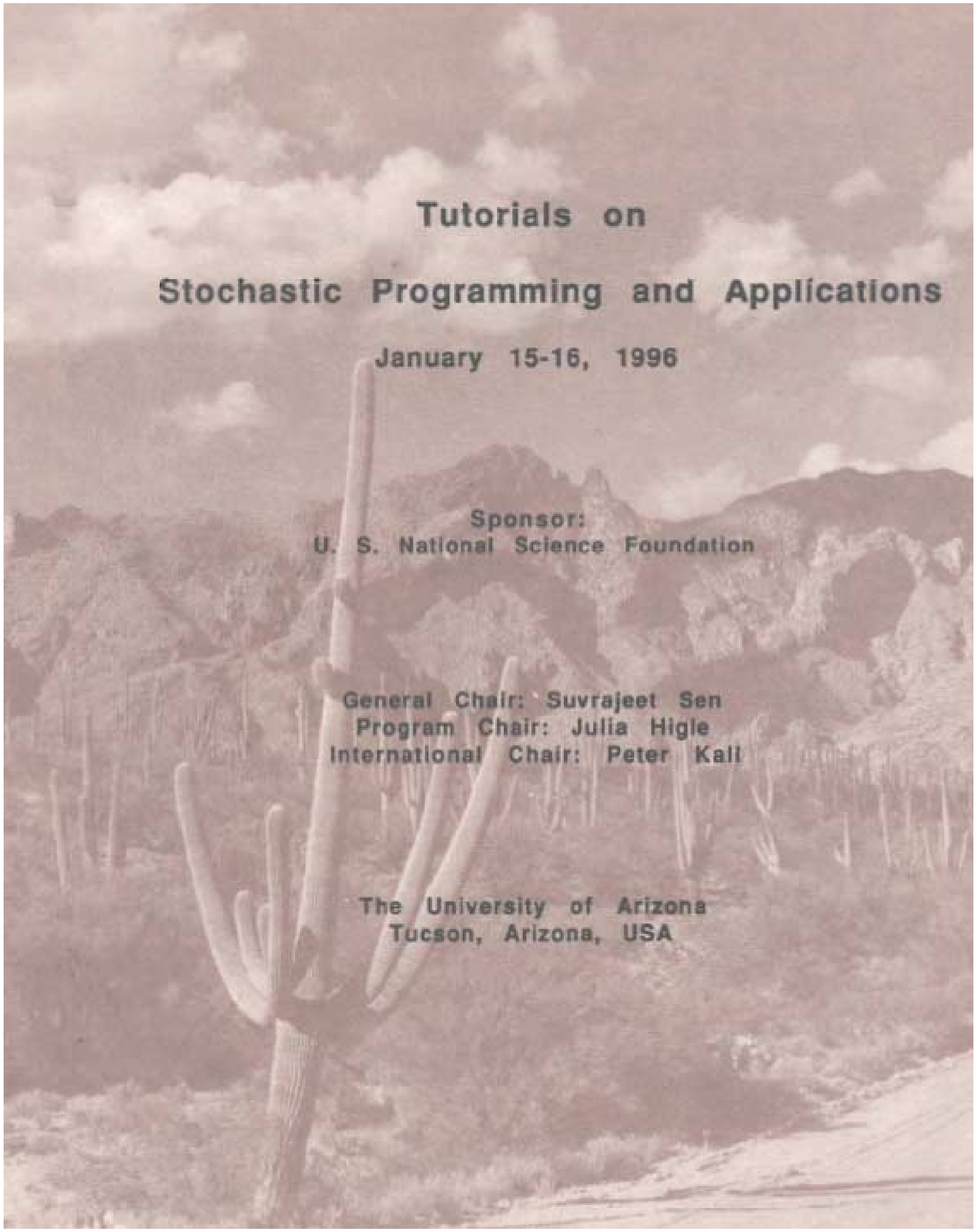
Stochastic Integer Programming
An Algorithmic Perspective

Shabbir Ahmed

sahmed@isye.gatech.edu

www.isye.gatech.edu/~sahmed

Georgia Institute of Technology
School of Industrial & Systems Engineering



Tutorials on

Stochastic Programming and Applications

January 15-16, 1996

Sponsor:
U. S. National Science Foundation

General Chair: Suvrajeet Sen
Program Chair: Julia Higle
International Chair: Peter Kall

The University of Arizona
Tucson, Arizona, USA

Outline

Two-stage SIP
- Formulation
- Challenges
- Simple Integer Recourse
    - Structural results
- General integer recourse
    - A few Decomposition algorithms

Multi-stage SIP
- Formulation
- Algorithms
    - Scenario decomposition
    - Polyhedral results

Two-Stage SIPs

- Decisions in two stages
  Stage 1 decision → Observe uncertainties → Stage 2 decision
  (“here and now”) (“recourse”)

- Known distribution
  The probability distribution of the uncertainties is known

- Exogenous uncertainties
  Stage 1 decisions do not affect the distribution

- Discrete/Combinatorial decisions

- Goal: Minimize cost of stage 1 decisions + Expected cost of stage 2 decisions

Examples

- Resource acquisition (Dempster et al.,1981,1983)

  Acquire machines → Observe processing times → Schedule jobs.

- Location-Routing (Laporte et al., 1994)

  Locate depots → Observe demand → Route vehicles.

- Ground Holding in Airline Operations (Ball et al., 2003)

  Schedule arrival/departure → Observe delays → Decide optimal holding pattern.

General Formulation

Decide $\mathbf{x}$ → Observe ω → Decide $\mathbf{y}(\omega)$

$$\begin{array}{ll} \min & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \mathbb{E}[Q(\mathbf{x},\omega)] \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \end{array}$$

$$\begin{array}{rll} Q(\mathbf{x},\omega) = & \min & \mathbf{q}(\omega)^T\mathbf{y} \\ & \text{s.t.} & W\mathbf{y} \geq \mathbf{h}(\omega) - T(\omega)\mathbf{x} \\ & & \mathbf{y} \in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2} \end{array}$$

Difficulty 1

- Evaluating the second-stage cost $Q(\mathbf{x}, \omega)$ for a fixed first-stage decision $\mathbf{x}$ and a particular realization ω of the uncertain parameters.
- Involves solving (possibly) NP-hard integer program
- E.g. Second stage: schedule jobs after observing processing requirements.
- Most SIP research assumes away this difficulty.

Difficulty 2

- Evaluating the expected second-stage cost for a fixed first-stage decision.
  - If the uncertain parameters have a continuous distribution:

    $$\mathbb{E}[Q(\mathbf{x},\omega)] = \int_{\omega\in\Omega} Q(\mathbf{x},\omega)dP(\omega)$$

    involves integrating the value function of an integer program and is in general impossible.
  - If the uncertain parameters have a discrete distribution:

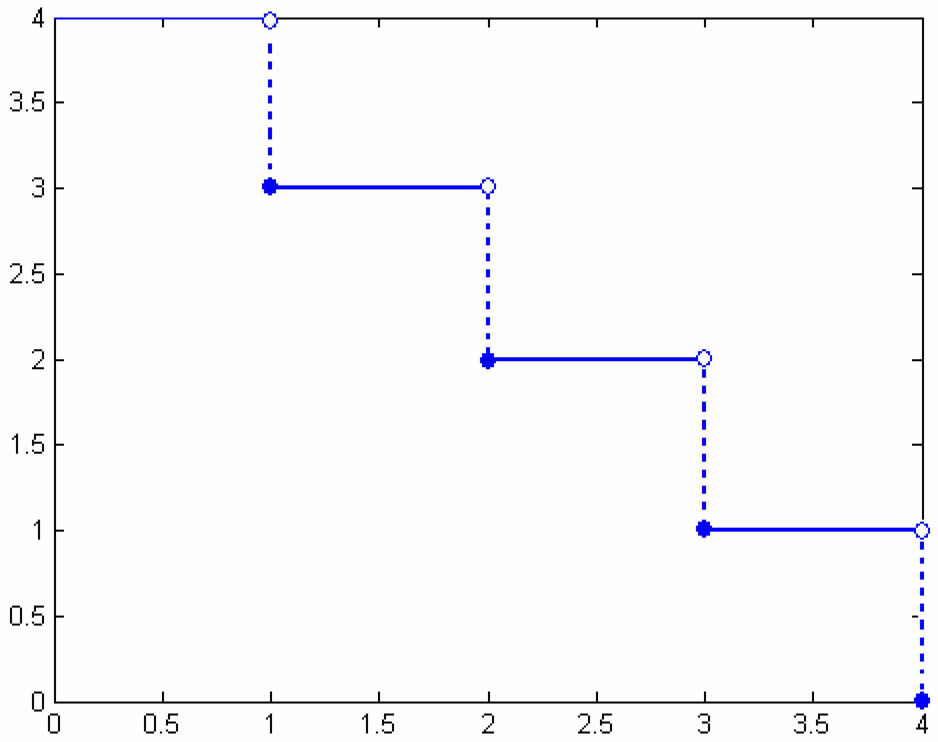
    $$\mathbb{E}[Q(\mathbf{x},\omega)] = \sum_{k=1}^{K} p_k Q(\mathbf{x},\omega^k)$$

    if $\omega \in \mathbb{R}^{100}$ and each component has 3 independent realizations, then $K = 3^{100} \approx 5 \times 10^{47}$!!

    involves solving a huge number of similar integer programs.

Difficulty 3

- $Q(\mathbf{x},\omega)$ is the value function of an integer program
- Is non-convex and discontinuous (lower-semicontinuous)



$$Q(x,4) = \min\{y : \; y \geq 4 - x, \; y \in \mathbb{Z}_+\}$$

Difficulty 3

- Optimizing $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \mathbb{E}[Q(\mathbf{x},\omega)]$, with respect to $\mathbf{x} \in X$.

Theorem (Stougie 1985; Schultz 1993,1995)

If $-\infty < Q(\mathbf{x},\omega) < +\infty$ for all $\mathbf{x}$ and ω, and $\mathbb{E}[|\omega|] < \infty$ then $\mathbb{E}[Q(\mathbf{x},\omega)]$ is real-valued and lower-semicontinuous on $\mathbb{R}^{n_1}$.

If , in addition, ω has an absolutely continuous density, then $\mathbb{E}[Q(\mathbf{x},\omega)]$ is continuous on $\mathbb{R}^{n_1}$.

