Design and Verify
— A New Scheme for Generating Cutting-Planes —

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Abstract

A cutting-plane procedure for integer programming (IP) problems usually involves invoking a black-box procedure (such as the Gomory-Chvátal (GC) procedure) to *compute* a cutting plane. In this paper, we describe an alternative paradigm of using the same cutting-plane black-box. This involves two steps. In the first step, we *design* an inequality $cx \leq d$, *independent* of the cutting plane black-box. In the second step, we *verify* that the designed inequality is a valid inequality by verifying that the set $P \cap \{x \in \mathbb{R}^n : cx \geq d+1\} \cap \mathbb{Z}^n$ is empty using cutting planes from the black-box. (Here P is the feasible region of the linear-programming relaxation of the IP.) We refer to the closure of all cutting planes that can be verified to be valid using a specific cutting plane black-box as the *verification closure* of the considered black box. This verification paradigm naturally leads to the question of how much extra strength one might hope to gain by having an oracle in place that provides us with potential cutting-planes and we are left with the task of verifying that its output is valid. We show that the verification closures are *almost admissible* operators, i.e., they are well-defined closures and share many properties that are common to known cutting planes closures. Moreover the verification closure of any admissible (i.e., 'reasonable') cutting planes scheme is at least as strong as the GC and N_0 closure, illustrating the power of verification. We then compare the strength of various regular closures (GC cuts, split cuts, and matrix cone cuts $\mathrm{N}_0, \mathrm{N}, \mathrm{N}_+$ cuts) with their verification versions and with each other. We show that the verification closure of an admissible cutting plane procedure is always stronger than the regular closure obtained from the admissible cutting plane procedure. We show that verification closure of both the GC procedure and the N_0 procedure are stronger than the split closure. A number of other similar results comparing regular and verification closures of different schemes are presented. We then provide lower bounds on the rank of verification cutting planes for known difficult infeasible $0/1$ instances, showing that while verification procedure is strong, it is not unrealistically so. Finally, we consider well-known and structured instances. We show that numerous families of inequalities with high GC, N_0, or N rank (such as clique inequalities) for the *stable set polytope* have a rank of 1 with respect to the verification closure of *any* admissible cutting-plane procedure. We show that for the *traveling salesman problem* the rank for the verification versions of $\mathrm{GC}, \mathrm{SC}, N_0, N$, and N_+ is in $\Theta(n)$ where n is the number of nodes. It is well-known that GC rank for *general polytopes in* $\mathbb{R}^2$ can be arbitrarily large. In contrast, we establish that the rank with respect to verification version of GC is 1.

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1 Introduction

Cutting planes are a crucial tool in solving Integer Programs (IPs). Often the only guiding principal (example: Kelley's Method [15]) used in deriving generic cutting-planes (like Gomory-Chvátal or split cuts) is that the incumbent fractional point must be separated. Therefore, cutting-planes are generated 'almost blindly', where we apply some black-box method to *constructively compute* valid cutting-planes and hope for the right set of cuts to appear that helps in proving optimality (or close significant portion of the gap). Now if we were somehow able to *deliberately design strong cutting-planes* that were tailor-made, for example to prove the optimality of good candidate solutions, then we could possibly speed up IP solver. This motivates a different paradigm to generate valid cutting-planes for integer programs: First we *design* a useful cutting-plane without considering its validity. Then, once the cutting-plane is designed, we *verify* whether it is valid.

For $n \in \mathbb{N}$, let $[n] = \{1, ..., n\}$ and for a polytope $P \subseteq \mathbb{R}^n$ denote its *integral hull* by $P_I := \operatorname{conv}(P \cap \mathbb{Z}^n)$. We now precisely describe these verification schemes (abbreviated as: $\mathbb{V}$-schemes). Let M be an *admissible* cutting-plane procedure (i.e., a valid and 'reasonable' cutting-plane system - we will formally define these) and let $\mathrm{M}(P)$ be the closure with respect to the family of cutting-planes obtained using M. For example, M could represent split cuts and then $\mathrm{M}(P)$ represents the split closure of P. Usually using cutting planes from a cutting plane procedure M, implies using valid inequalities for $\mathrm{M}(P)$ as cutting-planes. In the $\mathbb{V}$-scheme, we apply the following procedure: We design or guess the inequality $cx \leq d$ where $(c, d) \in \mathbb{Z}^n \times \mathbb{Z}$. To verify that this inequality is valid for P_I, we apply M to $P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}$ and check whether $\mathrm{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset$. If $\mathrm{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset$, then $cx \leq d$ is a valid inequality for P_I and we say it can be obtained using the $\mathbb{V}$-scheme of M.

At an abstract level, we might wonder how much we gain from having to only *verify* that a given inequality $cx \leq d$ is valid for P_I, rather than actually *computing* it. In fact at a first glance, it is not even clear that there would be any difference between computing and verifying. The strength of the verification scheme lies in the following inclusion that can be readily verified for admissible cutting plane procedures:

$$\mathrm{M}(P \cap \{cx \geq d+1\}) \subseteq \mathrm{M}(P) \cap \{cx \geq d+1\}. \tag{1}$$

The interpretation of this inclusion is that an additional inequality $cx \geq d+1$ appended to the description of P can provide us with crucial extra information when deriving new cutting-planes that is not available when considering P alone and then adding the additional inequality afterwards to the M-closure of P. In other words, (1) can potentially be a strict inclusion such that $\mathrm{M}(P \cap \{cx \geq d+1\}) = \emptyset$ while $\mathrm{M}(P) \cap \{cx \geq d+1\} \neq \emptyset$. This is equivalent to saying that we can *verify* the validity of $cx \leq d$, however we are not able to *compute* $cx \leq d$. To the best of our knowledge, the only paper discussing a related idea is [4], but theoretical and computational potential of this approach has not been further investigated.

The set obtained by intersecting all cutting-planes that can be verified to be valid using M will be called the $\mathbb{V}$-*closure of* M and denoted by $\partial\mathrm{M}(P)$. Formally,

$$\partial\mathrm{M}(P) := \bigcap_{\substack{(c,d) \in \mathbb{Z}^n \times \mathbb{Z} \\ \text{s.t. } \mathrm{M}(P \cap \{x \in \mathbb{R}^n \mid cx \geq d+1\}) = \emptyset}} \{x \in \mathbb{R}^n : cx \leq d\}. \tag{2}$$

Under mild conditions (1) implies $\partial\mathrm{M}(P) \subseteq \mathrm{M}(P)$ for all rational polytopes P. Since there exist inequalities that can be verified but not computed, this inclusion can be proper. We illustrate this in the next example.

