, 0) + \kappa^*(\overline{u}_2, 0) = \kappa^*(\overline{u}_1 + \overline{u}_2, 0)$ and $\kappa(\overline{u}_1, 0) + \kappa(\overline{u}_2, 0) > \kappa(\overline{u}_1 + \overline{u}_2, 0)$. Thus $E(\kappa^*) \supsetneq E(\kappa)$ which is contradiction since κ is facet-defining.

Now suppose ζ is facet-defining for 1DIIGP. Assume by contradiction that κ is not facet-defining for 2DIIGP. From Proposition 18 we know that there is a valid subadditive and minimal function $\kappa^* \neq \kappa$ such that $E(\kappa^*) \supsetneq E(\kappa)$. Let U be the closed triangle formed by $\{o1, a1, b1\}$. Let V be the closed quadrilateral formed by $\{o2, o3, a2, b2\}$. It is easy to see that $U + V$ is the union of the closed triangle $\{o1, c1, d1\}$ and the closed quadrilateral $\{o2, o3, c2, d2\}$. Also, for $u \in U$ and $v \in V$, we have $\kappa(u) + \kappa(v) = \kappa(u + v)$. Since κ is linear over U, $\kappa(\sum_{i \in S} u_i) = \sum_{i \in S} \kappa(u_i)$, whenever $u_i \in U$ and S is a finite set. Since κ^* satisfies the same equalities as κ, κ^* must satisfy these equalities. It can be verified that $\exists U', V' \in \mathbb{R}^2$ such that $\mathbb{P}(U') = U$ and $\mathbb{P}(V') = V$ and U' and V' satisfy the conditions of Proposition 24. Thus it follows from Proposition 24 that κ^* is linear over U, V, $U + V$ and has the same gradient over these sets.

Let d be the direction $(-1, 1)$. The value of κ^* for the line segment joining $o2$ and $o3$ must be zero as κ^* is linear over $U + V$ and $\kappa^*(o2) = \kappa^*(o3) = 0$. This proves that $\kappa^*(x, y) = 0$ whenever $(x, y) = nd(mod 1)$, where $n \in \mathbb{R}$.

Observe now that $\forall x \in I^2$, $\kappa(x) + \kappa(nd) = \kappa(x + nd)$, $\forall n \in \mathbb{R}$. Since, κ^* also satisfies these equalities,

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we have $\kappa^*(x) + \kappa^*(nd) = \kappa^*(x + nd)$. Since, $\kappa^*(nd) = 0$, we have that $\kappa^*(x) = \kappa^*(x + nd)$. Thus, κ^* is completely determined by its value along the x-axis.

Let $\theta : I^2 \to I^1$ be the function $\theta(x, y) = (x + y)(mod1)$ and let $\zeta^* : I^1 \to \mathbb{R}_+$ be the function $\zeta^*(x) = \kappa^*(x, 0)(mod1)$. It can be easily verified that ζ^* is subadditive and minimal for 1DIIGP. Now consider any equality that ζ satisfies, $\zeta(x_1) + \zeta(x_2) = \zeta(x_1 + x_2)$. By construction of κ, $\kappa(u_1, u_2) + \kappa(v_1, v_2) = \kappa(u_1 + v_1, u_2 + v_2)$ whenever $(u_1 + u_2)(mod1) = x_1$ and $(v_1 + v_2)(mod1) = x_2$. Then because κ^* satisfies all the equalities of κ, we conclude that $\kappa^*(u_1, u_2) + \kappa^*(v_1, v_2) = \kappa^*(u_1 + v_1, u_2 + v_2)$. Since $\kappa^*(u) = \kappa^*(u + nd)$, we can obtain that $\zeta^*(x_1) + \zeta^*(x_2) = \zeta^*(x_1 + x_2)$. Thus ζ^* satisfies all the equalities of ζ. Moreover, $E(\kappa^*) \supsetneq E(\kappa)$. Thus there exists $\overline{u}, \overline{v} \in I^2$ such that $\kappa^*(\overline{u}) + \kappa^*(\overline{v}) = \kappa^*(\overline{u} + \overline{v})$ and $\kappa(\overline{u}) + \kappa(\overline{v}) > \kappa(\overline{u} + \overline{v})$. Then $\zeta^*(\theta(\overline{u})) + \zeta^*(\theta(\overline{v})) = \zeta^*(\theta(\overline{u} + \overline{v}))$ and $\zeta(\theta(\overline{u})) + \zeta(\theta(\overline{v})) > \zeta(\theta(\overline{u} + \overline{v}))$. This is a contradiction to the fact that ζ is facet-defining. $\square$ An interesting observation here is that although ζ and κ are assumed to be piecewise linear functions, the proof of Theorem 34 uses only the fact that κ is piecewise linear in the region $\{o1, a1, b1\}$ and $\{o2, o3, a2, b2\}$. In other words, if the functions ζ and κ are piecewise linear around the origin, the result still holds.

Next we prove that any facet of 1DIIGP is also a facet of 2DIIGP. Formally we define the function π as follows.

Construction 35 *Given ζ a piecewise linear and continuous valid inequality for 1DIIGP with right-hand-side c, we construct the function π for 2DIIGP with right-hand-side (f_1, f_2) where $f_1 = c$ as $\pi(x, y) = \zeta(x)$.*

We prove that Construction 35 yields facets of $2DIIGP$. The following proposition is easily verified.

Proposition 36 *π is subadditive and minimal if and only if ζ is subadditive and minimal respectively.* $\square$

The proof of the next theorem uses some specific points from I^2 that are illustrated in Figure 6. Note that points described with the same letter are the same points in I^2. First let the line between $b1$ and $b2$ represent the first edge up to which π has the same gradient, i.e., π has the same gradient at all points in the quadrilateral formed by $\{o1, b1, b2, o3\}$. Let $a1$, $a2$ be the midpoints of $[o1, b1]$ and $[o3, b2]$. Observe that π has no edge between $a1$ and $a2$. Note that there may be more edges than shown in Figure 5 in the regions $\{b1, o2, o4, b2\}$. However, we do not need these edges in the proof of Theorem 37.
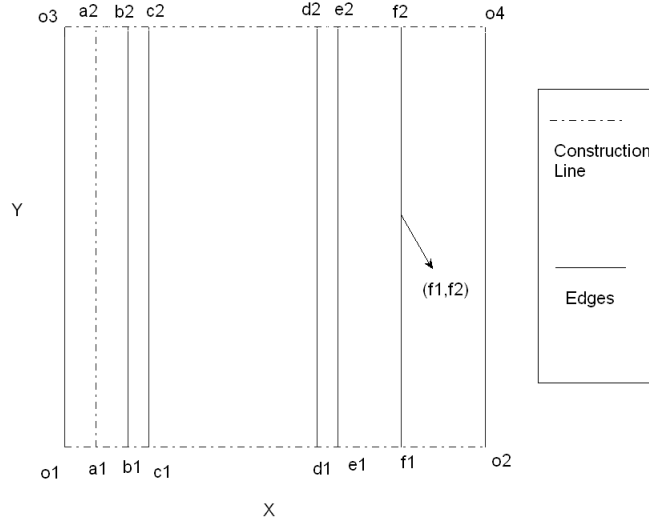

Figure 6: Edges of π

Theorem 37 π *is facet-defining for 2DIIGP iff* ζ *is facet-defining for 1DIIGP.*

Proof: First assume that π is facet-defining for 2DIIGP. Assume by contradiction that ζ is not facet-defining. Then there exists a valid subadditive and minimal function $\zeta^* \neq \zeta$ such that $E(\zeta^*) \supsetneq E(\zeta)$. Define $\pi^*(x,y) = \zeta^*(x)$. Since $\zeta^* \neq \zeta$, we have that $\pi^* \neq \pi$. Now, consider any equality satisfied by π, $\pi(x_1, y_1) + \pi(x_2, y_2) = \pi(x_1 + x_2, y_1 + y_2)$. By construction of π, we have that $\zeta(x_1) + \zeta(x_2) = \zeta(x_1 + x_2)$. Therefore $\zeta^*(x_1) + \zeta^*(x_2) = \zeta^*(x_1 + x_2)$. But it follows from the construction of π^* that $\pi^*(x_1, y_1) + \pi^*(x_2, y_2) = \pi^*(x_1 + y_1, x_2 + y_2)$. Thus π^* satisfies all the equalities of the π. Moreover since $E(\zeta^*) \supsetneq E(\zeta)$ there exists $\overline{u}_1, \overline{u}_2 \in I^1$ such that $\zeta^*(\overline{u}_1) + \zeta^*(\overline{u}_2) = \zeta^*(\overline{u}_1 + \overline{u}_2)$ and $\zeta(\overline{u}_1) + \zeta(\overline{u}_2) > \zeta(\overline{u}_1 + \overline{u}_2)$. Then by construction of π and π^*, we have that $\pi^*(\overline{u}_1, 0) + \pi^*(\overline{u}_2, 0) = \pi^*(\overline{u}_1 + \overline{u}_2, 0)$ and $\pi(\overline{u}_1, 0) + \pi(\overline{u}_2, 0) > \pi(\overline{u}_1 + \overline{u}_2, 0)$. We conclude from Proposition 18 that this is a contradiction to the fact that π is facet-defining.