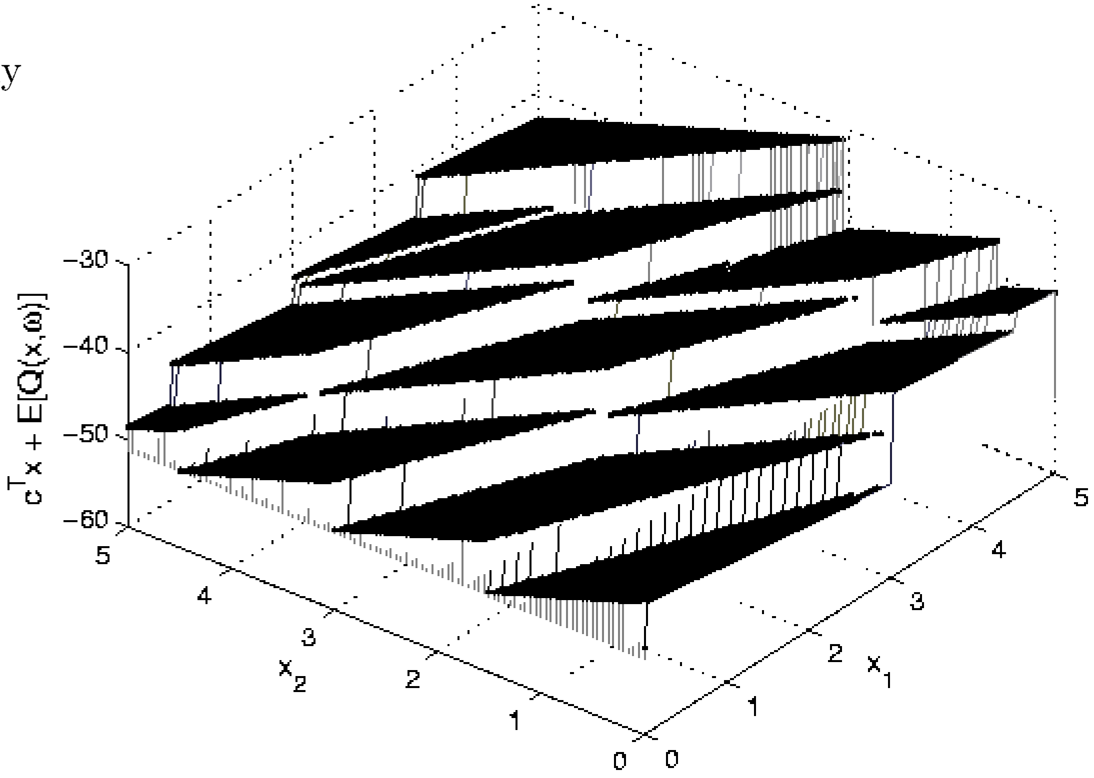
- In general, $\mathbb{E}[Q(\mathbf{x},\omega)]$ is non-convex and often discontinuous, and therefore, so is $f(\mathbf{x})$.

Non-convexity and Discontinuity

$$\min\{-1.5x_1 - 4x_2 + \mathbb{E}[Q(\mathbf{x},\omega)] : \ 0 \leq x_1 \leq 5, \ 0 \leq x_2 \leq 5\}$$

$$\begin{array}{lll} Q(\mathbf{x},\omega) = & \min & -16y_1 - 19y_2 - 23y_3 - 28y_4 \\ & \text{s.t.} & 2y_1 + 3y_2 + 4y_3 + 5y_4 \leq \omega_1 - \frac{1}{3}x_1 - \frac{2}{3}x_2 \\ & & 6y_1 + y_2 + 3y_3 + 2y_4 \leq \omega_2 - \frac{2}{3}x_1 - \frac{1}{3}x2 \\ & & y_1, y_2, y_3, y_4 \in \{0,1\} \end{array}$$

$$\omega \in \{5, 15\} \times \{5, 15\} \quad \text{uniformly}$$



Simple Integer Recourse (Stochastic RHS)

$$\begin{array}{ll} \min & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \mathbb{E}[Q(\mathbf{x},\omega)] \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \end{array}$$

$$\begin{array}{rll} Q(\mathbf{x},\omega) = & \min & (\mathbf{q}^+)^T\mathbf{y}_+ + (\mathbf{q}^-)^T\mathbf{y}_- \\ & \text{s.t.} & \mathbf{y}^+ \geq \omega - T\mathbf{x} \\ & & \mathbf{y}^- \geq -(\omega - T\mathbf{x}) \\ & & \mathbf{y}_+, \mathbf{y}_- \in \mathbb{Z}^{n_2} \end{array}$$

Dealing with the difficulties (SIR)

- No Difficulty 1

  $$Q(\mathbf{x},\omega) = \sum_{j=1}^{n_2} \Big(q_j^+ \lceil \omega_j - T_j\mathbf{x} \rceil^+ + q_j^- \lceil T_j\mathbf{x} - \omega_j \rceil^+ \Big)$$

  where $\lceil s \rceil^+ = \max\{0, \lceil s \rceil\}$.

- Let $g_j(z) = \mathbb{E}[\lceil \omega_j - z \rceil^+]$ and $h_j(z) = \mathbb{E}[\lceil z - \omega_j \rceil^+]$ then

  $$\mathbb{E}[Q(\mathbf{x},\omega)] = \sum_{j=1}^{n_2} \Big(q_j^+ g_j(T_j\mathbf{x}) + q_j^- h_j(T_j\mathbf{x}) \Big)$$

- Dealing with Difficulty 2: if we know how to evaluate the univariate functions $g_j(z)$ and $h_j(z)$, we are done.

Dealing with Difficulty 2 (SIR)

Theorem (Louveaux and van der Vlerk, 1993)

$$g(z) = \sum_{k=0}^{\infty} \Pr\{\omega > z + k\} \quad \text{and} \quad h(z) = \sum_{k=0}^{\infty} \Pr\{\omega < z - k\}$$

- In many cases, the above sums are easy to evaluate.
- Separability allows for the easy evaluation of $\mathbb{E}[Q(\mathbf{x}, \omega)]$.
- Difficulty 2 resolved.

Dealing with Difficulty 3 (SIR)

Let $\psi_j(z) = q_j^+ g(z) + q_j^- h(z) \quad j = 1, \ldots, n_2$ then

$$\mathbb{E}[Q(\mathbf{x}, \omega)] = \sum_{j=1}^{n_2} \Big(\psi_j(T_j \mathbf{x})\Big)$$

In general, $\psi(z)$ is not convex.

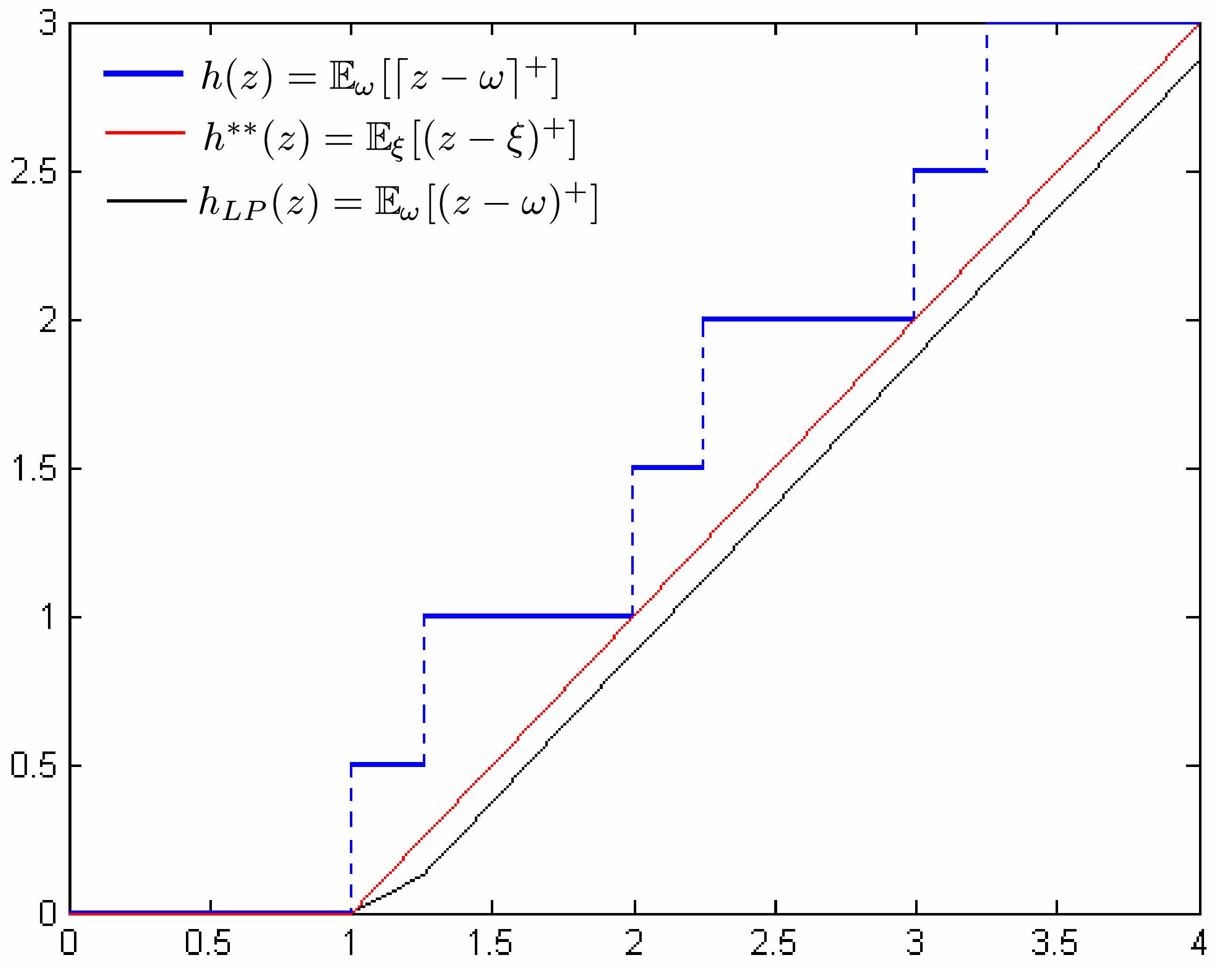
Theorem (Klein Haneveld et al., 1995)

There exists a random variable ξ such that

$$\psi^{**}(z) = q^+ \mathbb{E}_\xi \left[(\xi - z)^+\right] + q^- \mathbb{E}_\xi \left[(z - \xi)^+\right] + C$$

Here ψ^{**} denotes the convex hull of a function ψ over its entire domain.