Example 1. *Let* $\mathrm{SC}^i(P)$ *denote the* i*-th split closure of a polytope* P*. Also we denote* $\mathrm{SC}^1(P)$ *as* $\mathrm{SC}(P)$*. Consider the following family of polytopes [3] with* $n \in \mathbb{N}$*:*

$$A_n := \left\{ x \in [0,1]^n \mid \sum_{i \in I} x_i + \sum_{i \notin I} (1 - x_i) \geq \frac{1}{2} \quad \forall\, I \subseteq [n] \right\}. \tag{3}$$

Note that $(A_n)_I = \emptyset$ *and recall that it takes* n *rounds of split cuts to establish that* A_n *is infeasible [6]. For simplicity, consider the instance* $P := A_3$*. Then* $\mathrm{SC}^2(A_3) \neq \emptyset$ *and* $\mathrm{SC}^3(A_3) = \emptyset$*.*

We will show that the $\mathbb{V}$*-split closure of* A_3 *is the empty set, i.e.,* $\partial\mathrm{SC}(A_3) = \emptyset$*. We first design the inequality* $x_1 + x_2 + x_3 \geq 2$*. In order to show that the inequality* $x_1 + x_2 + x_3 \geq 2$ *is valid for* $\partial\mathrm{SC}(A_3)$ *we will establish that* $\mathrm{SC}(Q) = \emptyset$ *with* $Q := A_3 \cap \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1\}$*. It is easy to see that* $\max\{x_i \mid x \in Q\} < 1$ *for* $i \in [3]$ *and so we obtain that the split cuts* $x_i \leq 0$ *for* $i \in [3]$ *are valid for* $\mathrm{SC}(Q)$*. However,* $x_1 + x_2 + x_3 \geq \frac{1}{2}$ *is in the description of* Q*. Thus,* $\mathrm{SC}(Q) = \emptyset$*, and so* $x_1 + x_2 + x_3 \geq 2$ *can be obtained via the* $\mathbb{V}$*-split closure, i.e., it is valid for* $\partial\mathrm{SC}(A_3)$*. By symmetry, we also obtain that* $\partial\mathrm{SC}(A_3) \subseteq \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1\}$ *and so it follows that* $\partial\mathrm{SC}(A_3) = \emptyset$*.*

We note that rank of A_3 with respect to Gomory-Chvátal (GC) cuts [14, 2], Lift-and-project (LP) cuts [1], and Matrix cone cuts $(\mathrm{N}_0, \mathrm{N}, \mathrm{N}_+)$ [16] is also 3 but the $\mathbb{V}$-rank is 1 for any of these operators.

Outline and contribution. This paper conducts a systematic study of the strengths and weaknesses of the $\mathbb{V}$-schemes. In Section 2, we prove basic properties of the $\mathbb{V}$-closure. In order to present these results, we first describe general classes of reasonable cutting-planes in an abstract setting, using the class of so called *admissible cutting-plane procedures*, a machinery developed in [19]. We prove that $\partial\mathrm{M}$ is *almost admissible*, i.e. the $\mathbb{V}$-schemes satisfy many important properties that all known classes of admissible cutting-plane procedures including GC cuts, lift-and-project cuts, split cuts (SC), and $\mathrm{N}, \mathrm{N}_0, \mathrm{N}_+$ cuts satisfy. In Section 3, we show first that $\mathbb{V}$-schemes have natural inherent strength, i.e., even if M is an arbitrarily weak admissible cutting-plane procedure, $\partial\mathrm{M}(P)$ it is at least as strong as the GC and the N_0 closures. We then compare the strength of various regular closures (GC cuts, split cuts, and $\mathrm{N}_0, \mathrm{N}, \mathrm{N}_+$ cuts) with their $\mathbb{V}$-versions and with each other. For example, we show that $\partial\mathrm{GC}(P) \subseteq \mathrm{SC}(P)$ and $\partial\mathrm{N}_0(P) \subseteq \mathrm{SC}(P)$. The complete list of these results is illustrated in Figure 1. In Section 4, we study the rank of valid inequalities with respect to the $\mathbb{V}$-closures. Here we present upper and lower bounds on the $\mathbb{V}$-ranks of valid inequalities for a large class of 0/1 problems showing that the $\mathbb{V}$-closures are strong but not unrealistically so. In Section 5, we illustrate the strength of the $\mathbb{V}$-operation on specific structured problems. In particular, we show that facet-defining inequalities of monotone polytopes contained in $[0,1]^n$ have low rank with respect to any $\partial\mathrm{M}$ operator. We show that numerous families inequalities with high GC, N_0, or N rank [16] (such as clique inequalities) for the *stable set polytope* have a rank of 1 with respect to any $\partial\mathrm{M}$ with M being arbitrarily weak and admissible. We will also show that for the *traveling salesman problem* the rank for $\partial\mathrm{M}$ with $\mathrm{M} \in \{\mathrm{GC}, \mathrm{SC}, N_0, N, N_+\}$ is in $\Theta(n)$ where n is the number of nodes, i.e., the rank is $\Theta(\sqrt{\dim(P)})$ with P being the TSP-polytope. It is well-known that for the case of *general polytopes in* $\mathbb{R}^2$ the GC rank can be arbitrarily large. In contrast to this we will establish that the rank with respect to $\partial\mathrm{GC}$ is 1.

2 General properties of the $\mathbb{V}$-closure.

For the ease of presentation, we will only consider *rational* polytopes in the following definition, although they readily generalize to compact convex sets.

Definition 2.1 ([19])**.** *A cutting-plane procedure* M *defined for polytopes* $P := \{x \in [0,1]^n \mid Ax \le b\}$ *is* admissible *if the following holds:*

1. Validity*:* $P_I \subseteq \mathrm{M}(P) \subseteq P$.
2. Inclusion preservation*: If* $P \subseteq Q$*, then* $\mathrm{M}(P) \subseteq \mathrm{M}(Q)$ *for all polytopes* $P, Q \subseteq [0,1]^n$.
3. Homogeneity*:* $\mathrm{M}(F \cap P) = F \cap \mathrm{M}(P)$*, for all faces* F *of* $[0,1]^n$.
4. Single coordinate rounding*: If* $x_i \le \epsilon < 1$ *(or* $x_i \ge \epsilon > 0$*) is valid for* P*, then* $x_i \le 0$ *(or* $x_i \ge 1$*) is valid for* $\mathrm{M}(P)$.
5. Commuting with coordinate flips and duplications*:* $\tau_i(\mathrm{M}(P)) = \mathrm{M}(\tau_i(P))$*, where* τ_i *is either one of the following two operations: (i) Coordinate flip:* $\tau_i : [0,1]^n \to [0,1]^n$ *with* $(\tau_i(x))_i = (1 - x_i)$ *and* $(\tau_i(x))_j = x_j$ *for* $j \in [n] \setminus \{i\}$*; (ii) Coordinate Duplication:* $\tau_i : [0,1]^n \to [0,1]^{n+1}$ *with* $(\tau_i(x))_{n+1} = x_i$ *and* $(\tau_i(x))_j = x_j$ *for* $j \in [n]$.
6. Substitution independence*: Let* φ_F *be the projection onto the face* F *of* $[0,1]^n$*. Then* $\varphi_F(\mathrm{M}(P \cap F)) = \mathrm{M}(\varphi_F(P \cap F))$.
7. Short verification*: There exists a polynomial* p *such that for any inequality* $cx \le d$ *that is valid for* $\mathrm{M}(P)$ *there is a set* $I \subseteq [m]$ *with* $|I| \le p(n)$ *such that* $cx \le d$ *is valid for* $\mathrm{M}(\{x : a_i x \le b_i, i \in I\})$*. We call* $p(n)$ *the* verification degree *of* M.