Now suppose ζ is facet-defining for 1DIIGP. Assume by contradiction that π is not facet-defining. Then there exists a valid subadditive and minimal function $\pi^* \neq \pi$ such that $E(\pi^*) \supsetneq E(\pi)$. Let U be the closed quadrilateral formed by $\{o1, a1, a2, o3\}$. It is easy to see that $U + U$ is the closed quadrilateral $\{o1, b1, b2, o3\}$. Also, for $u \in U$ and $v \in U$, we have $\pi(u) + \pi(v) = \pi(u+v)$. Since π is linear over U, $\pi(\sum_{i \in S} u_i) = \sum_{i \in S} \pi(u_i)$, where $u_i \in U$ and S is a finite set. Since π^* satisfies the same equalities as π, π^* must satisfy these equalities. It can be verified that $\exists U' \in \mathbb{R}^2$ such that $\mathbb{P}(U') = U$ and U' satisfies the conditions of Proposition 24. Thus it follows from Proposition 24 that π^* is linear over $U + U$.

Let d be the direction $(0, 1)$. The value of π^* for the line segment joining $o1$ and $o3$ must be zero as π^* is linear over $U + U$ and $\pi^*(o1) = \pi^*(o3) = 0$. This proves that $\pi^*(x, y) = 0$ whenever $(x, y) = nd (mod 1)$, where $n \in \mathbb{R}$.

Observe that $\forall x \in I^2$, $\pi(x) + \pi(nd) = \pi(x + nd)$, where $n \in \mathbb{R}$. Since, π^* also satisfies these equalities, we have $\pi^*(x) + \pi^*(nd) = \pi^*(x + nd)$. Since, $\pi^*(nd) = 0$, we have that $\pi^*(x) = \pi^*(x + nd)$. Thus, π^* is completely determined by its value along the x-axis.

Let $\theta : I^2 \to I^1$ be the function $\theta(x, y) = x$ and let $\zeta^* : I^1 \to \mathbb{R}_+$ be the function $\zeta^*(x) = \pi^*(x, 0)$. It is easy to verify that ζ^* is subadditive and minimal for 1DIIGP. Now consider any equality that ζ satisfies, $\zeta(x_1) + \zeta(x_2) = \zeta(x_1 + x_2)$. By construction of π, $\pi(x_1, y_1) + \pi(x_2, y_2) = \pi(x_1 + x_2, y_1 + y_2)$ for any y_1 and y_2. Then since π^* satisfies all the equalities of π, $\pi^*(x_1, y_1) + \pi^*(x_2, y_2) = \pi^*(x_1 + x_2, y_1 + y_2)$. Since $\pi^*(x) = \pi^*(x + nd)$ we obtain that $\zeta^*(x_1) + \zeta^*(x_2) = \zeta^*(x_1 + x_2)$. Thus ζ^* satisfies all the equalities of ζ. Further because $E(\pi^*) \supsetneq E(\pi)$, there exists $\overline{u}_1, \overline{u}_2 \in I^2$ such that $\pi^*(\overline{u}_1) + \pi^*(\overline{u}_2) = \pi^*(\overline{u}_1 + \overline{u}_2)$ and $\pi(\overline{u}_1) + \pi(\overline{u}_2) > \pi(\overline{u}_1 + \overline{u}_2)$. Therefore $\zeta^*(\theta(\overline{u}_1)) + \zeta^*(\theta(\overline{u}_2)) = \zeta^*(\theta(\overline{u}_1 + \overline{u}_2))$ and $\zeta(\theta(\overline{u}_1)) + \zeta(\theta(\overline{u}_2)) > \zeta(\theta(\overline{u}_1 + \overline{u}_2))$. This is a contradiction since ζ is facet-defining for 2DIIGP. $\square$

Note that we can generalize Construction 31 using the automorphism and homomorphism results of Proposition 28 and Theorem 30. This corresponds to first multiplying the constraints of 2DIIGP by integers before adding them up and generating a one-dimensional cut. This yields the following extended construction.

Construction 38 *Given* ζ *a piecewise linear and continuous valid inequality for 1DIIGP with right-hand-side* c, *we construct the function* τ *for 2DIIGP with right-hand-side* (f_1, f_2) *where* $\lambda_1 f_1 + \lambda_2 f_2 = c$ *as* $\tau(x, y) = \zeta((\lambda_1 x + \lambda_2 y)(mod 1))$, *and* $\lambda_1, \lambda_2 \in \mathbb{Z}$ *and are not both zero.*

Any function derived using Construction 38 is the composition of Construction 31 or 35, with the rotation and reflection automorphisms and the multiplicative homomorphism λ. Propositions 25, 26, and Theorems 28, 30 thus give a proof that facet-defining inequalities for 2DIIGP can be obtained from facet-defining inequalities for 1DIIGP using Construction 38. This result is presented next.

Theorem 39 τ *is facet-defining for 2DIIGP iff* ζ *is facet-defining for 1DIIGP.* $\square$

It was computationally shown that aggregation of constraints before applying Gomory mixed integer cuts (GMICs) produces strong cuts; see Marchand and Wolsey [17]. The facet-defining result of Theorem 39 gives a theoretical foundation for the success of such aggregation procedures as it states that aggregating inequalities after weighing them with integers and then generating facets of 1DIIGP produces facets for 2DIIGP.

Now we prove that all two-gradient continuous and piecewise linear facets of 2DIIGP can be derived using Construction 38. To prove this theorem, we first show the following two preliminary results.

Lemma 40 *Assume that:*

1. $\nu_1, \nu_2 \in \mathbb{Z}$,

2. $gcd(\nu_1, \nu_2) = 1$,

3. $d_1\nu_1 + d_2\nu_2 = 0$, *where* $d_1, d_2 \in \mathbb{R}$.

Then $(\nu_1 x_1 + \nu_2 x_2)(mod1) \equiv 0$ *iff* $x_1 \equiv nd_1(mod1)$ *and* $x_2 \equiv nd_2(mod1)$ *where* $n \in \mathbb{Z}$.

Proof: First assume that $x_1 \equiv nd_1(mod1)$ and $x_2 \equiv nd_2(mod1)$. We have $x_1 = \alpha + nd_1$ and $x_2 = \beta + nd_2$, where $\alpha, \beta \in \mathbb{Z}$. Therefore $(\nu_1 x_1 + \nu_2 x_2) = \nu_1 \alpha + \nu_2 \beta \equiv 0(mod1)$.

Now assume $(\nu_1 x_1 + \nu_2 x_2)(mod1) = 0$, i.e., $(\nu_1 x_1 + \nu_2 x_2) = c$ where $c \in \mathbb{Z}$. All the solutions of this equation are of the form $(\tilde{x}_1 + nd_1, \tilde{x}_2 + nd_2)$ since $d_1\nu_1 + d_2\nu_2 = 0$. Also, because $gcd(\nu_1, \nu_2) = 1$, there exists integers $\tilde{x}_1, \tilde{x}_2$ such that $\nu_1\tilde{x}_1 + \nu_2\tilde{x}_2 = c$. Thus, all the solutions of $(\nu_1 x_1 + \nu_2 x_2) = c$ are of the form $(\tilde{x}_1 + nd_1, \tilde{x}_2 + nd_2)$ where $\tilde{x}_1, \tilde{x}_2 \in \mathbb{Z}$. Therefore $\tilde{x}_1 + nd_1 \equiv nd_1(mod1)$ and $\tilde{x}_2 + nd_2 \equiv nd_2(mod1)$ as $\tilde{x}_1, \tilde{x}_2 \in \mathbb{Z}$. □

Lemma 41 *Let* $\phi : I^2 \to \mathbb{R}$ *be a subadditive, valid and continuous function. If* $\exists\, d \equiv (d_1, d_2) \in \mathbb{R}^2$ *such that* $\phi(u) = \phi(u + nd)\ \forall u \in I^2$ *and* $\forall n \in \mathbb{R}$, *then* $\exists\, \nu_1, \nu_2 \in \mathbb{Z}$ *such that* $\nu_1 d_1 + \nu_2 d_2 = 0$ *where at least one of* ν_1 *and* ν_2 *is not zero.*

Proof: First note that if either d_1 or d_2 is zero, then the result is obvious. Therefore, assume $d_1 \neq 0$ and $d_2 \neq 0$.