Example



$\omega = 1$ w.p. $1/2$ and 1.25 w.p. $1/2$

$\xi = 1$ w.p. $1.$

Convexification

- Klein Haneveld et al (1995) give an algorithm for constructing the convex hull in case of discrete distributions.

Theorem (Klein Haneveld, 1995)

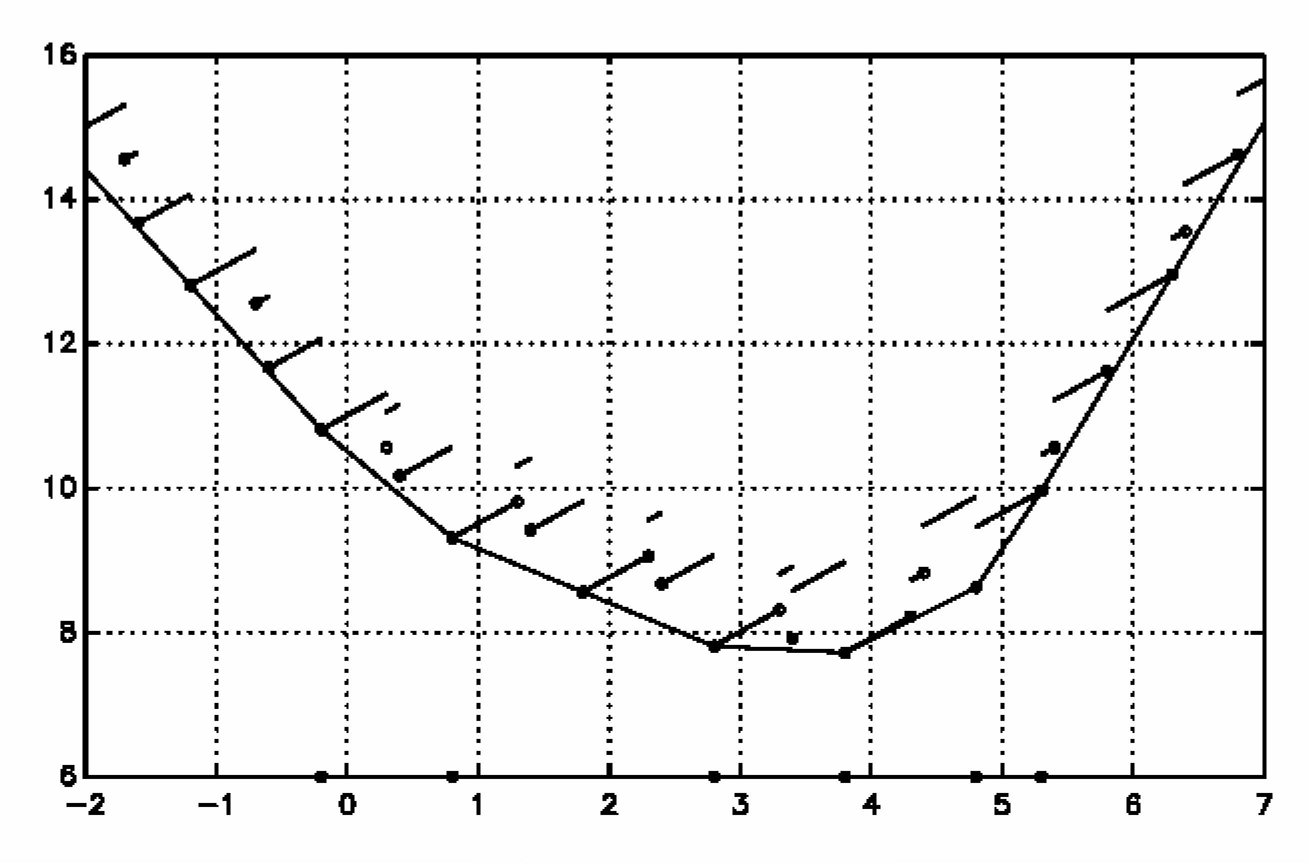
If T is of full row rank then

$$f^{**}(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \sum_{j=1}^{n_2}\left(\psi_j^{**}(T_j\mathbf{x})\right)$$

- In some cases $\min_{\mathbf{x}\in X} f(\mathbf{x}) \equiv \min_{\mathbf{x}\in X} f^{**}(\mathbf{x})$

- Then, we only need to solve a problem with continuous simple recourse

Example



$f(\mathbf{x})$ and $f^{**}(\mathbf{x})$

Klein Haneveld et al., 1995

Approximations

- Similar results: by perturbing the distribution, convex (continuous simple recourse type) approximations/lower bounding functions for the SIR function can be obtained.
  - Uniform error bound on the approximation

- Solve SIPs with SIR approximately or obtain lower bounds via solving continuous recourse models.
  - Can we use these within a branch and bound scheme?
  - Can we get convex hull/lower bound restricted to a subset of the domain?

General Mixed-Integer Recourse

- Assume that the recourse function is well-defined.

- Dealing with Difficulty 1
  - Assume that the second stage MIPs are “easily” solvable exactly.
  - Some literature on using approximations (e.g. Dempster 1983).

- Dealing with Difficulty 2
  - Approximate the distribution by a discrete distribution with a “manageable” number of realizations, e.g., by sampling.
  - Still need to solve several similar MIPs.

(One way of) Dealing with Difficulty 2
The Sample Average Approximation Method

- Need to solve $\min\{\mathbf{c}^T\mathbf{x} + \mathbb{E}[Q(\mathbf{x},\omega)] : \ \mathbf{x} \in X\}$

- Let S^ϵ be the set of ϵ-optimal solutions and v^* be the optimal objective value.

- Generate i.i.d samples $\{\omega^1, \ldots, \omega^N\}$ and solve the Sample Average Approximating (SAA) Problem:

$$\min\{\mathbf{c}^T\mathbf{x} + \frac{1}{N}\sum_{j=1}^{N} Q(\mathbf{x},\omega^j) : \ \mathbf{x} \in X\}$$

- Let S_N^δ be the set of δ-optimal solutions (with $\delta < \epsilon$) and v_N be the optimal objective value.

Convergence of the SAA Method

Theorem (Kleywegt et al. 2001)

If X is finite then $\lim_{N\to\infty} v_N = v^*$ and $\lim_{N\to\infty} \Pr\left(S_N^\delta \subseteq S^\epsilon\right) = 1$.

Moreover the convergence is exponentially fast.

The sample size needed to obtain an ϵ -optimal solution to the true problem with probability $1-\alpha$ is

$$N \propto \frac{1}{(\epsilon-\delta)^2} \log \frac{|X|}{\alpha}$$

Convergence of the SAA Method

Theorem (Ahmed and Shapiro, 2002)

If $X \subseteq \mathbb{R}^{n_1}$ is bounded (not necessarily finite) then the sample size needed to obtain an ϵ-optimal solution to the true problem with probability $1-\alpha$ is

$$N \propto \frac{n_1}{(\epsilon-\delta)^2} \log \frac{C}{\alpha(\epsilon-\delta)}$$

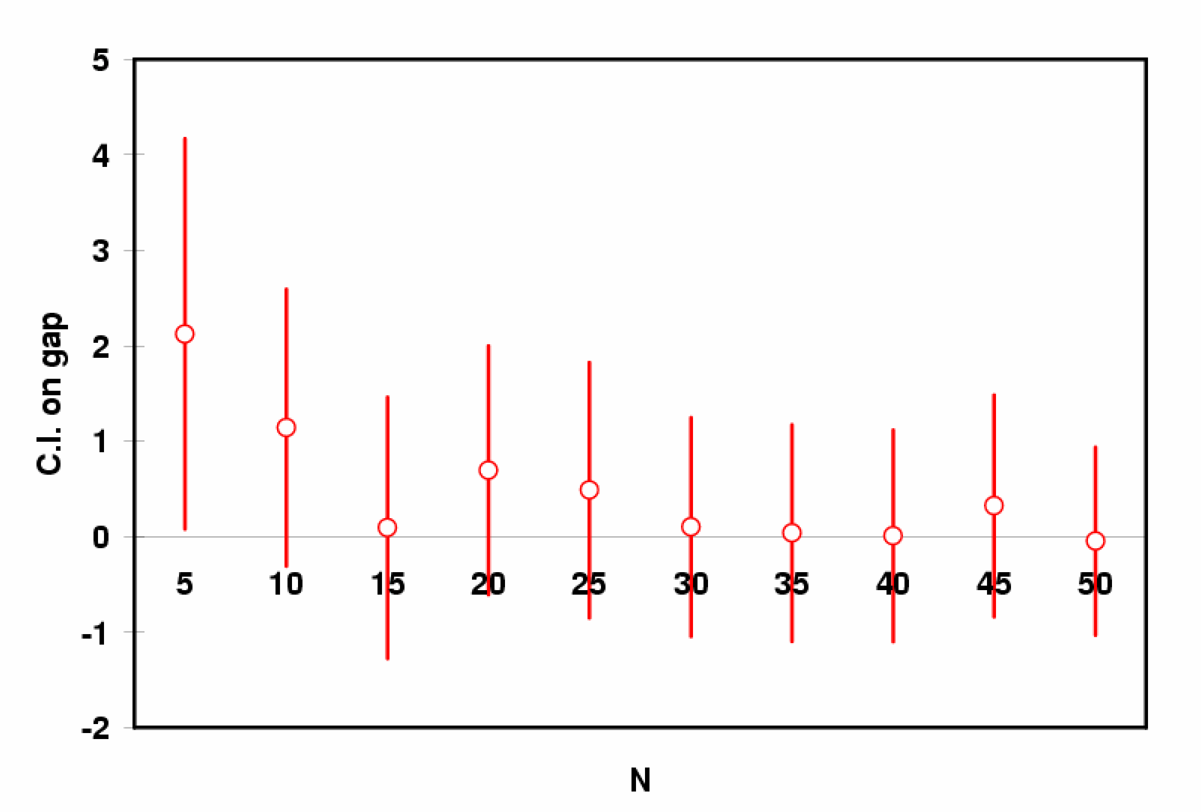
For pure integer recourse, only right hand-side uncertainty, and discrete distribution with K scenarios, the sample size needed to obtain an ϵ-optimal solution to the true problem with probability $1-\alpha$ is

$$N \propto \frac{C}{(\epsilon-\delta)^2} \log \frac{K n_1}{\alpha}$$

Practical SAA Method

- Select a sample size N, solve M independent SAA problems.
- Let v_N^i and $\mathbf{x}_N^i$ be the optimal value and optimal solution of the i-th SAA problem.
- A point estimate of a lower bound on v^* is given by $M^{-1} \sum_{i=1}^{M} v_N^i$ (Mak et al, 1999)
- A point estimate of the objective value of a candidate solution $\mathbf{x}_N^i$ is $\mathbf{c}^T \mathbf{x}_N^i + N'^{-1} \sum_{j=1}^{N'} Q(\mathbf{x}, \omega^j)$ where N' is a large sample.
- Then a point estimate of the optimality gap of the candidate solution $\mathbf{x}_N^i$ is obtained.
- Variability in the point estimates can be used to obtain confidence intervals.