If M *is defined for general rational polytopes* $P \subseteq \mathbb{R}^n$*, then we say* M *is admissible if (A.)* M *satisfies (1.)-(7.) when restricted to polytopes contained in* $[0,1]^n$ *and (B.) for general polytopes* $P \subseteq \mathbb{R}^n$*,* M *satisfies (1.), (2.), (7.) and Homogeneity is replaced by*

8. Strong Homogeneity: *If* $P \subseteq F^{\le} := \{x \in \mathbb{R}^n \mid ax \le b\}$ *and* $F = \{x \in \mathbb{R}^n \mid ax = b\}$ *where* $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}$*, then* $\mathrm{M}(F \cap P) = \mathrm{M}(P) \cap F$.

In the following, we assume that $\mathrm{M}(P)$ *is a closed convex set. If* M *satisfies all required properties for being admissible except (7.), then we say* M *is* almost admissible.

Requiring strong homogeneity in the general case leads to a slightly more restricted class than the requirement of homogeneity in the 0/1 case. We note here that almost all known classes of cutting-plane schemes such as GC cuts, lift-and-project cuts, split cuts, and N, N_0, N_+ are admissible (cf. [19] for more details). Observe that (1) in Section 1 follows from inclusion preservation.

Next we present a technical lemma (without proof due to lack of space) that we require for the main result of this section. We will use $\{\alpha x \le \beta\}$ as a shorthand for $\{x \in \mathbb{R}^n \mid \alpha x \le \beta\}$.

Lemma 2.2. *Let* Q *be a compact set contained in the interior of the set* $\{\beta x \le \zeta\}$ *with* $(\beta, \zeta) \in \mathbb{Z}^n \times \mathbb{Z}$ *and let* $(\alpha, \eta) \in \mathbb{Z}^n \times \mathbb{Z}$*. Then there exists a positive integer* τ *such that* Q *is strictly contained in the set* $\{(\alpha + \tau\beta)x \le \eta + \tau\zeta\}$.

We next show that ∂M satisfies almost all properties that we should except from a well-defined cutting-plane procedure. It can be verified that short verification (7.) also follows whenever $\partial\mathrm{M}(P)$ is a rational polyhedron. However, we do not need this property for the results in this paper.

Theorem 2.3. *Let* M *be an admissible cutting-plane procedure. Then* ∂M *is almost admissible. In particular,*

1. *For* $0/1$ *polytopes,* ∂M *satisfies properties (1.) to (6.).*
2. *If* M *is defined for general polytopes, then* ∂M *additionally satisfies property (8.).*

Proof. It is straightforward to verify (1.), (2.), and (4.) - (6.). The non-trivial part is property (8.) (or (3.) respectively). In fact it follows from the original operator M having this property. We will prove (8.); property (3.) in the case of $P \subseteq [0,1]^n$ follows *mutatis mutandis*.

First observe that $\partial \mathrm{M}(P \cap F) \subseteq \partial \mathrm{M}(P)$ and $\partial \mathrm{M}(P \cap F) \subseteq F$. Therefore, $\partial \mathrm{M}(P \cap F) \subseteq \partial \mathrm{M}(P) \cap F$. To verify $\partial \mathrm{M}(P \cap F) \supseteq \partial \mathrm{M}(P) \cap F$, we show that if $\hat{x} \notin \partial \mathrm{M}(P \cap F)$, then $\hat{x} \notin \partial \mathrm{M}(P) \cap F$. Observe first that if $\hat{x} \notin P \cap F$, then $\hat{x} \notin \partial \mathrm{M}(P) \cap F$. Therefore, we assume that $\hat{x} \in P \cap F$. Hence we need to prove that if $\hat{x} \notin \partial \mathrm{M}(P \cap F)$ and $\hat{x} \in P \cap F$, then $\hat{x} \notin \partial \mathrm{M}(P)$. Since $\hat{x} \notin \partial \mathrm{M}(P \cap F)$, there exists $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ such that $c\hat{x} > d$ and $\mathrm{M}(P \cap F \cap \{cx \geq d+1\}) = \emptyset$. By strong homogeneity of M, we obtain

$$\mathrm{M}(P \cap \{cx \geq d+1\}) \cap F = \emptyset. \tag{4}$$

Let $F^{\leq} = \{ax \leq b\}$ and $F = \{ax = b\}$ with $P \subseteq F^{\leq}$. Now observe that (4) is equivalent to saying that $\mathrm{M}(P \cap \{cx \geq d+1\})$ is contained in the interior of the set $\{ax \leq b\}$. Therefore by Lemma 2.2, there exists a $\tau \in \mathbb{Z}_+$ such that $\mathrm{M}(P \cap \{cx \geq d+1\})$ is contained in the interior of $\{(c+\tau a)x \leq d+1+\tau b\}$. Equivalently, $\mathrm{M}(P \cap \{cx \geq d+1\}) \cap \{(c+\tau a)x \geq d+1+\tau b\} = \emptyset$ which implies

$$\mathrm{M}(P \cap \{cx \geq d+1\}) \cap (P \cap \{(c+\tau a)x \geq d+1+\tau b\}) = \emptyset. \tag{5}$$

Since $P \subseteq F^{\leq}$, we obtain that

$$P \cap \{(c+\tau a)x \geq d+1+\tau b\} \subseteq P \cap \{cx \geq d+1\}. \tag{6}$$

Now using (6), (5) and inclusion preservation of M it follows $\mathrm{M}(P \cap \{(c+\tau a)x \geq d+1+\tau b\}) = \emptyset$. Thus $(c+\tau a)x \leq d+\tau b$ is a valid inequality for $\partial \mathrm{M}(P)$. Moreover note that since $\hat{x} \in P \cap F$, we have that $a\hat{x} = b$. Therefore, $(c+\tau a)\hat{x} = c\hat{x} + \tau b > d + \tau b$, where the last inequality follows from the fact that $c\hat{x} > d$. $\square$

3 Strength and comparisons of $\mathbb{V}$-closures.

In this section, we compare various regular closures and their $\mathbb{V}$-counterparts with each other. We first formally define possible relations between admissible closures and the notation we use.