If $\frac{d_1}{d_2}$ is rational, then the result is obvious. We now prove that $\frac{d_1}{d_2}$ cannot be irrational. Assume by contradiction that $\frac{d_1}{d_2}$ is irrational. By subadditivity and validity there exists a point of the form $(x, 0)$ or $(0, y)$ such that $\phi(x, 0) \neq 0$ or $\phi(0, y) \neq 0$. Without loss of generality, we assume that $\exists\, (x, 0) \in I^2$ such that $\phi(x, 0) = \delta > 0$. It follows from the lemma's assumption on (d_1, d_2) that

$$\phi((x + k\frac{d_1}{d_2})(mod1), 0) = \phi(x, 0), \tag{34}$$

where $k \in \mathbb{Z}$. Since $\frac{d_1}{d_2}$ is irrational, there is an infinite number of points of this form. Now we show that this implies that the function ϕ is discontinuous, yielding the required contradiction. To prove that ϕ is discontinuous, we show that there exists points arbitrarily close to the origin at which the value of ϕ is δ. Choose any $\epsilon > 0$. Since $\frac{d_1}{d_2}$ is irrational $\exists k \in \mathbb{Z}$ such that $0 < k\frac{d_1}{d_2}(mod1) < \epsilon$. Next observe that if $\frac{x}{kd_1/d_2(mod1)} = p \in \mathbb{Z}$, we have $(x - pk\frac{d_1}{d_2}(mod1), 0 - pk) \equiv (0, 0)$ and therefore $\delta = \phi(x, 0) = \phi(x - pk\frac{d_1}{d_2}(mod1), 0 - pk) = \phi(0, 0)$. This is a contradiction since $\phi(0, 0) = 0$. Therefore $\frac{x}{kd_1/d_2(mod1)} \notin \mathbb{Z}$. Thus, $0 < x - \left\lfloor \frac{x}{kd_1/d_2(mod1)} \right\rfloor kd_1/d_2(mod1) < \epsilon$. Finally, by (34), $\phi(x - \left\lfloor \frac{x}{kd_1/d_2(mod1)} \right\rfloor kd_1/d_2(mod1), 0) = \phi((x, 0)) = \delta$. Thus, there exists a point within a distance of ϵ from the origin where the value of the function ϕ is δ. □

Theorem 42 *Any continuous piecewise linear two-gradient facet of 2DIIGP can be derived from a facet of 1DIIGP using Construction 38.*

Proof: Let ϕ be a valid two-gradient continuous and piecewise linear function over I^2. Thus $\phi(x) = \alpha_i x + \beta_i\ \forall x \in P_i$, where $\cup_i P_i = I^2$. Since ϕ has two gradients, the set of polytopes P_i can be partitioned into two groups g_1 and g_2, such that $\alpha_i = \alpha_j$ if $i, j \in g_1$ or $i, j \in g_2$ and $\alpha_i \neq \alpha_j$ if $i \in g_1$ and $j \in g_2$. Let $S_1 = \cup_{i \in g_1} P_i$ and $S_2 = \cup_{i \in g_2} P_i$. Observe that $\mathbb{B}(\phi) = S_1 \cap S_2$. The proof is divided into four parts. First, we prove that two-gradient functions have no vertices and that all their edges are parallel. Second, we prove that any two-gradient function must have an edge passing through the origin. Third, we show that the function value is zero along this edge and consequently show that the function is of the form given in Construction 38. Fourth, we use Theorem 39 to show that the corresponding one-dimensional function ζ is facet-defining

for the 1DIIGP.

Assume by contradiction that there exist two edges l_1, l_2 along the unit vectors $u_1 \equiv (u_1^x, u_1^y)$, $u_2 \equiv (u_2^x, u_2^y)$ respectively that are not parallel. Let the slope of ϕ along l_i be σ_i. We have

$$\alpha_i^x u_1^x + \alpha_i^y u_1^y = \sigma_1, \tag{35}$$
$$\alpha_i^x u_2^x + \alpha_i^y u_2^y = \sigma_2, \tag{36}$$

for $i = 1, 2$. Since l_1 and l_2 are not parallel, the above systems of equations have unique and identical solutions, i.e., $\alpha_1 = \alpha_2$. This is a contradiction since the gradients in S_1 and S_2 are not equal. Therefore we conclude that all the edges of ϕ are parallel. As all the edges of ϕ are parallel, ϕ has no vertices. Let $d \equiv (d^x, d^y)$ be the direction of all the edges of ϕ.

Now assume by contradiction that the origin o does not belong to an edge of ϕ. Without loss of generality we may assume that o belongs to the interior of S_1. Since $\phi(o) = 0$, we must have the slope of ϕ in S_1 is zero in all directions (i.e., $\phi(x) = \beta_i \forall x \in P_i \subset S_1$). This is because if there is a direction $\tilde{d}$ with positive slope, then all points sufficiently close to o along the direction $-\tilde{d}$ in S_1 have a negative value of ϕ, which is not possible. Next we show that the slope along all directions for ϕ in S_2 is also zero which will provide the desired contradiction. Observe first that the slope of ϕ along d is zero. Thus, the slope of ϕ in S_2 along d is zero. Consider a point $a \in S_1 \cap S_2$. There exists a direction d_2 which is not parallel to d for which there exists $n', n > 0$, $n' < n$ such that $a + nd_2 = a$ and $a + n'd_2 \in \text{interior}(S_2)$. Since $\phi(a) = \phi(a + nd_2)$ and ϕ is continuous, piecewise linear and the slope of ϕ in S_1 along d_2 is zero, the slope along d_2 for S_2 is zero. Since d_2 is not parallel to d, this implies that the gradient of ϕ in S_2 is zero. This proves that the gradient of ϕ is identical for S_1 and S_2, which is the required contradiction. Hence o belongs to $S_1 \cap S_2 = \mathbb{B}(\phi)$.

Now consider the edge to which o belongs. By contradiction assume that the slope along this edge is nonzero for ϕ. Then there exists x' such that $\phi(x') < 0$, which is a contradiction. Thus, the slope of the function along the edge passing through o is 0. Since, $o \in S_1 \cap S_2$, the slope of ϕ in both S_1 and S_2 along the direction d is zero. Thus, $\forall x \in I^2$, $\phi(x) = \phi(x + nd)$, where $n \in \mathbb{R}$. Then by application of Lemma 41, $\exists \nu_1, \nu_2 \in \mathbb{Z}$, such that $\nu_1 d^x + \nu_2 d^y = 0$. Further we can always select ν_1 and ν_2 such that $gcd(\nu_1, \nu_2) = 1$. Then, by application of Lemma 40, we have that $\phi(x_1, y_1) = \phi(x_2, y_2)$ if $(\nu_1 x_1 + \nu_2 y_1)(mod 1) = (\nu_1 x_2 + \nu_2 y_2)(mod 1)$. Therefore, ϕ may be written as $\phi(x, y) = \zeta((\nu_1 x + \nu_2 y)(mod 1))$, where ζ is a two-slope continuous function over the one-dimensional group with, $\zeta(o) = 0$. It follows from Proposition 32 that ζ is subadditive. Furthermore, ζ is of the form given in Construction 38. Finally we deduce from Theorem 39 that ζ is facet-defining for 1DIIGP. $\square$

Next we use Theorem 42 to prove an extension of the Two-Slope Theorem of Gomory and Johnson [12] to 2DIIGP. In the Two-Slope Theorem, Gomory and Johnson prove that all piecewise linear subadditive continuous functions with exactly two slopes are facets of 1DIIGP. In the proof of Theorem 42 it was shown that if ϕ is a two-gradient piecewise linear and continuous function it must be of the form $\phi(x, y) = \zeta((\nu_1 x + \nu_2 y)(mod 1))$, where ζ has two slopes. This result together with Gomory and Johnson Two-Slope Theorem, implies the following corollary.

Corollary 43 *If $\phi : I^2 \to \mathbb{R}_+$ is a valid, continuous, minimal, piecewise linear function with exactly two gradients over I^2 then ϕ is facet-defining for 2DIIGP.* $\square$

7   A family of three-gradient facets.

Next we consider a family of functions ψ that have three gradients. These functions are nontrivial as they cannot be obtained from known families of facet-defining functions for the one-dimensional problem using Construction 38. Furthermore, they define facets for 2DIIGP. We first give a procedure to build these inequalities. This construction is illustrated in Figure 7.