Sample SAA Computation



Two-stage SIP with 10^8 scenarios.

Solving Similar IPs

- Still need to solve many “similar” MIPs.
- In case of stochastic linear programming, this problem is tackled using various warm-start strategies arising from exploiting LP duality.
- Unfortunately, a computationally useful IP duality theory is not yet mature.
- Two of the approaches in the SIP literature for pure integer second-stage:
  - Gröbner Basis (Schultz et al., 1998)
  - Value function construction (Kong et al., 2004)

Dealing with Difficulty 3

- Assume a finite ("manageable") number of scenarios.

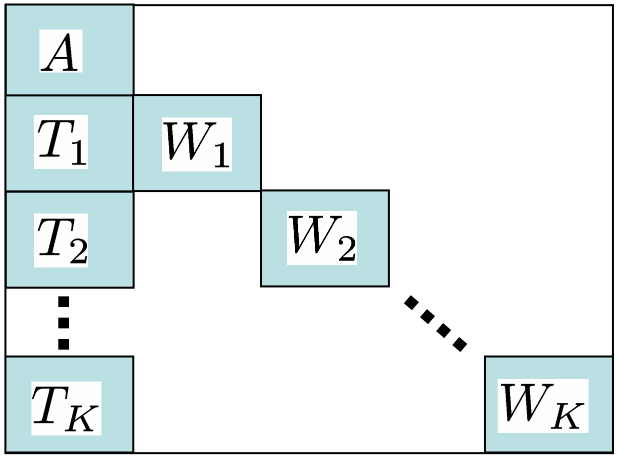
$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} + Q(\mathbf{x}) \\ \text{s.t.} & A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \end{array}$$

where $Q(\mathbf{x}) = \sum_{k=1}^{K} p_k Q_k(\mathbf{x})$

and $$Q_k(\mathbf{x}) = \begin{array}{ll} \min & \mathbf{q}_k^T\mathbf{y} \\ & W_k\mathbf{y} \geq \mathbf{h}_k - T_k\mathbf{x} \\ & \mathbf{y} \in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2} \end{array}$$

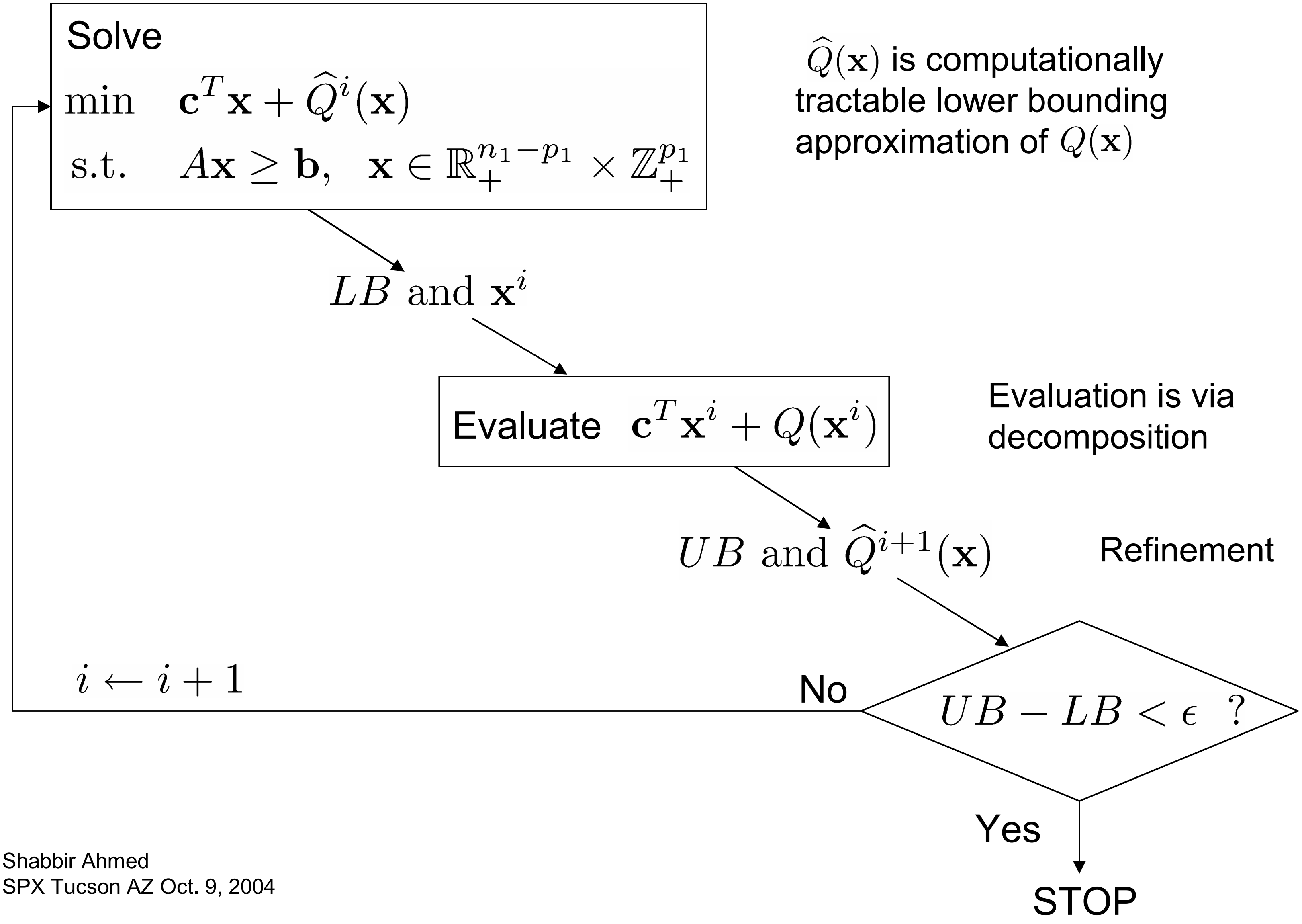
Deterministic Equivalent

$$\begin{array}{lll}
\min & \mathbf{c}^T\mathbf{x} + \sum_{k=1}^{K} p_k \mathbf{q}_k^T \mathbf{y}_k & \\
\text{s.t.} & A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} & \\
& T_k\mathbf{x} + W_k\mathbf{y}_k \geq \mathbf{h}_k & k = 1, \ldots, K \\
& \mathbf{y}_k \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2} & k = 1, \ldots, K
\end{array}$$



- Large-scale MIP.
- If not too many scenarios, use, e.g., CPLEX.
- Otherwise decompose

Two-Stage Decomposition



Solve

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} + \widehat{Q}^i(\mathbf{x}) \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \end{aligned}$$

$\widehat{Q}(\mathbf{x})$ is computationally tractable lower bounding approximation of $Q(\mathbf{x})$

LB and $\mathbf{x}^i$

Evaluate $\mathbf{c}^T\mathbf{x}^i + Q(\mathbf{x}^i)$

Evaluation is via decomposition

UB and $\widehat{Q}^{i+1}(\mathbf{x})$

Refinement

$UB - LB < \epsilon$?

No

$i \leftarrow i + 1$

Yes

STOP

Binary First-stage

Integer L-Shaped Method (Laporte & Louveaux, 1993)

Here $X = \{\mathbf{x}^1, \ldots, \mathbf{x}^M\} \subset \{0,1\}^{n_1}$ Let $L \leq Q(\mathbf{x}) \ \ \forall \ \mathbf{x} \in X$

Denote $J^m = \{j : \ x_j^m = 1\} \ \ m = 1, \ldots, M.$

Let

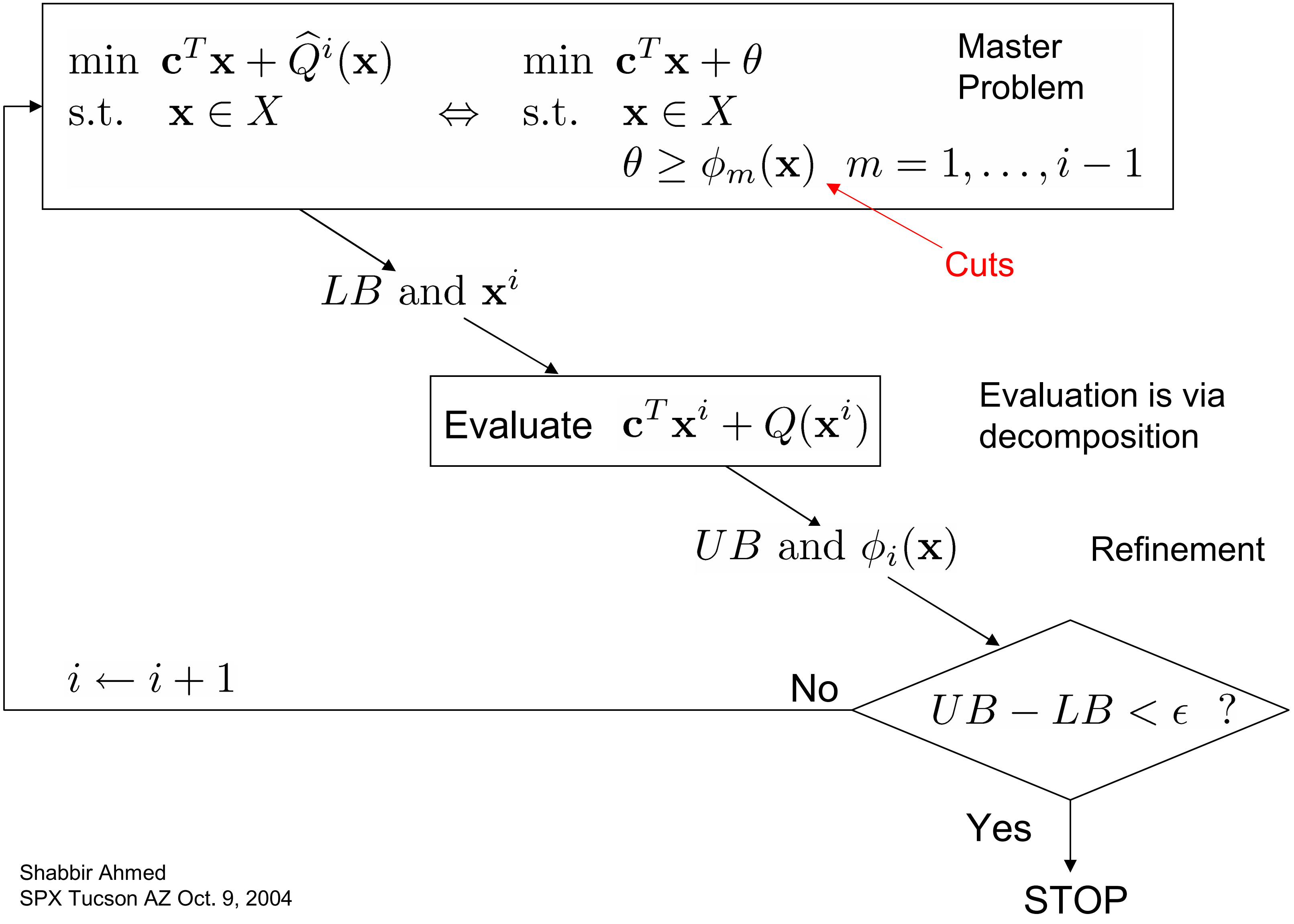
$$\phi_m(\mathbf{x}) = (Q(\mathbf{x}^m) - L)\Big(\sum_{j \in J^m} x_j - \sum_{j \notin J^m} x_j\Big) - (Q(\mathbf{x}^m) - L)(|J^m| - 1) + L$$