Definition 3.1. *Let* L, M *be almost admissible. Then*

1. L refines M*, if for all polytopes P we have $\mathrm{L}(P) \subseteq \mathrm{M}(P)$. We write:* L $\subseteq$ M*. It is indicated by empty arrow heads in Figure 1.*
2. L strictly refines M*, if* L *refines* M *and there exists a polytope P such that $\mathrm{L}(P) \subsetneq \mathrm{M}(P)$. We write:* L $\subsetneq$ M*. It is indicated by a filled arrow heads in Figure 1.*
3. L is incompatible with M*, if there exist polytopes P, Q such that $\mathrm{M}(P) \not\subseteq \mathrm{L}(P)$ and $\mathrm{M}(Q) \not\subseteq \mathrm{L}(Q)$. We write:* L $\perp$ M*. It is indicated with an arrow with circle head and tail in Figure 1.*

In each of the above definitions, if either one of L *or* M *is defined only for polytopes $P \subseteq [0,1]^n$, then we confine the comparison to this class of polytopes.*

We establish the relations depicted in Figure 1 in the rest of the section.

3.1 Strength of ∂M for arbitrary admissible cutting-plane procedures M

In order to show that ∂M refines M, we require the following technical lemma. The proof will be included in the full version of the paper; see [8] for a similar result. We use the notation $\sigma_P(\cdot)$ to refer to the support function of a set P, i.e., $\sigma_P(c) = \sup\{cx \mid x \in P\}$.

Lemma 3.2. *Let $P, Q \subseteq \mathbb{R}^n$ be compact convex sets. If $\sigma_P(c) \leq \sigma_Q(c)$ for all $c \in \mathbb{Z}^n$, then $P \subseteq Q$.*

Theorem 3.3. *Let* M *be admissible. Then* $\partial\mathrm{M} \subseteq \mathrm{M}$.

Proof. Let P be a polytope. Since $\mathrm{M}(P) \subseteq P$ and $\partial\mathrm{M}(P) \subseteq P$, both $\mathrm{M}(P)$ and $\partial\mathrm{M}(P)$ are bounded. Moreover since $\mathrm{M}(P)$ is closed by definition, and $\partial\mathrm{M}(P)$ is defined as the intersection of halfspaces (thus closed sets), we obtain that $\mathrm{M}(P)$ and $\partial\mathrm{M}(P)$ are both compact convex sets. Thus, by Lemma 3.2, it is sufficient to compare the support vectors of $\mathrm{M}(P)$ and $\partial\mathrm{M}(P)$ consisting only of integer vectors. Let $\sigma_{\mathrm{M}(P)}(c) = d$ for $c \in \mathbb{Z}^n$. We verify that $\sigma_{\partial\mathrm{M}(P)}(c) \leq \lfloor d \rfloor$. Observe that, $\mathrm{M}(P \cap \{cx \geq \lfloor d \rfloor + 1\}) \subseteq \mathrm{M}(P) \cap \{cx \geq \lfloor d \rfloor + 1\}$, where the inclusion follows from inclusion preservation of M. However note that since $cx \leq d$ is a valid inequality for $\mathrm{M}(P)$, we obtain that $\mathrm{M}(P) \cap \{cx \geq \lfloor d \rfloor + 1\} = \emptyset$. Thus, $\mathrm{M}(P \cap \{cx \geq \lfloor d \rfloor + 1\}) = \emptyset$ and so $cx \leq \lfloor d \rfloor$ is a valid inequality for $\partial\mathrm{M}(P)$. Equivalently we have $\sigma_{\partial\mathrm{M}(P)}(c) \leq \lfloor d \rfloor$, completing the proof. □

Under some mild conditions it can be verified that for M being admissible, we always have $\partial\mathrm{M} \subsetneq \mathrm{M}$; we defer this discussion to the full version of the paper. Note that Example 1 is an illustration of this fact for the case of GC, SC, N, N_o, N_+. We next show that even if M is chosen arbitrarily, $\partial\mathrm{M}$ is at least as strong as the GC closure and the N_0 closure.

Theorem 3.4. *Let* M *be admissible. Then* $\partial\mathrm{M} \subseteq \mathrm{GC}$ *and* $\partial\mathrm{M} \subseteq \mathrm{N}_0$ (*the latter holding for polytopes* $P \subseteq [0,1]^n$).

Sketch of proof. The proof of $\partial\mathrm{M} \subseteq \mathrm{GC}$ is similar to the proof of Theorem 3.3 and we skip it here due to lack of space. Now let P be a polytope with $P \subseteq [0,1]^n$. For proving $\partial\mathrm{M}(P) \subseteq N_0(P)$, recall that $N_0 = \bigcap_{i\in[n]} P_i$ with $P_i := \mathrm{conv}\left((P \cap \{x_i = 0\}) \cup (P \cap \{x_i = 1\})\right)$. Therefore let $cx \leq d$ with $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ be valid for P_i with $i \in [n]$ arbitrary. In particular, $cx \leq d$ is valid for $P \cap \{x_i = l\}$ with $l \in \{0,1\}$. Thus we can conclude that $P \cap \{cx \geq d+1\} \cap \{x_i = l\} = \emptyset$ for $i \in \{0,1\}$. Therefore $x_i > 0$ and $x_i < 1$ are valid for $P \cap \{cx \geq d+1\}$ and so by Property 4 of Definition 2.1, $x_i \leq 0$ and $x_i \geq 1$ are valid $\mathrm{M}(P \cap \{cx \geq d+1\})$. We obtain $\mathrm{M}(P \cap \{cx \geq d+1\}) = \emptyset$ and thus $cx \leq d$ is valid for $\partial\mathrm{M}(P)$. □

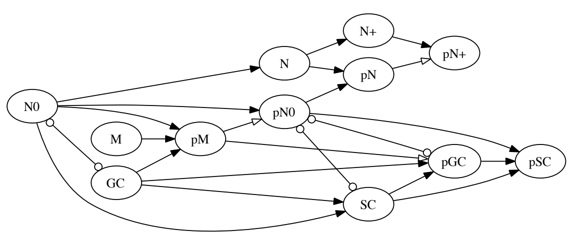

Figure 1: Direct and $\mathbb{V}$-operators and their relations. pL in the figure represents ∂L and M is an arbitrarily weak admissible system.

3.2 Comparing M and ∂M for M being GC, SC, N_0, N, or N_+

We will now compare various closures and their associated verification schemes. Due to space limitations many statements are without proof; they will be included in the full-length version of the paper. Our first result shows that the verification scheme of the Gomory-Chvátal procedure is at least as strong as split cuts.

Theorem 3.5. $\partial\mathrm{GC} \subseteq \mathrm{SC}$.