Construction 44 *Let $f_1 \neq 0$ and $f_2 \neq 0$. We divide I^2 into five polytopes $R1, R2, R3, R4, R5$ as shown in Figure 7. Polytopes $R1, R2, R3, R4, R5$ are defined by the points*
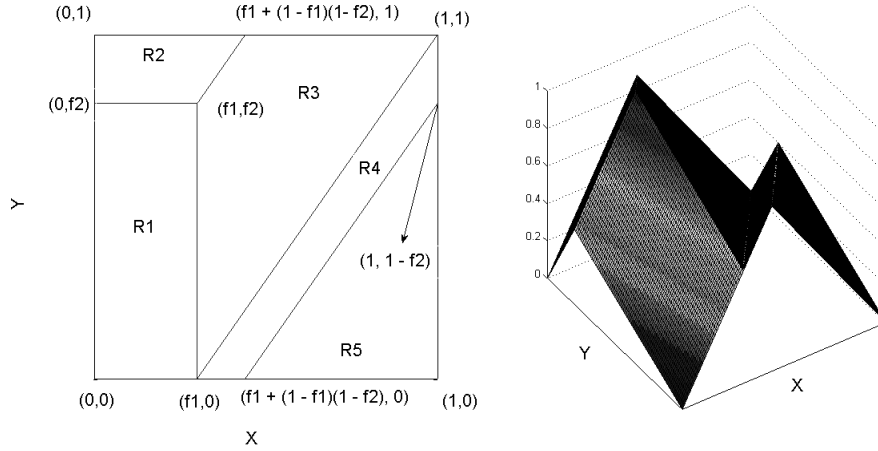
Figure 7: The three gradient facet-defining function with $f_1 = 0.3$, $f_2 = 0.8$

1. $(0,0), (f_1, 0)$, (f_1, f_2) and $(0, f_2)$,

2. $(0, f_2)$, (f_1, f_2), $(f_1 + (1 - f_1)(1 - f_2), 1)$ and $(0, 1)$,

3. $(f_1, 0), (1, 1), (f_1 + (1 - f_1)(1 - f_2), 1)$ and (f_1, f_2),

4. $(f_1, 0)$, $(f_1 + (1 - f_1)(1 - f_2), 0)$, $(1, 1 - f_2)$ and $(1, 1)$,

5. $(f_1 + (1 - f_1)(1 - f_2), 0)$, $(1, 0)$ and $(1, 1 - f_2)$

respectively. We construct ψ to be the only continuous piecewise linear function with $\psi(f_1, f_2) = 1$ and $\psi(0, 0) = 0$, whose gradients in $R2$ and $R4$ are equal and whose gradients in $R3$ and $R5$ are equal.

Observe from Construction 44 that we may write ψ as

$$\psi(u) = \begin{cases} \alpha_1^T u + \beta_1 & \forall u \in R1 \\ \alpha_2^T u + \beta_2 & \forall u \in R2 \\ \alpha_3^T u + \beta_3 & \forall u \in R3 \\ \alpha_2^T u + \beta_4 & \forall u \in R4 \\ \alpha_3^T u + \beta_5 & \forall u \in R5. \end{cases} \tag{37}$$

There are some relations between the vectors α_1, α_2 and α_3. First denote $\alpha_1 = (\sigma_1, \sigma_2)$. Since $R1$ and $R2$ share an edge parallel to the x-axis we conclude that $\alpha_2 = (\sigma_1, \sigma_3)$. Moreover, since $R1$ and $R3$ share an edge parallel to the $y-$axis we have that $\alpha_3 = (\sigma_4, \sigma_2)$. We next show by computing explicit values for σ_1, σ_2, σ_3 and σ_4 that there exists only one function that meets the requirements of Construction 44 for each value of (f_1, f_2).

The following proposition is easily verified.

Proposition 45 *The only possible values for σ_1, σ_2, σ_3 and σ_4 for Construction 44 are $\frac{1}{1 + f_1 - f_2}$, $\frac{1 - f_2}{f_2(1 + f_1 - f_2)}$, $-\frac{1}{1 + f_1 - f_2}$ and $-\frac{1 - f_2 + f_1 f_2}{f_2(1 - f_1)(1 + f_1 - f_2)}$ respectively.* $\square$

As a corollary to Proposition 45, we can derive that the scalars β_i are $\beta_1 = 0$, $\beta_2 = 1 - \sigma_1 f_1 - \sigma_3 f_2 = \frac{1}{1 + f_1 - f_2}$, $\beta_3 = (\sigma_1 - \sigma_4) f_1 = \frac{f_1}{f_2(1 - f_1)(1 + f_1 - f_2)}$, $\beta_4 = 0$, $\beta_5 = -\sigma_4 = \frac{1 - f_2 + f_1 f_2}{f_2(1 - f_1)(1 + f_1 - f_2)}$.

Note that the origin lies in each of the polytopes $R1, ..., R5$. Therefore we can also compute the function value at any given point by using the gradient of the function ψ in that polytope and the relative position
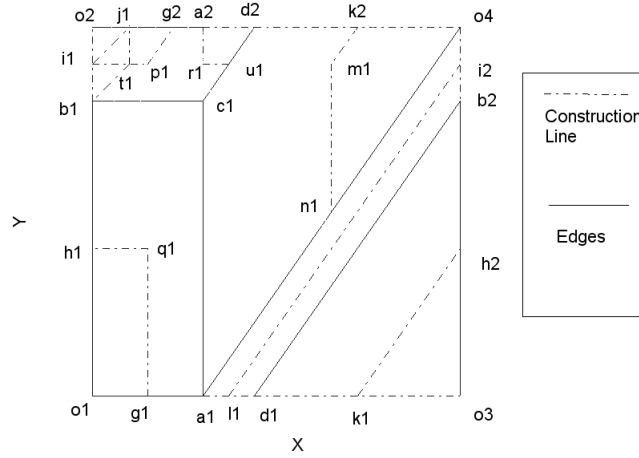
Figure 8: Edges of the three gradient function

of the point with respect to the origin. We will prove next that ψ is facet-defining for 2DIIGP. In order to do so, we first have to prove that ψ is symmetric and subadditive. To this end, we use the points described in Figure 8. Points described with the same letter are the same point in I^2. The points $o1, o2, o3, o4$ all correspond to the origin. The point $c1 \equiv (f_1, f_2)$ is the right-hand-side of 2DIIGP. The edges $[c1, d2, b1]$ and $[o4, a1]$ are parallel. Further we note that line segments $[b2, o4]$, $[a1, d1]$, $[d1, o3]$ are not edges since they belong to polytopes with the same gradient. So the edges of ψ are the line segments $[o1, a1]$, $[a1, c1]$, $[o1, b1]$, $[b1, c1]$, $[c1, d2, b1]$ and $[o4, a1]$. Although $d1$ belongs to the three polytopes $R2$, $R3$ and $R4$, it is not a vertex since ϕ has the same gradient over $R2$ and $R4$. Thus, the vertices of ψ are $o1, a1, b1, c1$.

Proposition 46 *The function ψ is symmetric.*

Proof: We consider the following four cases:

1. $x \le f_1$ and $y \le f_2$. Then $(f_1 - x, f_2 - y) \in R1$. Thus $\psi(x, y) + \psi(f_1 - x, f_2 - y) = \sigma_1(x) + \sigma_2(y) + \sigma_1(f_1 - x) + \sigma_2(f_2 - y) = \sigma_1 f_1 + \sigma_2 f_2 = 1$.

2. $x \le f_1$ and $y > f_2$. Then $(f_1 - x, f_2 - y + 1) \in R2$. Thus $\psi(x, y) + \psi(f_1 - x, f_2 - y + 1) = \sigma_1(x) - \sigma_3(1 - y) + \sigma_1(f_1 - x) - \sigma_3(1 - (f_2 - y + 1)) = \sigma_1 f_1 - \sigma_3(1 - f_2) = \frac{f_1}{1 + f_1 - f_2} + \frac{1 - f_2}{1 + f_1 - f_2} = 1$.

3. $x > f_1$ and $y \le f_2$. There are three subcases:

   (a) $(x, y) \in R3$. Then $(f_1 - x + 1, f_2 - y) \in R5$. Thus $\psi(x, y) + \psi(f_1 - x + 1, f_2 - y) = -\sigma_4(1 - x) - \sigma_2(1 - y) - \sigma_4(1 - (f_1 - x + 1)) + \sigma_2(f_2 - y) = \sigma_4(f_1 - 1) + \sigma_2(f_2 - 1) = \frac{1 - f_2 + f_1 f_2}{f_2(1 + f_1 - f_2)} - \frac{1 - 2f_2 + f_2^2}{f_2(1 + f_1 - f_2)} = 1$.

   (b) $(x, y) \in R4$. Then $(f_1 - x + 1, f_2 - y) \in R4$. Thus $\psi(x, y) + \psi(f_1 - x + 1, f_2 - y) = -\sigma_1(1 - x) - \sigma_3(1 - y) - \sigma_1(1 - (f_1 - x + 1)) - \sigma_3(1 - (f_2 - y)) = \sigma_1(f_1 - 1) + \sigma_3(f_2 - 2) = \frac{f_1 - 1}{1 + f_1 - f_2} - \frac{f_2 - 2}{1 + f_1 - f_2} = 1$.