Note $\phi_m(\mathbf{x}) \begin{cases} = Q(\mathbf{x}^m) & \text{if} \ \ \mathbf{x} = \mathbf{x}^m \\ \leq L & \text{if} \ \ \mathbf{x} \in X \setminus \{\mathbf{x}^m\} \end{cases}$

Then $Q(\mathbf{x}) = \max_{m=1,\ldots,M} \{\phi_m(\mathbf{x})\} \ \ \forall \ \mathbf{x} \in X$

and $\widehat{Q}^i(\mathbf{x}) = \max_{m=1,\ldots,i-1} \{\phi_m(\mathbf{x})\}$

Integer L-Shaped Method



$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} + \widehat{Q}^i(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in X \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} + \theta \\ \text{s.t.} & \mathbf{x} \in X \\ & \theta \geq \phi_m(\mathbf{x}) \quad m = 1, \ldots, i-1 \end{array}$$

Master Problem

Cuts

LB and $\mathbf{x}^i$

Evaluate $\mathbf{c}^T\mathbf{x}^i + Q(\mathbf{x}^i)$

Evaluation is via decomposition

UB and $\phi_i(\mathbf{x})$

Refinement

$UB - LB < \epsilon$?

No

$i \leftarrow i + 1$

Yes

STOP

Integer L-Shaped (Remarks)

- Master problem is a 0-1 MIP … solve by branch & bound.
- (Decomposed) MIP subproblems in the evaluation steps.
- Implementation: Do not B&B to optimality … branch-and-cut.
  - Add cuts whenever a binary solution is encountered in the B&B search.
- Cut quality depends on the quality of the lower bound.
  - Can be improved if more information is available on the value function.
- Other types of valid cuts can also be added.
  - Standard LP Benders cuts are valid, but weak.
- Finite termination guaranteed.
- Application: Stochastic vehicle routing (Laporte et al. 2002)

Disjunctive Decomposition

Sen and Higle (2000)

- Binary first-stage, Mixed-binary second-stage and fixed recourse.
- Goal
  - Avoid solving MIP subproblems during evaluation.
  - Exploit similarity of subproblems.

- Given $\mathbf{x}^i$, for each k solve the LP-relaxation. If solution is fractional, find a valid inequality $\alpha^T\mathbf{y} \geq \beta$ for the LP.

$$
\begin{aligned}
Q_k(\mathbf{x}^i) \geq \quad \min \quad & \mathbf{q}_k^T\mathbf{y} \\
& W\mathbf{y} \geq \mathbf{h}_k - T_k\mathbf{x}^i \\
& \alpha^T\mathbf{y} \geq \beta \\
& \mathbf{y} \in \mathbb{R}_+^{n_2}
\end{aligned}
$$

Only valid for current subproblem

The C3 Theorem

Sen and Higle (2000)

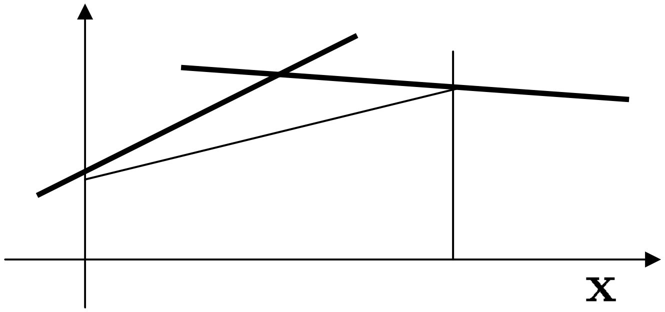
- There exists a function β such that the cut can be translated to be valid for the subproblem corresponding to any scenario and any first-stage solution.

$$\begin{aligned} Q_k(\mathbf{x}) \geq \quad \min \quad & \mathbf{q}_k^T \mathbf{y} \\ & W\mathbf{y} \geq \mathbf{h}_k - T_k \mathbf{x} \\ & \alpha^T \mathbf{y} \geq \beta(k, \mathbf{x}) \\ & \mathbf{y} \in \mathbb{R}_+^{n_2} \end{aligned}$$

- The cut-coefficients do not change (“common”).
- However, β is piece-wise linear and concave in $\mathbf{x}$.

Convexification

- Since only binary first-stage solutions are relevant, convexify (linearize) the concave function β .



- RHS linear in first-stage variables. Pass Benders cuts to master.

$$
\begin{aligned}
Q_k(\mathbf{x}) \geq \quad \min \quad & \mathbf{q}_k^T \mathbf{y} \\
& W\mathbf{y} \geq \mathbf{h}_k - T_k \mathbf{x} \\
& D\mathbf{y} \geq \mathbf{d}_k - E_k \mathbf{x} \\
& \mathbf{y} \in \mathbb{R}_+^{n_2}
\end{aligned}
$$

The D2 Algorithm (Remarks)

- As long as we have a separation scheme for finding “proper” valid inequalities for the subproblems, the algorithm terminates in a finite number of steps with the optimum.
- Application: Server location (Ntaimo and Sen, 2003)
- Sen and Sherali (2004) extend the approach to when the second-stage problems are partially solved by Branch-and-cut.

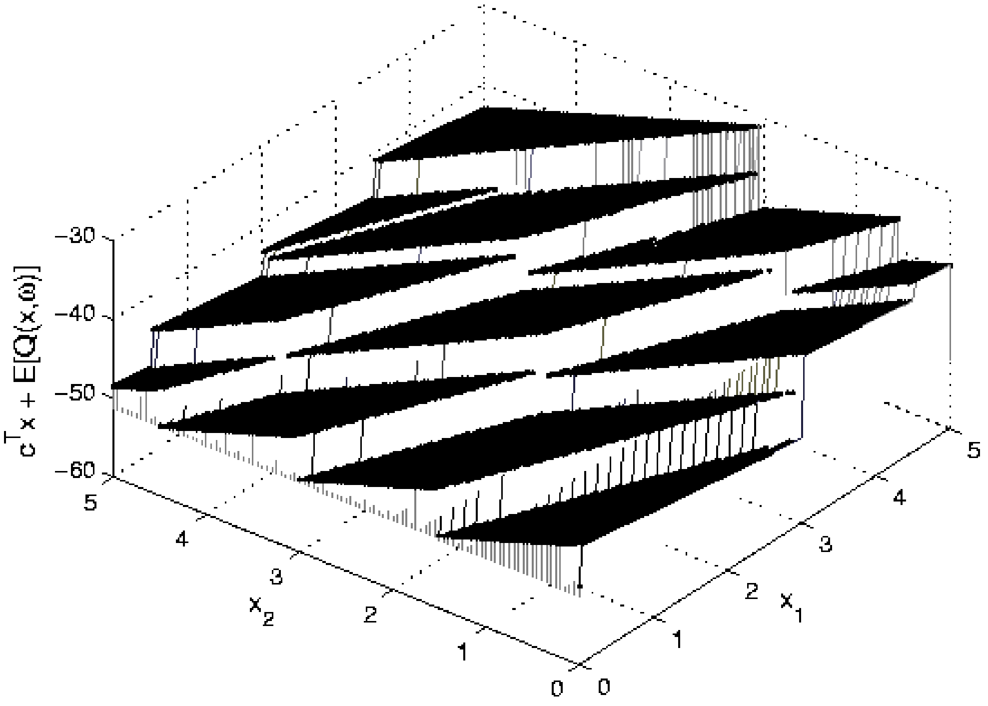
Mixed-Integer First Stage

- Binary first-stage ⇒ Need to evaluate/approximate value function only at binary solutions (finiteness inherent).
- Mixed-integer first-stage ⇒ Optimize a non-convex discontinuous objective over a (semi)-continuous domain.
- If Pure integer Second-stage:

Theorem (Schultz et al.1998)

The expected second-stage value function is piece-wise constant (over polyhedral regions), and an optimal solution to the problem lies at an extreme point of one of these polyhedra.

Pure Integer Second Stage



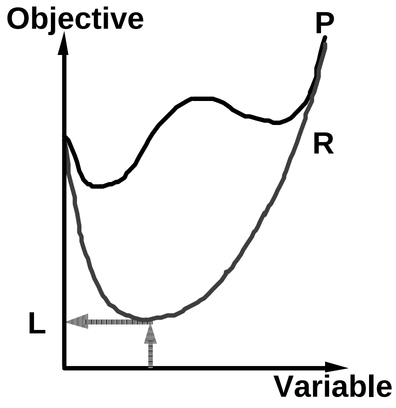
$$\min\{-1.5x_1 - 4x_2 + \mathbb{E}[Q(\mathbf{x},\omega)] :\ 0 \le x_1 \le 5,\ 0 \le x_2 \le 5\}$$

$$\begin{array}{lll} Q(\mathbf{x},\omega) = & \min & -16y_1 - 19y_2 - 23y_3 - 28y_4 \\ & \text{s.t.} & 2y_1 + 3y_2 + 4y_3 + 5y_4 \le \omega_1 - \frac{1}{3}x_1 - \frac{2}{3}x_2 \\ & & 6y_1 + y_2 + 3y_3 + 2y_4 \le \omega_2 - \frac{2}{3}x_1 - \frac{1}{3}x2 \\ & & y_1, y_2, y_3, y_4 \in \{0,1\} \end{array}$$

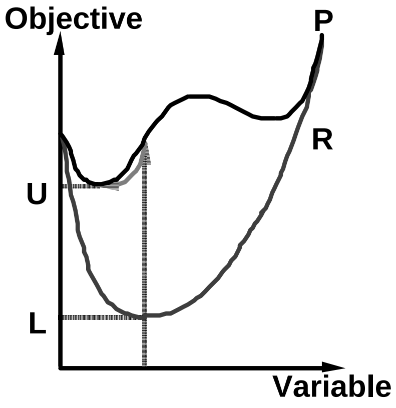
$$\omega \in \{5,15\} \times \{5,15\} \ \ \text{uniformly}$$

- Inherent finiteness.
- Algorithm: Check all these extreme points.
- Difficulties
  - Polyhedra not easy to characterize.
  - May be too many.
- Alternative: Use continuous branch and bound.