Proof. Consider $cx \leq d$ being valid for $P \cap \{\pi x \leq \pi_0\}$ and $P \cap \{\pi x \geq \pi_0 + 1\}$ with $c, \pi \in \mathbb{Z}^n$ and $d, \pi_0 \in \mathbb{Z}$. Clearly, $cx \leq d$ is valid for $SC(P)$ and it suffices to consider inequalities $cx \leq d$ with this property; all others are dominated by positive combinations of these. Therefore consider $P \cap \{cx \geq d+1\}$. By $cx \leq d$ being valid for the disjunction $\pi x \leq \pi_0$ and $\pi x \geq \pi_0 + 1$ we obtain that $P \cap \{cx \geq d+1\} \cap \{\pi x \leq \pi_0\} = \emptyset$ and $P \cap \{cx \geq d+1\} \cap \{\pi x \geq \pi_0 + 1\} = \emptyset$. This implies that $P \cap \{cx \geq d+1\} \subseteq \{\pi x > \pi_0\}$ and similarly $P \cap \{cx \geq d+1\} \subseteq \{\pi x < \pi_0 + 1\}$. We thus obtain that $\pi x \geq \pi_0 + 1$ and $\pi x \leq \pi_0$ are valid for $\mathrm{GC}(P \cap \{cx \geq d+1\})$. It follows $\mathrm{GC}(P \cap \{cx \geq d+1\}) = \emptyset$. Thus $cx \leq d$ is valid for $\partial\mathrm{GC}(P)$. □

Next we compare $\mathbb{V}$-schemes of two closures that are comparable. It is easy to see that switching to the verification schemes preserves inclusion:

Lemma 3.6. *Let* L, M *be admissible such that* $\mathrm{L} \subseteq \mathrm{M}$. *Then* $\partial\mathrm{L} \subseteq \partial\mathrm{M}$.

In order to prove strict refinement or incompatibility between $\mathbb{V}$-closures the following lemma is helpful. It establishes when strict refinement carries over to the $\mathbb{V}$-schemes.

Proposition 3.7. *Let* L, M *be admissible. If* $P \subseteq [0,1]^n$ *is a polytope with* $P_I = \emptyset$ *such that* $\mathrm{M}(P) = \emptyset$ *and* $\mathrm{L}(P) \neq \emptyset$, *then* $\partial\mathrm{L}$ *does not refine* $\partial\mathrm{M}$.

Sketch of proof. Let $G \subseteq [0,1]^n$ be a polytope. For $l \in \{0,1\}$, by $G_{x_{n+1}=l}$ we denote the polytope $S \subseteq [0,1]^{n+1}$ such that $S \cap \{x_{n+1} = l\} \cong G$ and S does not contain any other points. We can think of S arising from G by padding the coordinates of the vertices with l to the right. Consider the auxiliary polytope Q given as $Q := \mathrm{conv}\left(P_{x_{n+1}=1} \cup [0,1]^n_{x_{n+1}=0}\right)$. We next state a claim without proving it due to lack of space: The inequality $x_{n+1} \leq 0$ is valid for $\partial L(Q)$ if and only if $L(Q \cap \{x_{n+1} \geq 1\}) = \emptyset$ (and similarly for M). Observe that $Q \cap \{x_{n+1} \geq 1\} \cong P$ and by assumption, we have $\mathrm{M}(P) = \emptyset$ but $\mathrm{L}(P) \neq \emptyset$ and therefore $\partial\mathrm{M}(Q) \not\supseteq \partial\mathrm{L}(Q)$. □

Using the above proposition, we verify the various relationships depicted in Figure 1. In the following lemmata, polytopes are presented that help establish the strict inclusion via Proposition 3.7.

Lemma 3.8. $\partial N_0 \perp \partial\mathrm{GC}$ *via the two polytopes* $P_1 := \mathit{conv}\left([0,1]^3 \cap \{x_1 + x_2 + x_3 = 3/2\}\right) \subseteq [0,1]^3$ *and* $P_2 := \mathit{conv}\left(\{(\frac{1}{4}, \frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}, 1), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})\}\right) \subseteq [0,1]^3$.

The proof of the next lemma uses Proposition 3.7 and a result from [7].

Lemma 3.9. $\partial N_0 \perp \mathrm{SC}$ *via* $P_1 := A_3 \subseteq [0,1]^3$ *and* $P_2 := \mathit{conv}\left([0,1]^3 \cap \{x_1 + x_2 + x_3 = 3/2\}\right)$.

Using Proposition 3.7 and a modified version of an example presented in [16] we can show.

Lemma 3.10. $\partial N \subsetneq \partial N_0$.

The remaining relations in Figure 1 follow from Proposition 3.7 or Example 1.

4 Rank of valid inequalities with respect to $\mathbb{V}$-closures.

In this section, we establish several bounds on the rank of $\partial\mathrm{M}$ primarily for the case of polytopes $P \subseteq [0,1]^n$. Given a natural number k, we use the notation $\mathrm{M}^k(P)$ to be denote that k^{th} closure of P with respect to M.

Theorem 4.1 (Upper bound in $[0,1]^n$)**.** *Let* M *be admissible and* $P \subseteq [0,1]^n$ *be a polytope. Then* $rk_{\partial\mathrm{M}}(P) \leq n$.

Proof. As $\partial\mathrm{M} \subseteq \mathrm{N}_0$ and $\mathrm{rk}_{\mathrm{N}_0}(P) \leq n$ the result follows. $\square$

Note that in general the property of M being admissible, does not guarantee that the upper bound on rank is n. For example, the GC closure can have a rank strictly higher than n (cf. [12, 20]).

In quest for lower bounds on the rank of 0/1 polytopes, we note that among polytopes $P \subseteq [0,1]^n$ that have $P_I = \emptyset$, the polytope A_n has maximal rank (of n) for many admissible systems [18]. We will now establish that $\partial\mathrm{M}$ is not unrealistically strong by showing that it is subject to similar limitations. Recall that we do not require *short verification* (property (7.)) for $\partial(\mathrm{M})$ which is the basis for the lower bound in [19, Corollary 23] for admissible systems. We will show that the lower bound for $\partial\mathrm{M}$ is *inherited* from the original operator M. Let

$$F_n^k := \{x \in \{0, 1/2, 1\}^n \mid \text{ exactly } k \text{ entries equal to } 1/2\},$$

and let $A_n^k := \operatorname{conv}\left(F_n^k\right)$ be the convex hull of F_n^k; observe that $A_n^1 = A_n$ as defined in (3). With F being a face of $[0,1]^n$ let $I(F)$ denote the index set of those coordinate that are fixed by F.