   (c) $(x, y) \in R5$. This is similar to the first subcase since $(f_1 - x + 1, f_2 - y) \in R3$.

4. $x > f_1$ and $y > f_2$. There are three subcases:

   (a) $(x, y) \in R2$. Then $(f_1 - x + 1, f_2 - y + 1) \in R4$. Thus $\psi(x, y) + \psi(f_1 - x + 1, f_2 - y + 1) = \sigma_1 x - \sigma_3(1 - y) - \sigma_1(1 - (f_1 - x + 1)) - \sigma_3(1 - (f_2 - y + 1)) = \sigma_1 f_1 + \sigma_3(-1 + f_2) = \frac{f_1}{1 + f_1 - f_2} - \frac{-1 + f_2}{1 + f_1 - f_2} = 1$.

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(b) $(x, y) \in R3$. Then $(f_1 - x + 1, f_2 - y + 1) \in R3$. Thus $\psi(x, y) + \psi(f_1 - x + 1, f_2 - y + 1) = -\sigma_4(1 - x) - \sigma_2(1 - y) - \sigma_4(1 - (f_1 - x + 1)) - \sigma_2(1 - (f_2 - y + 1)) = \sigma_4(f_1 - 1) + \sigma_2(f_2 - 1) = \frac{1 - f_2 + f_1 f_2}{f_2(1 + f_1 - f_2)} - \frac{1 + f_2^2 - 2f_2}{f_2(1 + f_1 - f_2)} = 1$.

(c) $(x, y) \in R4$. This is similar to the first subcase since $(f_1 - x + 1, f_2 - y + 1) \in R2$.

$\square$

Proposition 47 *The function ψ is subadditive.*

Proof: The vertices of ψ are the points $o1$, $a1$, $b1$ and $c1$. Note also that there is at least one vertex on each edge. Therefore there is no need to add supplemental vertices. To prove that ψ is subadditive we know from Proposition 12 that it is sufficient to verify that

$$\psi(v_1) + \psi(v_2) \geq \psi(v_1 + v_2), \tag{38}$$

for all vertices $v_1, v_2 \in \mathbb{V}(\phi)$ and to verify that

$$\psi(e_1) + \psi(e_2) \geq \psi(v), \tag{39}$$

for all points e_1 and e_2 on edges of ψ such that $e_1 + e_2 = v$, $e_1, e_2 \in \mathbb{B}(\psi)$ and v is a vertex.

We first verify that (38) is satisfied. Observe that it is trivially satisfied if either v_1 or v_2 is the origin. Now consider the other cases.

1. We have $\psi(a1) + \psi(a1) = 2f_1\sigma_1$ and $\psi(2a1) = \psi(2f_1, 0)$. There are three subcases:

   (a) $f_1 \leq 0.5$ and $2f_1 \leq (1 - f_1)(1 - f_2) + f_1$. Then, $\psi(2a1) = \psi(2f_1, 0) = 2f_1\sigma_1 = 2\psi(f_1, 0) = 2\psi(a1)$.

   (b) $f_1 \leq 0.5$ and $2f_1 > (1 - f_1)(1 - f_2) + f_1$. Then, $\psi(2a1) = \psi(2f_1, 0) = (1 - f_2 + f_1 f_2)\sigma_1 + (2f_1 - (1 - f_2 + f_1 f_2))\sigma_4 = \frac{1 - f_2 - 2f_1 + 3f_1 f_2 - 2f_1^2 f_2}{(1 + f_1 - f_2)(f_2)(1 - f_1)}$. Also, $2\psi(a1) = 2f_1\sigma_1 = \frac{2f_1}{1 + f_1 - f2} = \frac{2f_1 f_2 - 2f_1^2 f_2}{(1 + f_1 - f_2)(f_2)(1 - f_1)}$. But $2f_1 f_2 - 2f_1^2 f_2 > 1 - f_2 - 2f_1 + 3f_1 f_2 - 2f_1^2 f_2$, since, $2f_1 > 1 - f_2 + f_1 f_2$. Therefore, $\psi(2a1) \geq 2\psi(a1)$.

   (c) $f_1 > 0.5$. Then $2f_1 - 1 < f_1$. Therefore, $\psi(2a1) = \psi(2f_1 - 1, 0) < \psi(f_1, 0) < 2\psi(f_1, 0) = 2\psi(a1)$ since $\sigma_1 > 0$.

2. $\psi(a1) + \psi(b1) = f_1\sigma_1 + f_2\sigma_2 = 1 = \psi(a1 + b1)$.

3. $\psi(a1) + \psi(c1) = f_1\sigma_1 + 1 > 1 \geq \psi(a1 + c1)$ as $\psi(x, y) \leq 1 \ \forall(x, y) \in I^2$.

4. $\psi(b1) + \psi(b1) = 2f_2\sigma_2$. There are two subcases:

   (a) $f_2 > 0.5$. Then $2f_2 - 1 < f_2$. Therefore $\psi(2b1) = \psi(0, 2f_2 - 1) < \psi(0, f_2) < 2\psi(b1)$.

   (b) $f_2 \leq 0.5$. Then $\psi(2b1) < \psi(b1) < 2\psi(b1)$. The first inequality is because $\sigma_3 < 0$.

5. $\psi(b1) + \psi(c1) = f_2\sigma_2 + 1 > 1 \geq \psi(b1 + c1)$ as $\psi(x, y) \leq 1 \ \forall(x, y) \in I^2$.

6. $\psi(c1) + \psi(c1) = 2 > \psi(c1 + c1)$ as $\psi(x, y) \leq 1 \ \forall(x, y) \in I^2$.

We next verify that (39) is satisfied. To simplify notation we denote $\Delta = (1 - f_1)(1 - f_2)$. It is easily seen that the edges of ψ are the line segments $[o1, b1]$, $[o1, a1]$, $[b1, c1]$, $[a1, c1]$, $[c1, d2, b2]$ and $[a1, o4]$. Further, by Proposition 10, we only need to verify (39) for edges that are not parallel to each other. Thus for each vertex, (except for the origin for which (39) is always satisfied), we examine the following combinations:

1. $e_1 \in [o1, a1]$, $e_2 \in [o1, b1]$.

   (a) $a1$: $e_1 = a1$, $e_2 = o1$. Thus subadditive.

   (b) $b1$: $e_1 = o1$, $e_2 = b1$. Thus subadditive.

(c) $c1$: $e_1 = a1$, $e_2 = b1$. Thus subadditive.

2. $e_1 \in [o1, a1]$, $e_2 \in [a1, c1]$.

   (a) $a1$: $e_1 = o1$, $e_2 = a1$. Thus subadditive.

   (b) $b1$: If $f_1 < 0.5$, there is no solution. If $f_1 \geq 0.5$, we have $e_1 = b1 - c1$, $e_2 = c1$. Then, $\psi(e_1) + \psi(c1) > 1 > \psi(b1)$.

   (c) $c1$: $e_1 = o1$, $e_2 = c1$. Thus, subadditive.

3. $e_1 \in [o1, a1]$, $e_2 \in [c1, d1, b2]$.

   (a) $a1$: There is no solution. Suppose by contradiction there is one, then $e_2 = d1$. In such a case, $e_1 = a1 - d1$ must belong to the line segment between $o1$ and $a1$. However, $a1 - d1 = -\Delta = -1 + f_1 + f_2 - f_1 f_2 \equiv f_1 + f_2 - f_1 f_2 > f_1$, a contradiction. Thus $e_1 \notin [o1, a1]$.

   (b) $b1$: $e_1 = o1$, $e_2 = b2$; $e_1 = (1 - f1, 0)$, $e_2 = c1$, if $f_1 \geq 1/2$. Thus subadditive.

   (c) $c1$: $e_1 = o1$, $e_2 = c1$; $e_1 = a1$, $e_2 = b2$. Thus subadditive.

4. $e_1 \in [o1, a1]$, $e_2 \in [a1, o4]$.

   (a) $a1$: $e_1 = o1$, $e_2 = a1$; $e_1 = a1$, $e_2 = o$. Thus subadditive.

   (b) $b1$: We must have $e_2 = ((1 - \Delta), f_2)$. Then $e1 = (\Delta, 0)$. If $e_1 \in [o1, a1]$, we have $\psi(e_1) + \psi(e_2) = \Delta \sigma_1 - \Delta \sigma_1 - (1 - f_2)\sigma_3 = f_2 \sigma_2 = \psi(b1)$.

   (c) $c1$: No solution. Suppose there is, then we must have $e2 = ((1 - \Delta), f_2)$. Then $e1 = (f_1 + \Delta, 0) \notin [o1, a1]$.