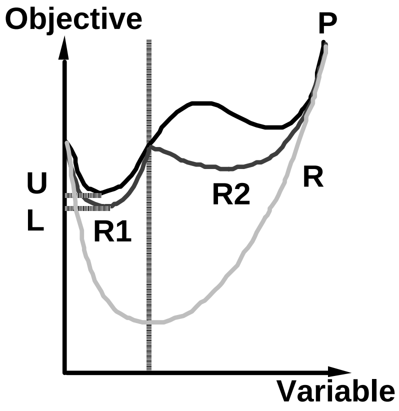
Continuous Branch & Bound



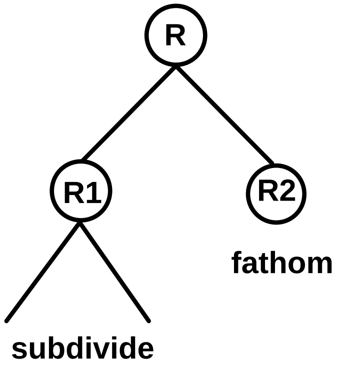
(a) Lower bounding



(b) Upper bounding

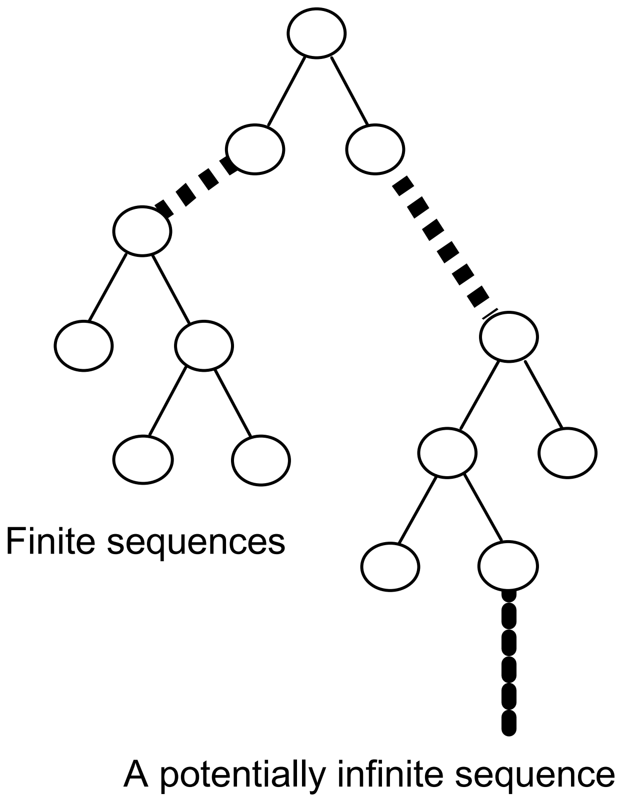


(c) Domain subdivision



(d) Search tree

Finiteness Issue

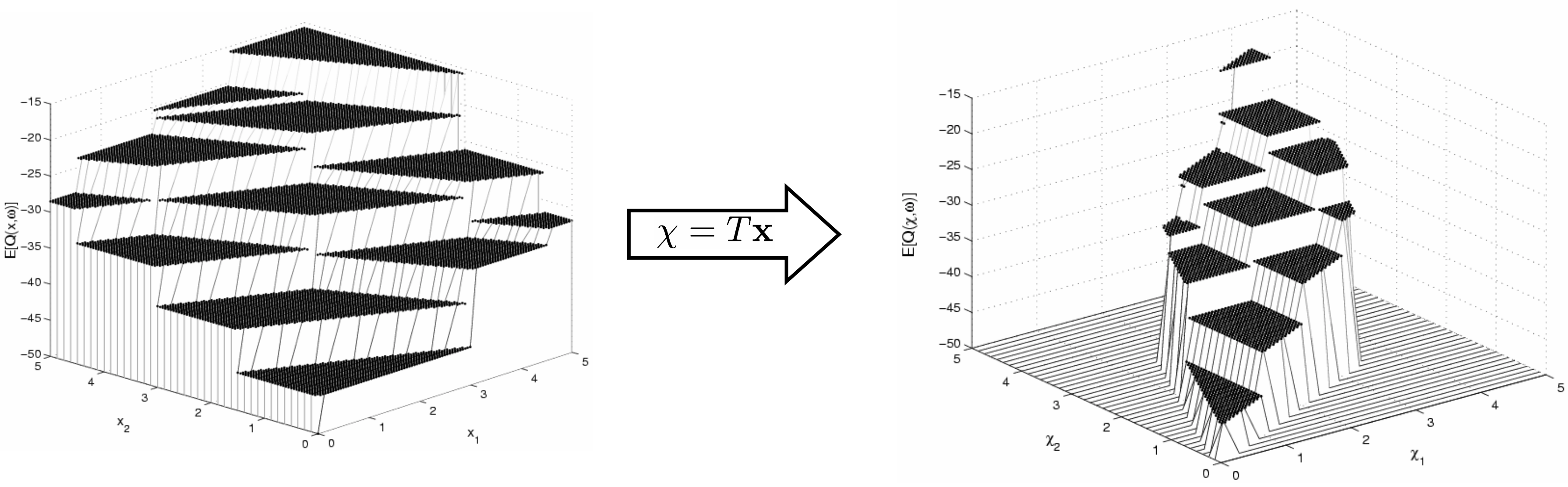


- The most common branching scheme is rectangular partitioning - branching along a variable axis.
- The resulting partitions are rectangular.
- The polyhedral shaped discontinuous pieces cannot be isolated by a finite number of rectangular partitions.
- There will be some partitions containing discontinuities.
- These will require infinite partitioning for bounds to converge.

Pure Integer Second Stage

B&B in the tender space (Ahmed et al., 2004)

- Fixed technology matrix.
- Solve the problem in the space of the tender variables.
- Discontinuous are orthogonal to the tender axes.



$\chi = T\mathbf{x}$

$$\min\{\mathbf{c}^T\mathbf{x} + \sum_{k=1}^{K} p_k Q_k(\mathbf{x}) : \ x \in X\}$$

$$Q_k(\mathbf{x}) = \min\{q_k^T\mathbf{y} : \ W_k\mathbf{y} \geq \mathbf{h}_k + T\mathbf{x}, \ \mathbf{y} \in \mathbb{Z}^{n_2}\}$$

$$\min\{\phi(\chi) + \sum_{k=1}^{K} p_k \psi_k(\chi) : \ \chi \in \Xi\}$$

$$\phi(\chi) = \min\{\mathbf{c}^T\mathbf{x} : \ \mathbf{x} \in X, \ T\mathbf{x} = \chi\}$$

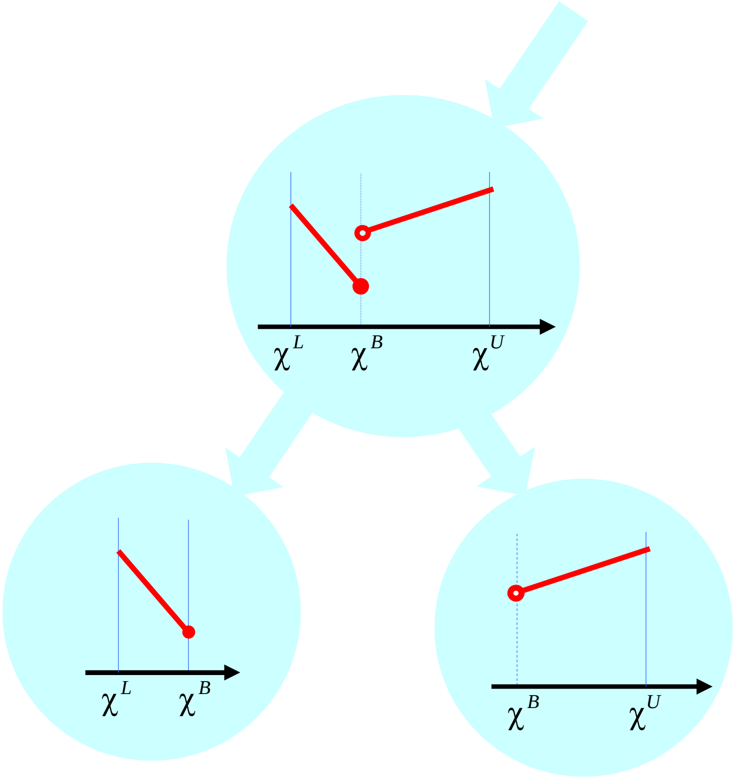
$$\psi_k(\chi) = \min\{\mathbf{q}_k^T\mathbf{y} : \ W_k\mathbf{y} \geq \mathbf{h}_k + \chi, \ \mathbf{y} \in \mathbb{Z}^{n_2}\}$$