Lemma 4.2. *Let* M *be admissible and let* $\ell \in \mathrm{N}$ *such that* $A_n^{k+\ell} \subseteq \mathrm{M}(A_n^k)$ *for all* $n, k \in \mathrm{N}$ *with* $k + \ell \leq n$*. If* $n \geq k + 2\ell + 1$*, then* $A_n^{k+2\ell+1} \subseteq \partial\mathrm{M}(A_n^k)$*.*

Proof. Let $P := A_n^k$ and let $cx \leq d$ with $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$ be valid for $\partial\mathrm{M}(P)$. Without loss of generality we assume $\mathrm{M}(P \cap \{cx \geq d+1\}) = \emptyset$, i.e., $cx \leq d$ is one of the defining inequalities. We claim that

$$A_{k+\ell}^k \cong A_n^k \cap F \not\subseteq P \cap \{cx \geq d+1\} \tag{7}$$

for all $(k+\ell)$-dimensional faces F of $[0,1]^n$. Assume by contradiction that $A_n^k \cap F \subseteq P \cap \{cx \geq d+1\}$. As $A_{k+\ell}^{k+\ell} \subseteq \mathrm{M}(A_{k+\ell}^k)$ by assumption we obtain $\emptyset \neq A_{k+\ell}^{k+\ell} \subseteq \mathrm{M}(A_{k+\ell}^k) \subseteq \mathrm{M}(P \cap \{cx \geq d+1\}$ which contradicts the validity of $cx \leq d$ over $\partial\mathrm{M}(P)$. Without loss of generality we can further assume that $c \geq 0$ and $c_i \geq c_j$ whenever $i \leq j$ by applying coordinate flips and permutations.

Next we claim that for all $(k+\ell)$-dimensional faces F of $[0,1]^n$ the point v^F defined as

$$v_i^F := \begin{cases} \in \{0,1\} \text{ according to } F, \text{ for all } i \in I(F) \\ 0, \text{ if } c_i \text{ is one of the } \ell \text{ largest coefficients of } c \text{ with } i \notin I(F) \\ 1/2, \text{ otherwise} \end{cases} \tag{8}$$

for $i \in [n]$ must not be contained in $P \cap \{cx \geq d+1\}$, i.e., $cv^F < d+1$ and so $cv^F \leq d + 1/2$. Note that $v^F \in P$ and observe that $v^F := \mathrm{argmin}_{x \in F_n^k \cap F} cx$. Therefore, if $v^F \in P \cap \{cx \geq d+1\}$, then $A_n^k \cap F \subseteq P \cap \{cx \geq d+1\}$ which in turn contradicts (7). This claim holds in particular for those faces F fixing coordinates to 1.

Finally, we claim that $A_n^{k+2\ell+1} \subseteq P \cap \{cx \leq d\}$. It suffices to show that $cv \leq d$ for all $v \in F_n^{k+2\ell+1}$ and we can confine ourselves to the worst case v given by

$$v_i := \begin{cases} 1, \text{ if } i \in [n - (k + 2\ell + 1)] \\ 1/2, \text{ otherwise.} \end{cases}$$

Observe that $cv \geq cw$ holds for all $w \in F_n^{k+2\ell+1}$. Let F be the $(k+\ell)$-dimensional face of $[0,1]^n$ obtained by fixing the first $n-(k+\ell)$ coordinates to 1. Then

$$\begin{aligned} cv &= \sum_{i=1}^{n-(k+2\ell+1)} c_i + \frac{1}{2} \sum_{i=n-(k+2\ell+1)+1}^{n} c_i \\ &\leq \sum_{i=1}^{n-(k+\ell)} c_i - \frac{1}{2} c_{n-(k+\ell)} + \sum_{i=n-(k+\ell)+1}^{n-k} 0 + \frac{1}{2} \sum_{i=(n-k)+1}^{n} c_i \\ &= cv^F - \frac{1}{2} c_{n-(k+\ell)} \leq d + \frac{1}{2} - \frac{1}{2} c_{n-(k+\ell)}. \end{aligned}$$

In case $c_{n-(k+\ell)} \geq 1$ it follows that $cv \leq d$. Therefore consider the case $c_{n-(k+\ell)} = 0$. Then we have that $c_i = 0$ for all $i \geq n-(k+\ell)$. In this case cv^F is integral and $cv^F < d+1$ implies $cv^F \leq d$. So $cv \leq cv^F \leq d$ follows which completes the proof. □

Using Lemma 4.2 we can establish the following lower bound for A_n.

Theorem 4.3 (Lower bound for A_n)**.** *Let* M *be admissible and let* $\ell \in \mathrm{N}$ *such that* $A_n^{k+\ell} \subseteq \mathrm{M}(A_n^k)$ *for all* $n, k \in \mathrm{N}$ *with* $k+\ell \leq n$. *If* $n \geq k+2\ell+1$, *then* $rk_{\partial \mathrm{M}}(A_n) \geq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor$.

Proof. We will show the $A_n^{1+k(2\ell+1)} \subseteq (\partial \mathrm{M})^k(A_n)$ as long as $n \geq k+2\ell+1$. The proof is by induction on k. Let $k=1$, then $A_n^{1+2\ell+1} \subseteq \partial\mathrm{M}(A_n^1) = \partial\mathrm{M}(A_n)$ by Lemma 4.2. Therefore consider $k > 1$. Now $(\partial\mathrm{M})^k(A_n) = \partial\mathrm{M}((\partial\mathrm{M})^{k-1}(A_n)) \supseteq \partial\mathrm{M}(A_n^{1+(k-1)(2\ell+1)}) \supseteq A_n^{1+k(2\ell+1)}$, where the first inclusion follows by induction and the second by Lemma 4.2 again. Thus $(\partial\mathrm{M})^k(A_n) \neq \emptyset$ as long as $1+k(2\ell+1) \leq n$, which is the case as long as $k \leq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor$ and we can thus conclude $\mathrm{rk}_{\partial\mathrm{M}}(A_n) \geq \left\lfloor \frac{n-1}{2\ell+1} \right\rfloor$. □

For $\mathrm{M} \in \{\mathrm{GC}, \mathrm{SC}, N_0, N, N_+\}$ we have that $\ell = 1$ [19] and therefore obtain the following corollary.

Corollary 4.4. *Let* $\mathrm{M} \in \{\mathrm{GC}, N_0, N, N_+, \mathrm{SC}\}$ *and* $n \in \mathrm{N}$ *with* $n \geq 4$. *Then* $rk_{\partial\mathrm{M}}(A_n) \geq \left\lfloor \frac{n-1}{3} \right\rfloor$.

We can also derive an upper bound on the rank of A_n which is a consequence of [19, Lemma 5].

Lemma 4.5 (Upper bound for A_n)**.** *Let* M *be admissible and* $n \in \mathrm{N}$. *Then* $rk_{\partial\mathrm{M}}(A_n) \leq n-2$.

5 $\mathbb{V}$-cuts for well-known and structured problems.