5. $e_1 \in [o1, b1]$, $e_1 \in [b1, c1]$.

   (a) $a1$: Then we must have $e_2 = c1$. If there is e_1, such that $e_1 + c1 = a1$, then $\psi(e_1) + \psi(e_2) > 1 > \psi(a1)$.

   (b) $b1$: $e_1 = o1$ and $e_2 = b1$. Thus subadditive.

   (c) $c1$: $e_1 = o1$ and $e_2 = c1$. Thus subadditive.

6. $e_1 \in [o1, b1]$, $e_2 \in [c1, d1, b2]$.

   (a) $a1$: Then we must have $e_2 = c1$. Thus, if there is any $e1$ such that $e_1 + e_2 = a1$, we obtain, $\psi(e_1) + \psi(e_2) \geq 1 > \psi(a1)$.

   (b) $b1$: Then $e_1 = o1$ and $e_2 = b2$. Thus subadditive.

   (c) $c1$: Then $e_1 = o1$ and $e_2 = c1$. Thus subadditive.

7. $e_1 \in [o1, b1]$, $e_2 \in [a1, o4]$.

   (a) $a1$: Then $e_1 = o1$ and $e_2 = a1$. Thus subadditive.

   (b) $b1$: Then $e_1 = b1$ and $e_2 = o4$. Thus subadditive.

   (c) $c1$: Then $e_1 = b1$ and $e_2 = a1$. Thus subadditive.

8. $e_1 \in [b1, c1]$, $e_2 \in [a1, c1]$.

   (a) $a1$: We must have $e_1 = b1$. Then $e_2 = (f_1, (1 - f_2))$. If feasible, $\psi(e_1) + \psi(e_2) = f_2 \sigma_2 + f_1 \sigma_1 + (1 - f_2)\sigma_2 = f_1 \sigma_1 + \sigma_2 > f_1 \sigma_1 = \psi(a1)$.

   (b) $b1$: We must have $e_2 = a1$. If $e1 = ((1 - f_1), f_2) \in [b1, c1]$, we have a solution. Then, we have $\psi(e_1) + \psi(e_2) = f_1 \sigma_1 + (1 - f_1)\sigma_1 + f_2 \sigma_2 > f_2 \sigma_2 = \psi(b1)$.

(c) $c1$: $e_1 = b1$ and $e_2 = a1$. Thus subadditive.

9. $e_1 \in [b1, c1]$, $e_2 \in [c1, d1, b2]$.

    (a) $a1$ There are two subcases:

        i. $f_2 < 0.5$: Clearly the y coordinate of e_2 must be $1 - f_2$. Therefore $e2$ must be $(1 - 2f_2 + 2f_1 f_2, 1 - f_2)$. Thus, $e_1 = (f_1 + 2f_2 - 2f_1 f_2, f_2)$. Then $\psi(e_1) + \psi(e_2) = (2 + f_1)\sigma_1$. On the other hand $\psi(a1) = f_1 \sigma_1$.

        ii. $f_2 \geq 0.5$: Clearly the x_2 coordinate of e_2 must be $1 - f_2$. Therefore $e2$ must be $((f_1 + 2\Delta), (1 - f_2))$. Also, we must have $(f_1 + 2\Delta) < 1$. Under these conditions, $e_1 = (1 - 2\Delta), f_2)$. Then $\psi(e_1) + \psi(e_2) = (1 - 2\Delta)\sigma_1 + f_2\sigma_2 - (1 - f_1 - 2\Delta)\sigma_1 - f_2\sigma_3 = f_1\sigma_1 + f_2\sigma_2 + f_2\sigma_1$. On the other hand $\psi(a1) = f_1\sigma_1$.

    (b) $b1$: Clearly, $e2 = d1$. If $(1 - f_1 - \Delta, f_2) \in [b1, c1]$, then $e1 = (1 - f_1 - \Delta, f_2)$. Since $\sigma_1 > 0$, we have that $\psi(e_1) + \psi(e_2) > \sigma_1 + f_2\sigma_2 \geq f_2\sigma_2 > \psi(b1)$.

    (c) $c1$: Clearly, $e_2 = d1$. Now, if $(1 - \Delta, f_2) \in [b1, c1]$, then we have $e1 = (1 - \Delta, f_2)$. Then $\psi(e_1) + \psi(e_2) = (1 - \Delta)\sigma_1 + f_2\sigma_2 + (f_1 + \Delta)\sigma_1 = 1 + \sigma_1 > 1 = \psi(c1)$.

10. $e_1 \in [b1, c1]$, $e_2 \in [a1, o4]$.

    (a) $a1$: Clearly, we must have that $e_2 = (f_1 + \Delta, 1 - f_2)$. Thus when feasible, $e_1 = (1 - \Delta, f_2)$. Then $\psi(e_1) + \psi(e_2) = (1 - \Delta)\sigma_1 + f_2\sigma_2 - (1 - f_1 - \Delta)\sigma_1 - f_2\sigma_3 = (1 + f_1)\sigma_1 > f_1\sigma_1 > \psi(a1)$.

    (b) $b1$: $e_1 = b_1$ and $e_2 = o4$. Thus subadditive. Also, we can have $e_1 = (1 - f_1, f_2)$ and $e_2 = (f_1, 0)$. Then $\psi(e_1) + \psi(e_2) = s_1 + s_2 f_2$. On the other hand $\psi(b1) = s_2 f_2$.

    (c) $c1$: $e_1 = c1$ and $e_2 = o4$. Or, $e_1 = b1$ and $e_2 = a1$. Thus subadditive.

11. $e_1 \in [a1, c1]$, $e_2 \in [c1, d2, b2]$.

    (a) $a1$: Clearly $e_2 = b2$. Then $e_1 = (f_1, 1 - f_2)$. But $\psi(e_1) > \psi(a1)$.

    (b) $b1$: Clearly the x_1 coordinate of e_2 must be $1 - f_1$. If $f_1 > 0.5$, there is no solution. There are a few subcases we have to consider:

        i. $1 - f_1 > f_1 + \Delta$ (Or $\frac{f_1}{1 - f_1} < f_2$). Then, we have $e_2 = (1 - f_1, \frac{1 - 2f_1 - \Delta}{1 - f_1})$ and $e_1 = (f_1, \frac{f_1}{1 - f_1})$. Therefore, $\psi(e_1) + \psi(e_2) = -(1 - (1 - f_1))\sigma_1 - (1 - \frac{1 - 2f_1 - \Delta}{1 - f_1})\sigma_3 + f_1\sigma_1 + \frac{f_1}{1 - f_1}\sigma_2 = \frac{\Delta f_2 + f_1}{(1 - f_1)f_2}\sigma_1$. On the other hand $\psi(b_1) = f_2\sigma_2 = \frac{\Delta f_2}{(1 - f_1)f_2}\sigma_1$. Thus subadditive.

        ii. $1 - f_1 \leq f_1 + \Delta$. Then we have $e_2 = (1 - f_1, \frac{1 - 2f_1}{1 - f_1} + f_2)$ and $e_1 = (f_1, \frac{f_1}{1 - f_1})$. Therefore, $\psi(e_1) + \psi(e_2) = (1 - f_1)\sigma_1 - (1 - \frac{1 - 2f_1}{1 - f_1} - f_2)\sigma_3 + f_1\sigma_1 + \frac{f_1}{1 - f_1}\sigma_2 = \frac{f_2 - f_2^2 + f_1 + f_1 f_2^2 - f_1 f_2}{(1 - f_1)f_2}\sigma_1$. Thus, $\psi(e_1) + \psi(e_2) - \psi(b1) = \frac{f_1}{f_2(1 - f_1)} > 0$.

    (c) $c1$: $e_1 = a_1$ and $e_2 = b_2$. Thus subadditive.

12. $e_1 \in [a1, c1]$, $e_2 \in [a1, o4]$. Thus subadditive.

    (a) $a1$: $e_1 = a_1$ and $e_2 = o4$.

    (b) $b1$: Then we have $e_2 = (1 - f_1, \frac{1 - 2f_1}{1 - f_1})$. There are two subcases:

        i. $f_1 > \Delta$. Then $e_1 = (f_1, \frac{f_1 - \Delta}{1 - f_1})$. Therefore, $\psi(e_1) + \psi(e_2) = -(1 - (1 - f_1))\sigma_1 - (1 - \frac{1 - 2f_1}{1 - f_1})\sigma_3 + f_1\sigma_1 + \frac{f_1 - \Delta}{1 - f_1}\sigma_2 = \frac{f_1 - \Delta + f_2\Delta}{(1 - f_1)f_2}\sigma_1$. On the other hand $\psi(b_1) = f_2\sigma_2 = \frac{\Delta f_2}{(1 - f_1)f_2}\sigma_1$. Thus subadditive.

        ii. $f_1 \leq \Delta$. Then $e_1 = (f_1, \frac{1 - \Delta}{1 - f_1})$. Therefore, $\psi(e_1) + \psi(e_2) = -(1 - (1 - f_1))\sigma_1 - (1 - \frac{1 - 2f_1}{1 - f_1})\sigma_3 + f_1\sigma_1 + \frac{1 - \Delta}{1 - f_1}\sigma_2 = \frac{f_1 + f_2\Delta}{(1 - f_1)f_2}\sigma_1$. Thus subadditive.