The B&B Algorithm

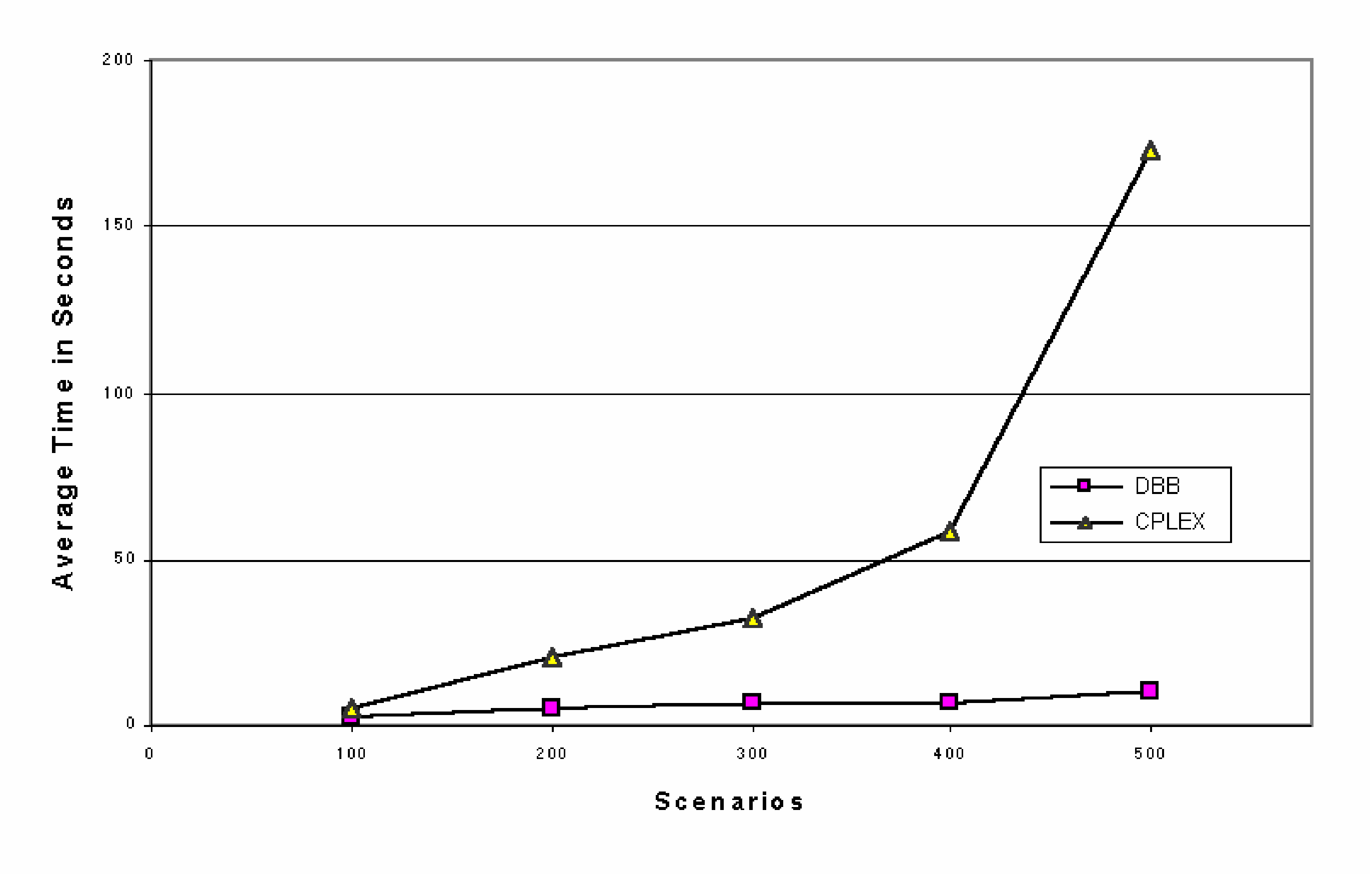
- Lower bounding: Second-stage value function is non-decreasing and lsc.

$$\psi_k(\chi) \geq \psi_k(\chi^L + \epsilon) \quad \forall\, \chi \in [\chi^L, \chi^U]$$

- Upper bounding: Function evaluation.
- Branching: Partition along the discontinuities. Maintains rectangular partitions.
- Finite convergence to global optima.
- Improved lower bounding methods.



Sample Computation



- Test set: Capacity acquisition-assignment problems.
- Ahmed and Garcia, 2003.

More on Pure Integer Second Stage

- Hemmecke and Schultz, 2003
  - Rhs uncertainty. Pure integer first stage.
  - IP Test Sets (Computational Algebra).
- van der Vlerk, 2004
  - Rhs uncertainty.
  - Convex lower bounds by changing the distribution.
- Kong et al., 2004
  - Rhs uncertainty. Pure integer first stage.
  - Construct and optimize value function.
- Kong et al. 2004
  - Conditions for total unimodularity.
  - Benders: MIP master may be better than LP master.

Multi-Stage SIPs

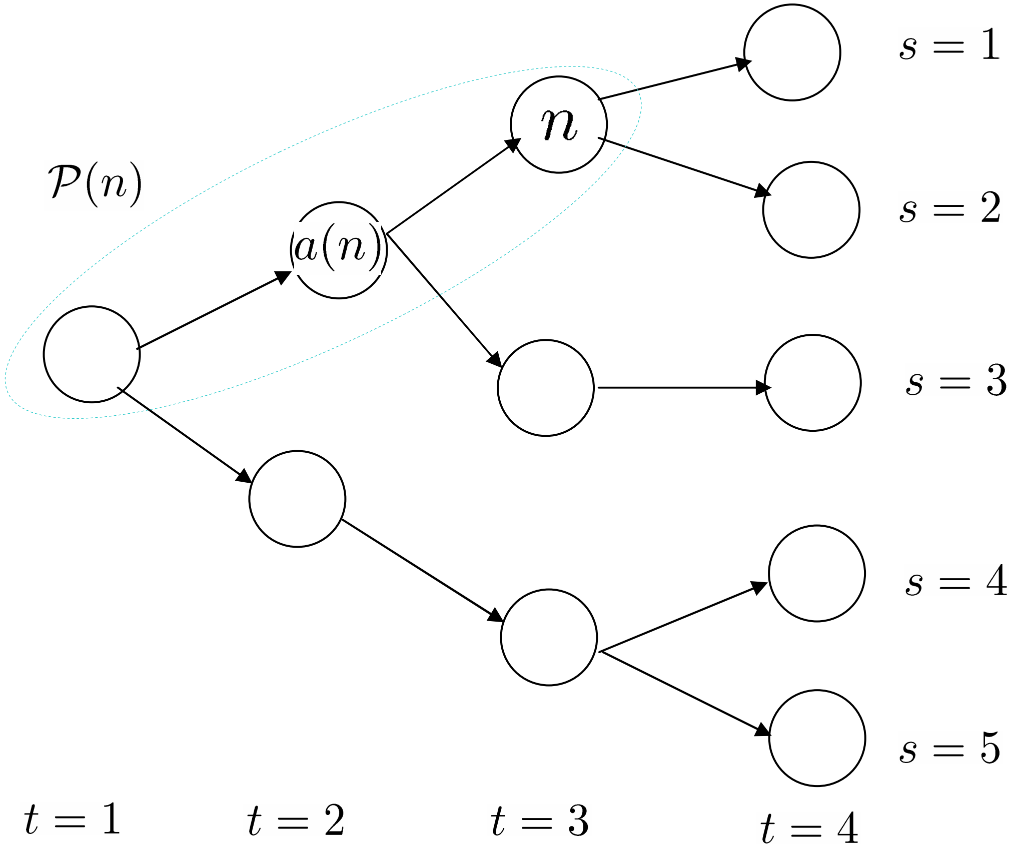
- Decisions in multiple (but, a finite number of) stages

Stage 1 decision → Observe uncertainties → Stage 2 decision → Observe uncertainties → Stage 3 decision → …..

- Example Applications:
  - Unit commitment (Takriti et al. 1996, Caroe & Schultz, 1999).
  - Capacity/Production planning (Ahmed et al. 2001, Lulli and Sen, 2002).
  - Asset liability management (Drijver et al., 2000).
- Challenges: Same as before … but now in multiple-folds!

The Scenario Tree

- Assuming finite support, the evolution of the uncertain parameters can be modeled as a scenario tree.

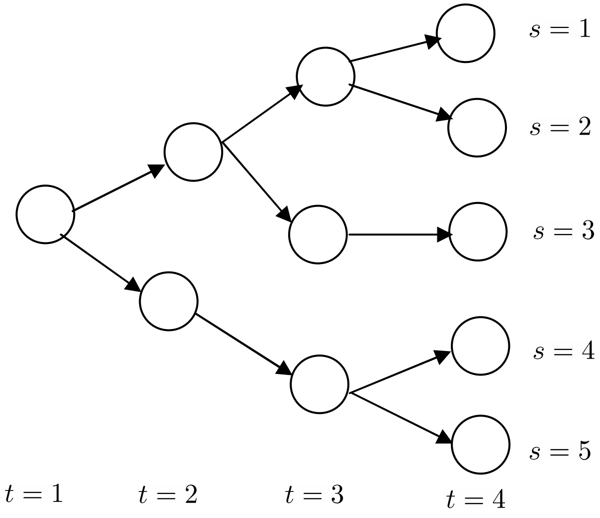


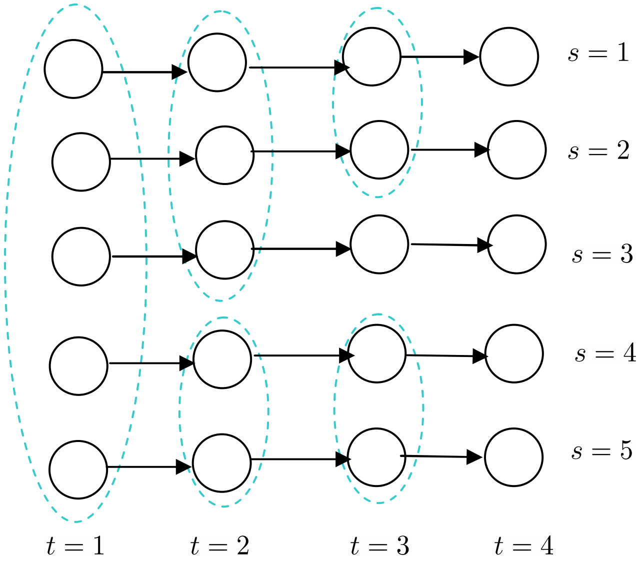
A Formulation

- Deterministic equivalent formulation.
- Solution at a node depends on the solutions in ancestor nodes.
- Index according to node – a tree formulation.

$$
\begin{aligned}
\min \quad & \sum_{n \in \mathcal{T}} p_n \mathbf{c}_n^T \mathbf{x}_n \\
\text{s.t.} \quad & \sum_{m \in \mathcal{P}(n)} A_{mn} \mathbf{x}_m \geq \mathbf{h}_n \qquad && n \in \mathcal{T} \\
& \mathbf{x}_n \in X_n \subseteq \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \qquad && n \in \mathcal{T}
\end{aligned}
$$

Scenario Decomposition





The Scenario Formulation

$$\begin{aligned}
\min \quad & \sum_{s=1}^{S} \sum_{t=1}^{T} p_s \mathbf{c}_t^s \mathbf{x}_t^s \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} A_{\tau,t}^s \mathbf{x}_\tau^s \geq \mathbf{h}_t^s \qquad \forall\ t, \quad \forall\ s \\
& \mathbf{x}_t^s \in X_t^s \subseteq \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \qquad \forall\ t, \quad \forall\ s \\
& H\mathbf{x} = 0
\end{aligned}$$

F^s

Non-anticipativity constraints

Lagrangian Relaxation

$$(D) \quad \max_{\lambda} \quad L(\lambda) = \quad \sum_{s=1}^{S} \left(\min_{\mathbf{x}^s \in F^s} \quad \sum_{t=1}^{T} p_s \mathbf{c}_t^s(\lambda) \mathbf{x}_t^s \right)$$

- Any feasible solution to the Lagrangian dual provides a lower bound to the true optimal value.
- Evaluating the dual function requires solving one deterministic problem per scenario (Decomposition).
- Dual involves maximizing a concave non-smooth function … can be solved using non-smooth optimization techniques.
- Difficulty: Many dual multipliers.
- Difficulty: Duality Gap $v_D < v_P$.