We will first establish a useful lemma which holds for any $\partial\mathrm{M}$ with M being admissible. The lemma is analogous to Lemma 1.5 in [16].

Lemma 5.1. *Let* M *be admissible and let* $P \subseteq [0,1]^n$ *be a polytope with* $(c,d) \in \mathbb{Z}_+^{n+1}$. *If* $cx \leq d$ *is valid for* $P \cap \{x_i = 1\}$ *for every* $i \in [n]$ *with* $c_i > 0$, *then* $cx \leq d$ *is valid for* $\partial\mathrm{M}(P)$.

Proof. Clearly, $cx \leq d$ is valid for P_I; if $x \in P \cap \mathbb{Z}^n$ non-zero, then there exists an $i \in [n]$ with $x_i = 1$, otherwise $cx \leq d$ is trivially satisfied. We claim that $cx \leq d$ is valid for $\partial\mathrm{M}$. Let $Q := P \cap \{cx \geq d+1\}$ and observe that $Q \cap \{x_i = 1\} = \emptyset$ for any $i \in [n]$ with $c_i > 0$. Therefore by coordinating rounding $\mathrm{M}(Q) \subseteq \bigcap_{i \in [n]: c_i > 0} \{x_i = 0\}$. By definition of Q we also have that $\mathrm{M}(Q) \subseteq \{cx \geq d+1\}$. Since $c \geq 0$ and $d \geq 0$ it follows that $\mathrm{M}(Q) = \emptyset$ and the claim follows. □

5.1 Monotone polytopes

The following theorem is a direct consequence of Lemma 5.1 and follows in a similar fashion as Lemma 2.7 in [5] or Lemma 2.14 in [16].

Theorem 5.2. *Let* M *be admissible. Further, let $P \subseteq [0,1]^n$ be a polytope and $(c,d) \in \mathbb{Z}_+^{n+1}$ such that $cx \leq d$ is valid for $P \cap F$ whenever F is an $(n-k)$-dimensional face of $[0,1]^n$ obtained by fixing coordinates to* 1*. Then $cx \leq d$ is valid $(\partial \mathrm{M})^k(P)$.*

We call a polytope $P \subseteq [0,1]^n$ *monotone* if $x \in P$, $y \in [0,1]^n$, and $y \leq x$ (coordinate-wise) implies $y \in P$. We can derive the following corollary from Theorem 5.2 which is the analog to Lemma 2.7 in [5].

Corollary 5.3. *Let* M *be admissible and let $P \subseteq [0,1]^n$ be a monotone polytope with $\max_{x \in P_I} ex = k$. Then $rk_{\partial \mathrm{M}}(P) \leq k+1$.*

Proof. Observe that since P is monotone, so is P_I and thus P_I possesses an inequality description $P = \{x \in [0,1]^n \mid Ax \leq b\}$ with $A \in \mathbb{Z}_+^{m \times n}$ and $b \in \mathbb{Z}_+^n$ for some $m \in \mathrm{N}$. Therefore it suffices to consider inequalities $cx \leq d$ valid for P_I with $c, d \geq 0$. As $\max_{x \in P_I} ex = k$ and P is monotone, we claim that $P \cap F = \emptyset$ whenever F is an $n-(k+1)$ dimensional face of $[0,1]^n$ obtained by fixing $k+1$ coordinates to 1. Assume by contradiction that $x \in P \cap F \neq \emptyset$. As $P \cap F$ is monotone, setting all fractional entries of x to 0 is contained in $P_I \cap F$ which is a contradiction to $\max_{x \in P_I} ex = k$. Therefore $cx \leq d$ is valid for all $P \cap F$ with F being an $n-(k+1)$ dimensional face of $[0,1]^n$ obtain by fixing $k+1$ coordinates to 1. The result follows now with Theorem 5.2. ☐

5.2 Stable set polytope

Given a graph $G := (V, E)$, the fractional stable set polytope of G is given by $\mathrm{FSTAB}(G) := \{x \in [0,1]^n \mid x_u + x_v \leq 1 \ \forall (u,v) \in E\}$.

Theorem 5.4. *Clique Inequalities, odd hole inequalities, odd anti-hole inequalities, and odd wheel inequalities are valid for $\partial \mathrm{M}(FSTAB(G))$ with* M *being an admissible operator.*

Sketch of proof. We illustrate the proof for the case of clique inequalities. The other cases are similar. Let $H(U,E)$ be an induced clique. Then the clique inequality is $\sum_{u \in U} x_u \leq 1$. Now for every vertex v in U fixing $x_v = 1$, the system $P^0 = \{x \in [0,1]^{|U|} \mid x_u + x_v \leq 1 \ \forall (u,v) \in E\}$, implies that $x_u = 0$ for $u \neq v$. Thus, the clique inequality is valid for $P^0 \cap \{x \mid x_v = 1\} \ \forall v \in V$. Now by Lemma 5.1 the result follows. ☐

5.3 The traveling salesman problem

So far we have seen that transitioning from a general cutting-plane procedure M to its $\mathbb{V}$-scheme ∂M can result in a significantly lower rank for valid inequalities, potentially making them accessible in a small number of rounds. However, there are also examples where this is not the case. We will now show that the rank of (the relaxation of) the traveling salesman polytope remains high, even when using $\mathbb{V}$-schemes of strong operators such as SC or N_+. For $n \in \mathbb{N}$, let $G = (V, E)$ be the complete graph on n vertices and $H_n \subseteq [0,1]^n$ be the polytope given by (see [5] for more details)

$$\begin{aligned} x(\delta(\{v\}) &= 2 && \forall\, v \in V \\ x(\delta(W)) &\geq 2 && \forall\, \emptyset \subsetneq W \subsetneq V \\ x_e &\in [0,1] && \forall e \in E. \end{aligned}$$

Note that the (ambient) dimension of H_n is $\Theta(n^2)$. We obtain the following statement which is the analog to [5, Theorem 4.1]. A similar result for the admissible systems M in general can be found in full-length version of [19].

Theorem 5.5. *Let* $\mathrm{M} \in \{\mathrm{GC}, N_0, N, N_+, \mathrm{SC}\}$*. For* $n \in \mathbb{N}$ *and* H_n *as defined above we have* $rk_{\partial\mathrm{M}}(H_n) \in \Theta(n)$*. In particular* $rk_{\partial\mathrm{M}}(H_n) \in \Theta(\sqrt{dim(P)})$*.*

Sketch of proof. As shown in [3] or [5, Theorem 4.1] H_n contains a copy of $A_{\lfloor n/8 \rfloor}$. The lower bound follows with Corollary 4.4 and the upper bound with Corollary 5.3 as shown in [5]. □

The same result can be shown to hold for the asymmetric TSP problem (see [3] and[5]).