(c) $c1$: $e_1 = c1$ and $e_2 = o4$. Thus subadditive.

$\square$

We conclude from Theorem 14 that, ψ is minimal since ψ is subadditive and symmetric. Next we describe some of the other points represented in Figure 8 that will be helpful in proving that ψ is facet-defining for 2DIIGP. The point $g1$ is the center of $[o1, a1]$. The point $h1$ is the center of $[o1, b1]$. The line segment $[g1, q1]$ is parallel to the line segment $[a1, c1]$. Similarly, the line segment $[h1, q1]$ is parallel to $[b1, c1]$. The point $i1$ is the center of $[o2, b1]$. The distance between $j1$ and $o2$ is $\Delta/2$. The line segment $[i1, j1]$ is parallel to $[c1, d2]$. The point $v1$ is the center of $[o2, d2]$. The line segment $[p1, i1]$ is parallel to the line segment $[o2, d2]$. Similarly, $[v1, p1]$ is parallel to $[c1, d2]$. The point $k2$ is the center of $[d2, o4]$ and the point $l1$ is the center of the line segment $[a1, d1]$. Line segments $[a2, r1]$ and $[j1, t1]$ are parallel to $[o2, b1]$, while $[r1, u1]$ is parallel to $[a2, d2]$. Finally, $[t1, b1]$ is parallel to $[c1, d2]$.

Theorem 48 *ψ is a facet of 2DIIGP.*

Proof: We assume by contradiction that ψ is not facet-defining for 2DIIGP. From Theorem 17 we conclude that there exists a solution ψ^* to the system of equations $E(\psi)$ that is different from ψ. Let U be the closed rectangle $\{o1, g1, q1, h1\}$. Then $U + U$ is the closed rectangle $\{o1, a1, c1, b1\}$. It is easy to verify that all requirements of Proposition 24 are satisfied. Therefore we conclude that ψ^* is linear over U and $U + U \equiv R1$.

Next we let U be the closed quadrilateral $\{o2, g2, p1, i1\}$. Then $U + U$ is the closed quadrilateral $\{o2, d2, c1, b1\} \equiv R2$. It is easy to verify that all the requirements of Proposition 24 are satisfied. Therefore we conclude that ψ^* is linear over the quadrilateral $\{o2, d2, c1, b1\}$.

Then we consider U to be the closed triangle $\{o2, i1, j1\}$, and V to be the closed quadrilateral $\{a1, l1, i2, o4\}$. Then $U + V$ is the union of the closed quadrilaterals, $\{o2, b1, t1, j1\}$, $\{a2, r1, u1, d2\}$ and $\{a1, d1, b2, o4\}$. It is easy to verify that all the requirements of Proposition 24 are satisfied. Therefore we conclude that ψ^* is linear over U, V and $U + V$ and has the same the same gradient over these sets. This implies that ψ^* has the same gradient on the closed quadrilaterals $\{o2, b1, c1, d2\} \equiv R2$ and $\{a1, d1, b2, o4\} \equiv R4$.

Let U be the union of the closed triangle $\{k1, o3, h2\}$ and the closed quadrilateral $\{o4, k2, m1, n1\}$. Then $U + U$ is the union of the closed polytopes $\{d1, o3, b2\}$ and $\{d2c1a1o4\}$, i.e., the union of $R3$ and $R5$. All the requirements of Proposition 24 are satisfied. We therefore conclude that ψ^* is linear and has the same gradient over $R3$ and $R5$.

Denote the slope of ψ in $R1$ to be σ_1^* and σ_2^* along the x-axis and y-axis respectively. Since $\psi^*(f_1, f_2) = 1$, we must have

$$f_1\sigma_1^* + f_2\sigma_2^* = 1. \tag{40}$$

Now since the line segments $[o1, a1]$ and $[b1, c1]$ belong to both $R1$ and $R2$, the slope of ψ^* along the x-axis in $R2$ must be σ_1^*. Let the slope of ψ^* along the y-axis in $R2$ be σ_3^*. Then we must have

$$f_2\sigma_2^* + (1 - f_2)\sigma_3^* = 0. \tag{41}$$

Similar argument leads to the equations,

$$(f_1 + \Delta)\sigma_1^* + (1 - f_1 - \Delta)\sigma_4^* = 0 \tag{42}$$
$$(1 - f_1)\sigma_1^* + \sigma_3^* = (1 - f_1)\sigma_4^* + \sigma_2^* \tag{43}$$

where σ_4^* is the slope of ψ^* along the x-axis in $R3$ and $R5$. Since Proposition 45 states that (40) - (43) has a unique solution we conclude that $\psi^* = \psi$, a contradiction. $\square$

Next, we give an example in which the above three-gradient facet of the 2DIIGP yields a facet of the convex hull of the feasible region of an integer program.

Example 49 *Consider the system*

$$-4x + 3y \geq 10 \tag{44}$$
$$-2x + 4y \geq 16. \tag{45}$$
$$x, y \in \mathbb{Z}_+ \tag{46}$$

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After introducing nonnegative integer slack variables s_1 and s_2, and performing simplex iterations, we obtain the following tableau

$$x + 0.4s_1 - 0.3s_2 \quad = \quad 0.8 \tag{47}$$
$$y + 0.2s_1 - 0.4s_2 \quad = \quad 4.4. \tag{48}$$

We now obtain the GMIC from both the rows and an automorphism of the three-gradient cut, namely, $\psi \circ \varphi$ where $\varphi(x, y) = (1 - y, x)$. This cut is $0.5s_1 + 0.25s_2 \geq 1$ or equivalently $-x + y \geq 4$. This inequality is facet-defining. The two GMICs are

$$0.5s_1 + 0.8750s_2 \geq 1 \quad \Rightarrow \quad -3.75x + 5y \geq 20$$
$$0.5s_1 + 0.6667s_2 \geq 1 \quad \Rightarrow \quad -4x + 5y \geq 20.$$

It can be verified that the GMICs are not facet-defining for this example. □

8 Mixed Integer Extension.

In the previous sections, we described facet-defining inequalities for 2DIIGP. We now extend these results to the mixed integer case.

Definition 50 *A pair of functions $\phi : I^2 \to \mathbb{R}_+$ and $\mu_\phi : J^2 \to \mathbb{R}_+$ defines a valid inequality for 2DMIIGP, where ϕ satisfies 2, 3, 4 of Definition 3 along with the conditions*

1. *μ_ϕ is a nonnegative function,*

2. *$\sum_{u \in I^2} \phi(u)t(u) + \sum_{w \in J^2} \mu_\phi(w)s(w) \geq 1 \ \forall (t, s) \in$ 2DMIIGP.*

We next give a theorem of Johnson [14] that allows us to generate coefficients for continuous variables in MIIGP of any dimension. We present the result for the two dimensional case.

Theorem 51 *Assume $r \in I^2$ where $r \neq (0, 0)$, then the pair of functions $\phi : I^2 \to \mathbb{R}_+$ and $\mu_\phi : J^2 \to \mathbb{R}_+$ defines a minimal valid inequality for the convex hull of feasible points of 2DMIIGP if and only if*

1. *ϕ is subadditive*

2. *ϕ satisfies the symmetry condition*

3. *$\mu_\phi(u) = lim_{h \to 0^+} \frac{\phi(\mathbb{P}(hu))}{h}$.* □

Thus, if a function ϕ is subadditive and minimal for 2DIIGP, and its directional derivatives exist at $(0, 0)$, then it can be used to generate minimal inequalities for 2DMIIGP.

We now use the result of Theorem 51 to extend the facets τ given in Construction 38 into facets for 2DMIIGP by computing the coefficients μ_τ of the continuous variables. Note that τ was obtained from ζ, a facet for 1DIIGP. We therefore denote $\zeta^+ = \lim_{h \to 0^+} \frac{\zeta(h)}{h}$ and $\zeta^- = \lim_{h \to 0^+} \frac{\zeta(-h)}{h}$.