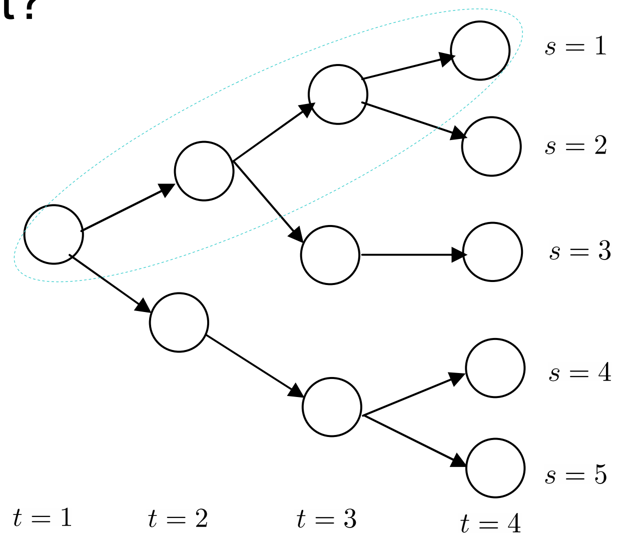
Dual Decomposition

Caroe and Schultz (1999)

- Use Lagrangian dual as the lower bounding scheme within a branch and bound algorithm.
- Branch to enforce non-anticipativity.
- Finite termination in case of pure integer solutions.
- Applicable to two-stage stochastic integer programs.
- Application: Unit commitment problem.

Polyhedral Results

- Given valid inequalities for a deterministic MIP, can we find a valid inequality for the stochastic counterpart?
- Generating "tree" inequalities from "path" inequalities.
- Branch and cut schemes for deterministic equivalent.
- Tighten subproblems within decomposition based branch and cut schemes.



Example: Uncapacitated Lot-sizing

$$\begin{array}{lll} \min & \sum_{t=1}^{T}(\alpha_t x_t + \beta_t y_t + \delta_t s_t) & \\ \text{s.t.} & s_{t-1} + x_t = d_t + s_t & t = 1, \ldots, T \\ & x_t \leq M y_t & t = 1, \ldots, T \\ & x_t \geq 0, \quad y_t \in \{0,1\} & t = 1, \ldots, T. \end{array}$$

Given $\ell \in \{1, \ldots, T\}$ and $S \subseteq \{1, \ldots, \ell\}$, let $\overline{S} = \{1, \ldots, \ell\} \setminus S$, and $d_{ij} = \sum_{t=i}^{j} d_t$. Then, the following (ℓ, S)-inequality is valid

$$\sum_{t \in S} x_t + \sum_{t \in \overline{S}} d_{t\ell} y_t \geq d_{1\ell}.$$

- Common substructure in many production planning problems
- (I,S) inequalities sufficient to describe convex hull.
- Exponential family … polynomially separable.

Stochastic Uncapacitated Lot-sizing

Guan et al. (2004)

$$\begin{array}{lll} \min & \sum_{n \in \mathcal{T}} (\alpha_n x_n + \beta_n y_n + \delta_n s_n) & \\ \text{s.t.} & s_{a(n)} + x_n = d_n + s_n & n \in \mathcal{T} \\ & x_n \leq M y_n & n \in \mathcal{T} \\ & x_n \geq 0, \quad y_n \in \{0,1\} & n \in \mathcal{T}. \end{array}$$

- Given any subset of the nodes, the corresponding (l,S) inequalities are valid.
- These inequalities can be “combined” to generate a new family of inequalities.
- Necessary and sufficient conditions for the inequalities to be facet-defining.
- Excellent performance within branch & cut.
- The combining idea is quite general and can be applied to other SIPs.

Concluding Remarks

- Would have liked to talk about Approximation Algorithms for SIP.

- Survey articles on SIP:
  - Klein Haneveld and van der Vlerk (1998)
  - Louveaux and Schultz (2003)
  - Schultz et al. (1995)
  - Sen (2004)

- WWW Resources:
  - SP Community Page: http://www.stoprog.org
  - Bibliography (2003): http://mally.eco.rug.nl/biblio/SIP.HTML
  - Test Problems: http://www.isye.gatech.edu/~sahmed/siplib/

References

S. Ahmed and A. Shapiro. The sample average approximation method for stochastic programs with integer recourse. Optimization Online, http://www.optimization-online.org, 2002.

S. Ahmed and R. Garcia. Dynamic Capacity Acquisition and Assignment under Uncertainty. *Annals of Operations Research*, 124:267-283, 2003

S. Ahmed, M. Tawarmalani, and N. V. Sahinidis. A finite branch and bound algorithm for two-stage stochastic integer programs. *Mathematical Programming*, 100:355-377, 2004.

M. O. Ball, R. Hoffman, A. R. Odoni, and R. Rifkin. A stochastic integer program with dual network structure and its application to the ground-holding problem. *Operations Research*, 51(1):167-171, 2003.

C. C. Carøe and R. Schultz. Dual decomposition in stochastic integer programming. *Operations Research Letters*, 24(1-2):37-45, 1999.

M.A.H. Dempster, M.L. Fisher, L. Jansen, B.J. Lageweg, J.K. Lenstra, and A.H.G. Rinnooy Kan. Analytical evaluation of hierarchical planning systems. *Operations Research*, 29:707-716, 1981.

M.A.H. Dempster, M.L. Fisher, L. Jansen, B.J. Lageweg, J.K. Lenstra, and A.H.G. Rinnooy Kan. Analysis of heuristics for stochastic programming: results for hierarchical scheduling problems. *Mathematics of Operations Research*, 8:525-537, 1983.

R. Hemmecke and R. Schultz. Decomposition of test sets in stochastic integer programming. *Mathematical Programming,* 94:323-341, 2003.

Y. Guan, S. Ahmed and G.L. Nemhauser. A branch-and-cut algorithm for the stochastic uncapacitated lot-sizing problem. Stochastic Programming E-Print Series, 2004.

W.K. Klein Haneveld, L. Stougie and M.H. van der Vlerk. On the convex hull of the simple integer recourse objective function , *Annals of Operations Research*, 56:209-224, 1995.

W.K. Klein Haneveld and M. H. van der Vlerk. Stochastic integer programming: general models and algorithms. *Ann. Oper. Res.*, 85:39-57, 1999.

A.J. Kleywegt, A. Shapiro, and T. Homem-de-Mello. The Sample Average Approximation Method for Stochastic Discrete Optimization. *SIAM Journal on Optimization*, 12:479-502, 2001.

N. Kong, A.J. Schaefer and B. Hunsaker. Two-Stage Integer Programs with Stochastic Right-Hand Sides: A Superadditive Dual Approach, Technical Report, University of Pittsburgh, 2004.

F.V. Louveaux and R. Schultz. Stochastic Integer Programming. Chapter 4 in Handbooks in Operations Research and Management Science, Vol. 10, Stochastic Programming (Ruszczynski and Shapiro etd.), Elsevier, 2003.

References

G. Laporte and F.V. Louveaux. The integer L-shaped method for stochastic integer programs with complete recourse. *Operations Research Letters*, 13:133-142, 1993.

G. Laporte, F.V. Louveaux, and L. van Hamme. Exact solution of a stochastic location problem by an integer L-shaped algorithm. *Transportation Science*, 28(2):95-103, 1994.

G. Laporte, F.V. Louveaux, and L. van Hamme. An integer L-shaped algorithm for the capacitated vehicle routing problem with stochastic demands. *Operations Research*, 50:415-423, 2002.

F. Louveaux and M.H. van der Vlerk. Stochastic programming with simple integer recourse, *Mathematical Programming*, 61(3):301-325, 1993.

W.K. Mak, D.P. Morton and R.K. Wood. Monte Carlo Bounding Techniques for Determining Solution Quality in Stochastic Programs. *Operations Research Letters,* 24:47-56, 1999.

L. Ntaimo and S. Sen. The Million-Variable “March” for Stochastic Combinatorial Optimization. To appear in the *Journal of Global Optimization*, 2004.

R. Schultz. Continuity properties of expectation functions in stochastic integer programming. *Mathematics of Operations Research*, 18(3):578-589, 1993.

R. Schultz. On structure and stability in stochastic programs with random technology matrix and complete integer recourse. *Mathematical Programming*, 70:73-89, 1995.

R. Schultz, L. Stougie, and M.H. van der Vlerk. Two-stage stochastic integer programming: a survey. *Statistica Neerlandica*, 50(3):404-416, 1996.

R. Schultz, L. Stougie and M.H. van der Vlerk. Solving stochastic programs with complete integer recourse: a framework using Groebner Bases , *Mathematical Programming*, 83(2):229-252, 1998.

S. Sen and J. Higle. The C3 Theorem and a D2 Algorithm for Large Scale Stochastic Mixed-Integer Programming: Set Convexification. Technical Report, University of AZ. 2003.

S. Sen and H. D. Sherali. Decomposition with Branch-and-Cut Approaches for Two Stage Stochastic Mixed-Integer Programming. Technical Report, University of AZ. 2004.

S. Sen. Algorithms for Stochastic Mixed-Integer Programming Models. Technical Report, University of AZ. 2004.

L. Stougie. *Design and analysis of algorithms for stochastic integer programming*, PhD Thesis, Center for Mathematics and Computer Science, Amsterdam, 1987.

M. H. van der Vlerk. Convex approximations for complete integer recourse models. *Mathematical Programming*, 99(2):297-310, 2004.