5.4 General polytopes in $\mathbb{R}^2$

The GC rank of valid inequalities for polytopes in $\mathbb{R}^2$ can be arbitrarily high; see example in [17]. However, ∂GC is significantly stronger and all valid inequalities for polytopes in $\mathbb{R}^2$ have a ∂GC rank of 1.

Theorem 5.6. *Let* P *be a polytope in* $\mathbb{R}^2$*. Then* $\partial\mathrm{GC}(P) = P_I$*.*

Sketch of proof. The proof is divided into various cases based on the dimension of P_I. Due to space limitations we only present the proof for the case when $\dim(P_I) = 2$. We will illustrate that every facet-defining inequality can be obtained using the ∂GC operator. In this case, every facet-defining inequality $cx \leq d$ satisfies at least two integer points belonging to P_I. Let $Q := P \cap \{x \in \mathbb{R}^2 \mid cx \geq d\}$. Then observe that: (i) Q is a lattice-free polytope; (ii) exactly one side of Q contains multiple integer points. This is the side of Q given by the inequality $cx \geq d$. Other sides of Q contain no integer point. Let T be a maximal lattice-free convex set containing Q. By (ii), $cx \geq d$ defines a face of T that contains two or more integer points. Therefore T is a type 1 or type 2 maximal lattice-free triangle; see [11]. Since T is a triangle of type 1 or type 2, it is contained in two sets of the form $\{\pi^1_0 \leq \pi^1 x \leq \pi^1_0 + 1\}$ and $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$ where $\pi^1, \pi^2 \in \mathbb{Z}^2$ and $\pi^1_0, \pi^2_0 \in \mathbb{Z}$; see [9]. Moreover, $\pi^1 = c$ and $\pi^1_0 = d$. Therefore $Q \cap \{cx \geq d+1\} \subseteq T \cap \{cx \geq d+1\} \subseteq \{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$. Moreover, since the integer points belonging to the boundary of Q satisfy the condition $cx = d$, we obtain that integer points that satisfy $cx \geq d + 1$ and lie on the boundary of the set $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$ do not belong to Q. Now by using convexity of Q and the location of integer points in $P \cap \{cx = d\}$, we can verify that $Q \cap \{cx \geq d + 1\}$ lies in the interior of the set $\{\pi^2_0 \leq \pi^2 x \leq \pi^2_0 + 1\}$. Therefore $\mathrm{GC}(Q \cap \{cx \geq d + 1\}) = \emptyset$. However, since $Q \cap \{cx \geq d + 1\} = P \cap \{cx \geq d + 1\}$, we can obtain the facet-defining inequality $cx \leq d$ using the ∂GC operator applied to P. □

6 Concluding remarks

In this paper, we consider a new paradigm for generating cutting-planes. Rather than *computing* a cutting-plane we suppose that the cutting-plane is given, either by a *deliberate construction* or guessed in some other way and then we *verify* its validity using a regular cutting-plane procedure. We have shown that cutting-planes obtained via the verification scheme can be very strong, significantly exceeding the capabilities of the regular cutting-plane procedure. This superior strength is illustrated, for example, in Theorem 3.3, Theorem 3.5, Figure 1, Theorem 4.1, Lemma 4.5, Theorem 5.2, Theorem 5.4, Theorem 5.5 and Theorem 5.6. On the other hand, we also show that the verification scheme is not unrealistically strong, as illustrated by Theorem 4.3 and Theorem 5.5.

References

[1] E. Balas, S. Ceria, and G. Cornuéjols, *A lift-and-project cutting plane algorithm for mixed integer 0-1 programs*, Mathematical Programming **58** (1993), 295–324.

[2] V. Chvátal, *Edmonds polytopes and a hierarchy of combinatorial problems*, Discrete Mathematics **4** (1973), 305–337.

[3] V. Chvátal, W. Cook, and M. Hartmann, *On cutting-plane proofs in combinatorial optimization*, Linear Algebra and its Applications **114** (1989), 455–499.

[4] W. Cook, C. R. Coullard, and Gy. Turan, *On the complexity of cutting plane proof*, Mathematical Programming **47** (1990), 11–18.

[5] W. Cook and S. Dash, *On the matrix cut rank of polyhedra*, Mathematics of Operations Research **26** (2001), 19–30.

[6] G. Cornúejols and Y. Li, *Elementary Closures for Integer Programs*, Operations Research Letters **28** (2001), 1–8.

[7] G. Cornuéjols and Y. Li, *On the rank of mixed 0-1 polyhedra*, Mathematical Programming **91** (2002), 391–397.

[8] D. Dadush, S. S. Dey, and J. P. Vielma, *The Chvátal-Gomory Closure of Strictly Convex Body*, `http://www.optimization-online.org/DB_HTML/2010/05/2608.html`, 2010.

[9] S. Dash, S. S. Dey, and O. Günlük, *Two dimensional lattice-free cuts and asymmetric disjunctions for mixed-integer polyhedra*, `http://www.optimization-online.org/DB_HTML/2010/03/2582.html`, 2010.

[10] S. S. Dey and J.-P. P. Richard, *Linear programming-based lifting and its application to primal cutting plane algorithms*, INFORMS Journal on Computing **21** (2009), 137–150.

[11] S. S. Dey and L. A. Wolsey, *Two row mixed integer cuts via lifting*, Mathematical Programming **124** (2010), 143–174.

[12] F. Eisenbrand and A. S. Schulz, *Bounds on the Chvátal rank of polytopes in the 0/1-cube*, Combinatorica **23** (2003), 245–262.

[13] M. Fischetti, A. Lodi, and D. Salvagnin, *Just MIP it!*, Annals of Informations Systems **10** (2009).

[14] R. E. Gomory, *Outline of an algorithm for integer solutions to linear programs*, Bulletin of the American Mathematical Society **64** (1958), 275–278.

[15] J. E. Kelley, *The cutting plane method for solving convex programs*, Journal of the SIAM **8** (1960), 703– 712.

[16] L. Lovász and A. Schrijver, *Cones of matrices and set-functions and 0-1 optimization*, SIAM Journal on Optimization **1** (1991), 166–190.

[17] G. L. Nemhauser and L. A. Wolsey, *Integer and combinatorial optimization*, Wiley-Interscience, 1988.

[18] S. Pokutta and A. S. Schulz, *Characterization of integer-free 0/1 polytopes with maximal rank*, Working paper.

[19] ———, *On the rank of generic cutting-plane proof systems*, Integer Programming and Combinatorial Optimization, 14th International IPCO Conference, Lausanne, Switzerland, June 9-11, 2010, Proceedings, Lecture Notes in Computer Science, Springer (F. Eisenbrand and B. Shepherd, eds.), 2010, pp. 450– 463.

[20] S. Pokutta and G. Stauffer, *A new lower bound technique for the Gomory-Chvátal procedure*, `http://www.optimization-online.org/DB_HTML/2010/09/2748.html`, 2010.