Proposition 52 *For τ given in Construction 38*

$$\mu_\tau(v_1, v_2) = \begin{cases} (\lambda_1 v_1 + \lambda_2 v_2)\zeta^+ & \text{if } \lambda_1 v_1 + \lambda_2 v_2 \geq 0 \\ (\lambda_1 v_1 + \lambda_2 v_2)\zeta^- & \text{if } \lambda_1 v_1 + \lambda_2 v_2 \leq 0. \end{cases} \tag{49}$$

Proof: The proof is identical for both the cases. Therefore we only prove the result for the first one. For sufficiently small $h > 0$, $\tau(\mathbb{P}(vh)) = \zeta(h(\lambda_1 v_1 + \lambda_2 v_2))$. Therefore by application of Theorem 51, $lim_{h \to 0^+} \frac{\tau(\mathbb{P}(vh))}{h} = (\lambda_1 v_1 + \lambda_2 v_2)\zeta^+$. □

In Proposition 52, we observe that the coefficients of continuous variables of μ_τ are obtained through aggregation. This result is interesting because one of the main computational difficulties with one-dimensional

group cuts is that the coefficients of their continuous variables can be weak. Andersen, Cornuéjols, and Li [2] show that significant computational improvements can be obtained by aggregating constraints so as to reduce the coefficients of continuous variables in the resulting cuts. The result of Proposition 52 presents a group-theoretical basis for this result.

In the following proposition, we show how ψ given in Construction 44 can be extended to yield a family of facets for 2DMIIGP as shown in the next proposition.

Proposition 53 *For ψ given in Construction 44*

$$
\mu_\psi(v_1, v_2) = \begin{cases}
v_1\sigma_1 + v_2\sigma_2 & \text{if } v_1 \geq 0, v_2 \geq 0 \\
v_1\sigma_1 + v_2\sigma_3 & \text{if } v_1 \geq 0, v_2 \leq 0 \\
v_1\sigma_4 + v_2\sigma_2 & \text{if } v_1 \leq 0, v_2 \geq 0 \\
v_1\sigma_1 + v_2\sigma_3 & \text{if } v_1 < 0, v_2 < 0 \text{ and } |v_1| \leq |v_2|(1-f_1) \\
v_1\sigma_4 + v_2\sigma_2 & \text{if } v_1 < 0, v_2 < 0 \text{ and } |v_1| \geq |v_2|(1-f_1)
\end{cases}
\tag{50}
$$

where σ_i represents the slopes obtained in Proposition 45.

Proof: The proof is similar for the five cases. Therefore we only prove the result for the first one. For sufficiently small h, $\psi(\mathbb{P}(vh)) = hv_1\sigma_1 + hv_2\sigma_2$. Thus $lim_{h\to 0^+} \frac{\psi(\mathbb{P}(vh))}{h} = v_1\sigma_1 + v_2\sigma_2$. $\qquad\square$

It appears from Proposition 53 that the three-gradient facet ψ yields diverse coefficients for continuous variables. In particular we will show that it can generate coefficients for continuous variables that are not dominated by GMICs from the individual rows. This is significant because it can be shown that GMIC generates the strongest possible coefficients for continuous variables among all facets of 1DMIIGP. Although this result seems to be well-known and was proven in Cornuéjols, Li, and Vandenbussche [4] for homomorphisms of the GMIC, we did not find it explicitly written for the general case. We therefore include its proof for completeness in Proposition 54.

Proposition 54 *Among all the facets of 1DMIIGP, the coefficients of continuous variables are strongest in GMIC.*

Proof: The cut coefficients for continuous variables in a GMIC for 1DMIIGP with right-hand-side $f(\neq 0)$ are,

$$
\mu_{GMIC}(u) = \begin{cases}
\frac{u}{f} & u \geq 0 \\
-\frac{u}{1-f} & u \leq 0.
\end{cases}
\tag{51}
$$

We prove the result for continuous variables with positive coefficients. The proof for the other case is similar. Assume by contradiction that (π, μ_π) is a facet of 1DMIIGP such that $\mu_\pi(1) = \sigma < \frac{1}{f} = \mu_{GMIC}(1)$. Using Theorem 51, we have that

$$
\mu_\pi(1) = \lim_{h\to 0^+} \frac{\pi(\mathbb{P}(h))}{h} = \sigma.
\tag{52}
$$

Choose $\epsilon > 0$ such that $\sigma + \epsilon < \frac{1}{f}$. Since the limit in (52) exists, $\exists\ \tilde{\delta} > 0$ such that $\frac{\pi(\delta)}{\delta} < \sigma + \epsilon\ \forall 0 < \delta < \tilde{\delta}$. Choose $\delta^* \in [0, \tilde{\delta})$ such that $\frac{f}{\delta^*} = n \in \mathbb{Z}$. Clearly, $\pi(\delta^*) < \delta^*(\sigma + \epsilon) < \frac{\delta^*}{f}$. Also, by subadditivity of π, we have $n\pi(\delta^*) \geq \pi(n\delta^*) = \pi(f) = 1$, or $\pi(\delta^*) \geq 1/n = \frac{\delta^*}{f}$, which is the desired contradiction. $\qquad\square$

We illustrate now the fact that coefficients of continuous variables from μ_ψ can be strictly better than the coefficients generated from the GMICs of each of the individual constraints. In Figure 9, we represent the region where μ_ψ dominates both the GMICs and the region where it is dominated by GMICs for 2DMIIGP with right-hand-side $(0.3, 0.8)$. We see that the region where the μ_ψ dominates the GMICs is large, which is very encouraging. This implies that, (ψ, μ_ψ) may be useful in generating stronger cuts for mixed integer programs.

Next we present an example of a simple MIP for which the three-gradient yields a facet-defining inequality. It is interesting to note that in this example the GMICs generated from the individual constraints do not yield facet-defining inequalities of the MIP.
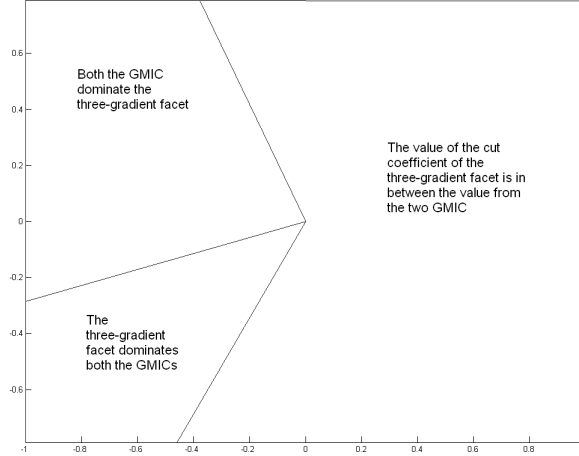
Figure 9: Comparison of coefficients of continuous variables in ψ and GMIC

Example 55 *Consider the system*

$$-0.3077x - 0.3077y \leq -2.2462 \tag{53}$$
$$1.3846x - 0.6154y \leq -0.0923 \tag{54}$$
$$x, y \in \mathbb{Z}_+ \tag{55}$$

We introduce the nonnegative continuous slack variables s_1 and s_2 and after performing simplex iterations, we obtain the tableau

$$x - s_1 + 0.5s_2 = 2.2 \tag{56}$$
$$y - 2.25s_1 - 0.5s_2 = 5.1. \tag{57}$$

The three-gradient cut is $1.1364s_1 + 0.9091s_2 \geq 1$, which becomes $-x + y \geq 4$. This inequality is facet-defining for the convex hull of points represented by (53), (54) and (55). The two GMICs are

$$1.25s_1 + 2.5s_2 \geq 1 \quad \Rightarrow \quad -1.6x + y \geq 2.1$$
$$2.5s_1 + 0.5556s_2 \geq 1 \quad \Rightarrow \quad y \geq 6.$$

It can be verified that the GMICs are not facet-defining in this example. □

9   Conclusion.

In this paper, we laid the foundation for the study of 2DMIIGP. We developed tools to analyze piecewise linear functions defined over I^2. We presented tools to prove that a function is subadditive and presented a two-dimensional version of Gomory and Johnson's Interval Lemma. We presented two different constructions that yield the first known families of facet-defining inequalities for 2DMIIGP. The first construction uses valid inequalities of 1DMIIGP and uses them as building block for creating inequalities for the two-dimensional group problem. This family of facets for 2DMIIGP gives a theoretical explanation for the success of constraints aggregation in practical implementations. The second construction we present has three gradients and yields facet-defining inequalities of 2DMIIGP that cannot be obtained directly from 1DMIIGP using the first construction. This construction is interesting because it yields coefficients for continuous variables that are not dominated by those of the GMICs obtained from the individual rows of 2DMIIGP. We believe that

obtaining stronger coefficients for continuous variables is the main improvement in the cuts that are based on higher-dimensional group relaxations.

This paper opens the way for future research on the group problem. Research directions include (1) the evaluation of the computational potential of the derived inequalities inside branch-and-cut frameworks, (2) the derivation of other families of facets for 2DMIIGP and finally (3) the extension of the tools and constructions proposed in this paper to $n-$dimensional group problems.

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